Rend. Sem. Mat. Univ. Pol. Torino
Vol. 59, 2 (2001)
Liaison and Rel. Top.

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GORENSTEIN POINTS IN $\mathbb{P}^3$

Abstract. After the structure theorem of Buchsbaum and Eisenbud [1] on Gorenstein ideals of codimension 3, much progress was made in this area from the algebraic point of view; in particular some characterizations of these ideals using $h-$vectors (Stanley [9]) and minimal resolutions (Diesel [3]) were given. On the other hand, the Liaison theory gives some tools to exploit, but, at the same time, it requires one to find, from the geometric point of view, new Gorenstein schemes.
The works of Geramita-Migliore [5] and Migliore-Nagel [6] present some constructions for Gorenstein schemes of codimension 3; in particular they deal with points in $\mathbb{P}^3$.
Starting from the work of Migliore and Nagel, we study their constructions and we give a new construction for points in $\mathbb{P}^3$: given a specific subset of a plane complete intersection, we add a "suitable" set of points on a line not in the plane and we obtain an aG zeroscheme that is not complete intersection. We emphasize the interesting fact that, by this method, we are able to "visualize" where these points live.

1. Introduction

It is well known, by the structure theorem of Buchsbaum and Eisenbud [1] and by the results of Diesel [3], what are all the possible sets of graded Betti numbers for Gorenstein artinian ideals of height 3. Geramita and Migliore, in their paper [5], show that every minimal free resolution which occurs for a Gorenstein artinian ideal of codimension 3, also occurs for some reduced set of points in $\mathbb{P}^3$, a stick figure curve in $\mathbb{P}^4$ and more generally a "generalized" stick figure in $\mathbb{P}^n$. On the other hand, Stanley [9] characterized the h-vectors of all the Artinian Gorenstein quotients of $k[x_0, x_1, x_2]$, showing that their h-vectors are SI-sequences and, viceversa, every SI-sequence $(1, h_1, \dots, h_{s-1}, 1)$, where $h_1 \leq 3$, is the h-vector of some Artinian Gorenstein scheme of codimension less than or equal to 3. In Section 2 we will see how Nagel and Migliore [6] found reduced sets of points in $\mathbb{P}^3$ which have h-vector $(1, 3, h_2, \dots, h_{s-2}, 3, 1)$.

In this case, the points in $\mathbb{P}^3$ solving the problems can be found as the intersection of two nice curves (stick figures) which have good properties. It is, however, very hard to see where these points live! We try to make the set of points found by these construction more visible.

In the last section we give some examples: we take a set of points, which come from Nagel-Migliore's construction (i.e. a reduced arithmetically Gorenstein zeroscheme not a Complete Intersection) and we study where this set lives. In particular, we have a nice description of Gorenstein point sets whose h-vector are of the form $(1, 3, 4, 5, \dots, n-1, n, n, \dots, n, n-1, \dots, 5, 4, 3, 1)$.

This allowed us to determine, in a way which is independent of the previous constructions, particular configurations of points which are reduced arithmetically Gorenstein zeroschemes not complete intersection.

2. Gorenstein points in $\mathbb{P}^3$ from the h-vector

In this section we will see how Nagel and Migliore find a reduced arithmetically Gorenstein zeroscheme in $\mathbb{P}^3$ (i.e. a reduced Gorenstein quotient of $k[x_0, x_1, x_2, x_3]$ of Krull dimension 1) with given h-vector.

We start with some basic definitions that we find in [6] and in [9].

DEFINITION 1. *Let $H = (h_0, h_1, \dots, h_i, \dots)$ be a finite sequence of non-negative integers. Then H is called an O-sequence if $h_0 = 1$ and $h_{i+1} \leq h_i^{<i>}$ for all i.*

By the Macaulay theorem we know that the O-sequences are the Hilbert functions of standard graded k-algebras.

DEFINITION 2. *Let $h = (1, h_1, \dots, h_{s-1}, 1)$ be a sequence of non-negative integers. Then h is an SI-sequence if:*

- *$h_i = h_{s-i}$ for all $i = 0, \dots, s$,*
- *$(h_0, h_1 - h_0, \dots, h_t - h_{t-1}, 0, \dots)$ is an O-sequence, where t is the greatest integer $\leq \frac{s}{2}$.*

Stanley [9] characterized the h-vectors of all graded Artinian Gorenstein quotients of $k[x_0, x_1, x_2]$, showing that these are SI-sequence and any SI-sequence, with $h_1 = 3$, is the h-vector of some Artinian Gorenstein quotient of $k[x_0, x_1, x_2]$.

Now we can see how Nagel and Migliore [6] find a reduced arithmetically Gorenstein zeroscheme in $\mathbb{P}^3$ with given $h-$vector. This set of points will result from the intersection of two arithmetically Cohen-Macaulay curves in $\mathbb{P}^3$, linked by a Complete Intersection curve which is a stick figure.

DEFINITION 3. *A* **generalized stick figure** *is a union of linear subvarieties of $\mathbb{P}^n$, of the same dimension d, such that the intersection of any three components has dimension at most $d - 2$ (the empty set has dimension -1).*

In particular, sets of reduced points are stick figure, and a stick figure of dimension $d = 1$ is nothing more than a reduced union of lines having only nodes as singularities.

So, let

$$h = (h_0, h_1, \dots, h_s) = (1, 3, h_2, \dots, h_{t-1}, h_t, h_t, \dots, h_t, h_{t-1}, \dots, h_2, 3, 1)$$

be a SI-sequence, and consider the first difference

$$\Delta h = (1, 2, h_2 - h_1, \dots, h_t - h_{t-1}, 0, 0, \dots, 0, h_{t-1} - h_t, \dots, -2, -1)$$

Define two sequences $a = (a_0, \dots, a_t)$ and $g = (g_0, \dots, g_{s+1})$ in the following way:

$$a_i = h_i - h_{i-1} \;\; for \; 0 \leq i \leq t$$

and

$$g_i = \begin{cases} i+1 & \text{for } 0 \le i \le t \\ t+1 & \text{for } t \le i \le s-t+1 \\ s-i+2 & \text{for } s-t+1 \le i \le s+1 \end{cases}$$

We observe that $a_1 = g_1 = 2$, a is a O-sequence since h is a SI-sequence and g is the h-vector of a codimension two Complete Intersection. So, we would like to find two curves C and X in $\mathbb{P}^3$ with h-vector respectively a and g. In particular it is easy to see that g is the h-vector of a Complete Intersection, X, of two surfaces in $\mathbb{P}^3$ of degree $t+1$ and $s-t+2$.

We can get X as a stick figure by taking as the equation of those surfaces two forms which are the product, respectively, of $A_0, \ldots, A_t$ and $B_0, \ldots, B_{s-t+1}$, all generic linear forms. Nagel and Migliore [6] proved that the stick figure (embedded in X), determined by the union of a_i consecutive lines in $A_i = 0$ (always the first in $B_0 = 0$), is an aCM scheme C with h-vector a. In this way, if we consider C', the residual of C in X, the intersection of C and C' is an aG scheme Y of codimension 3. This is also a reduced set of points because X, C and C' are stick figures and it has the desired h-vector by the following theorem:

THEOREM 1. *Let C, C', X, Y be as above. Let $g = (1, c, g_2, \ldots, g_s, g_{s+1})$ be the h-vector of X, let $a = (1, a_1, \ldots, a_t)$ and $b = (1, b_1, \ldots, b_l)$ be the h-vectors of C and C', then*

$$b_i = g_{s+1-i} - a_{s+1-i}$$

for $i \ge 0$. Moreover the sequence $d_i = a_i + b_i - g_i$ is the first difference of the h-vector of Y.

So we have to show that $d_i = h_i - h_{i-1}$:

- For $0 \le i \le t$ we have $d_i = a_i = h_i - h_{i-1}$
- For $t+1 \le i \le s-t$ we have $d_i = b_i - g_i = 0$
- For $s-t+1 \le i \le s+1$ we have $d_i = b_i - g_i = -a_{s+1-i} = -(h_{s+1-i} - h_{s-i}) = h_i - h_{i-1}$

REMARK 1. Theorem 1 says, for example, that there exists no cubic through the 8 points of a Complete Intersection of two cubics, but not through the nine. In fact, if we consider a reduced Complete Intersection zeroscheme X in $\mathbb{P}^2$ given by two forms of degree a and b, the $h-$vector of $X \setminus \{P\}$ is $(1, 2, 3, \ldots, a-1, a, a, \ldots, a, a, a-1, \ldots, 3, 2)$, whatever point P we cut off.

EXAMPLE 1. Let $h = (1, 3, 4, 3, 1)$ be a SI-sequence. Consider the first difference of h, i.e. $\Delta h = (1, 2, 1, -1, -2, -1)$.

So, $g = (1, 2, 3, 3, 2, 1)$ is the h-vector of X, stick figure which is the Complete Intersection of $F_1 = \prod_{i=0}^{2} A_i$ and $F_2 = \prod_{i=0}^{3} B_i$, where A_i and B_i are general linear forms.

Now, we call $P_{i,j}$ the intersection between $A_i = 0$ and $B_j = 0$. Then $C = P_{0,0} \cup P_{1,0} \cup P_{1,1} \cup P_{2,0}$ is the scheme which has h-vector $a = (1, 2, 1)$.

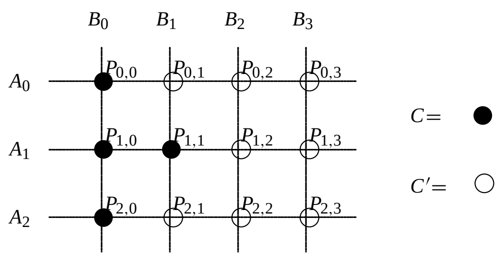

Figure 1

So, it is clear that the residual C' of C in X is the union of the lines of X which aren't components in C. Then the reduced set of points Y with h-vector $(1, 3, 4, 3, 1)$ consists of 12 points which exactly are:

- 3 points on $P_{0,0}$, intersection between $P_{0,0}$ and $P_{0,1}$, $P_{0,2}$ and $P_{0,3}$
- 2 points on $P_{1,0}$, intersection between $P_{1,0}$ and $P_{1,2}$, $P_{1,3}$
- 4 points on $P_{1,1}$, intersection between $P_{1,1}$ and $P_{1,2}$, $P_{1,3}$, $P_{0,1}$ and $P_{2,1}$
- 3 points on $P_{2,0}$, intersection between $P_{2,0}$ and $P_{2,1}$, $P_{2,2}$ and $P_{2,3}$

EXAMPLE 2. Let $h = (1, 3, 5, 3, 1)$. With the previous notations, we have that the first difference of h is $\Delta h = (1, 2, 2, -2, -2, -1)$, so $g = (1, 2, 3, 3, 2, 1)$. Hence, we can take a stick figure X which is a Complete Intersection between a cubic and a quartic.

Therefore, as above, we get a subscheme of X with h-vector $(1, 2, 2)$.

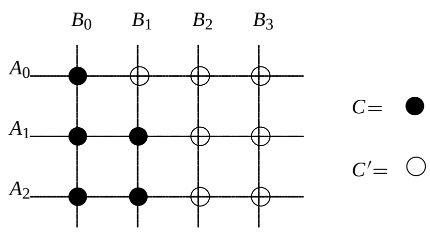

Figure 2

In this way, the intersection between C and the residual C' gives the reduced set of 13 points with the expected h-vector.

3. Gorenstein Sets of points not complete intersection

In this paragraph, we start visualizing some sets of points which result from the Migliore-Nagel construction. This construction has given an idea of how to build particular sets of points in $\mathbb{P}^3$ which are arithmetically Gorenstein zeroschemes and not Complete Intersections. For this purpose, we start from a careful analysis of Examples 1 and 2.

EXAMPLE 3. In example 1 we can see that the set Y of 12 points which realizes the h-vector $h = (1, 3, 4, 3, 1)$, has the following configuration: 3 points on $P_{0,0}$ (the intersection between $P_{0,0}$ and $P_{0,1}$, $P_{0,2}$, $P_{0,3}$), 2 points on $P_{1,0}$ (the intersection between $P_{1,0}$ and $P_{1,2}$, $P_{1,3}$), 3 points on $P_{2,0}$ (the intersection between $P_{2,0}$ and $P_{2,1}$, $P_{2,2}$ and $P_{2,3}$), 4 points on $P_{1,1}$ (intersection between $P_{1,1}$ and $P_{0,1}$, $P_{2,1}$, $P_{1,2}$, and $P_{1,3}$). So, we denote these points by

$$P_{i,j}^{k,l} = P_{i,j} \cap P_{k,l}.$$

We focus our attention on the plane B_0, where we consider 9 points: the intersections of the lines $P_{i,0}$ with the planes B_1,B_2,B_3.

So we have three triplets of points which are collinear, but also the triplets of the form $\{P_{i,j}^{i,l}\}$ $i = 1, 2, 3$ are collinear, because they live in the intersection between B_0 and B_i, $i = 1, 2, 3$. These points, except $P = P_{1,0}^{1,1}$, are in Y. Now, we consider $P_{1,1}$: this line is through P and is not in B_0. The remaining 4 points are the intersection between $P_{1,1}$ and A_0, A_2, B_2, B_3 and they are different from P. The union of all these points, except P, is our Gorenstein set Y.

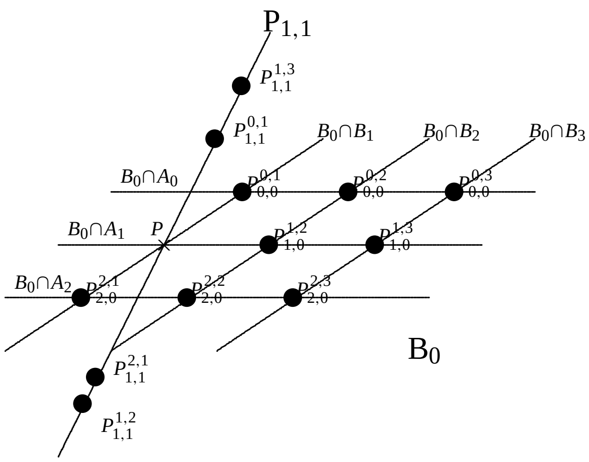

Figure 3

So, from that analysis we get a guess to construct a more visible Gorenstein set of 12 points. We start from a plane B_0 with 9 points which satisfies some relation of collinearity (as in Figure 3), we cut off a point, and we choose a line $r = P_{1,1}$ through this point and not in the plane. Notice that this is equivalent to say that we choose the planes A_1 and B_1. It is easy to see that we can choose the points on this line r randomly. This is due to the fact that, at this point of the Migliore-Nagel construction, each of the planes A_0, A_2, B_2, B_3, are defined by three collinear points (for example, A_0 is the plane through $P_{0,0}^{0,1}$, $P_{0,0}^{0,2}$, and $P_{0,0}^{0,3}$). In other words, if we start from Figure 3, the 9 points don't fix uniquely the planes A_0, A_2, B_2, and B_3, but they define 4 pencils of planes in which we can choose the previous planes.

EXAMPLE 4. Now, let's analyze Example 2 (a set of 13 points) and try to visualize this set as before. Here we have:

- 3 points on $P_{0,0}$, intersection between $P_{0,0}$ and $P_{0,1}$, $P_{0,2}$ and $P_{0,3}$
- 2 points on $P_{1,0}$, intersection between $P_{1,0}$ and $P_{1,2}$, $P_{1,3}$
- 3 points on $P_{1,1}$, intersection between $P_{1,1}$ and $P_{0,1}$, $P_{1,2}$, $P_{1,3}$
- 2 points on $P_{2,0}$, intersection between $P_{2,0}$ and $P_{2,2}$, $P_{2,3}$
- 3 points on $P_{2,1}$, intersection between $P_{2,1}$ and $P_{0,1}$ $P_{2,2}$, $P_{2,3}$

As in the previous example,we consider the 9 points in the plane B_0, but this time we have to cut off two points: $P := P^{1,1}_{1,0}$ and $Q := P^{2,1}_{2,0}$. After we take the lines $r := P_{1,1}$ and $s := P_{2,1}$ respectively through P and Q, we have to fix three points on each line: $P^{0,1}_{1,1}$, $P^{1,2}_{1,1}$, $P^{1,3}_{1,1}$ and $P^{0,1}_{2,1}$, $P^{2,2}_{2,1}$, $P^{2,3}_{2,1}$.

This time, we cannot randomly choose all the six points: in fact these points are given by the intersections of r and s with the planes A_0, B_2 and B_3. So if we randomly choose three points (for example in r), then the planes A_0, B_2 and B_3 are fixed, and the points on s too. The result appears as in the figure below:

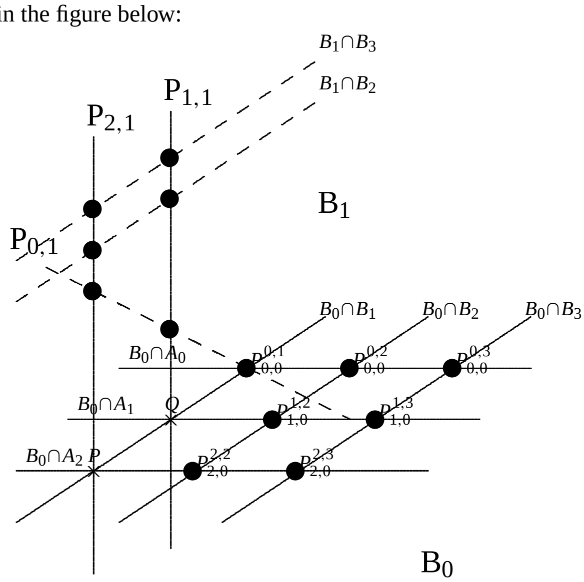

Figure 4

If we look carefully at the plane B_0 of the two examples, the 9 points are a Complete Intersection in $\mathbb{P}^3$ defined by three generators f, g, h where $\deg(f) = 1$, $\deg(g) = \deg(h) = 3$ and both g and h are products of three linear forms.

Obviously we can generalize this idea to bigger sets. We had to observe that, following Nagel-Migliore, we can always "picture" a Gorenstein set of points in $\mathbb{P}^3$, but we can do it with more or less freedom. This freedom depends on Y, or better, on its h-vector h_Y. In fact, we can say that if our SI-sequence is of the form $h_Y = (1, 3, 4, ..., t, t, t, t, ...4, 3, 1)$, with the hypothesis that all the entries of Δh, except $h_1 - h_0 = 2$, are equal to 0 or 1, then it is possible to find a particular plane Complete Intersection X of points and, after taking a line through a point $P \in X$ (and cut off this point) and a correct number of points different from P on the line, we obtain a Gorenstein set of points $Y \subset \mathbb{P}^3$ with h-vector h.

Now, suppose that the hypothesis on H_Y are verified. The next question is the following: is it possible to substitute the generators g, h by g', h' not products of linear forms ?

So we tried to take a generic complete intersection X of the form $(1, 3, 3)$; as before, we cut off a point P and we choose a set W of 4 points over a general line through P, not in the plane. Working with the h-vectors of X, P and W, we are able to prove that $Y = (X \cup W) \setminus \{P\}$ is again a Gorenstein set of Points, not a Complete Intersection, with h-vector $(1, 3, 4, 3, 1)$.

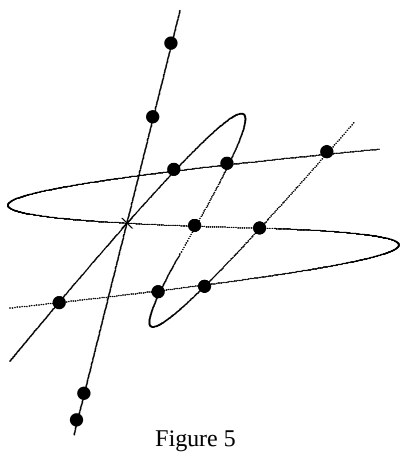

Figure 5

This fact gave us the idea for another generalization: what happens if we take a Complete Intersection of the form $(1, a, b)$ minus a point, and a set of points over a line through this point? Do we obtain a Gorenstein set of points?

We notice that this time, however, we don't start from an h-vector, but we search a new method to construct Gorenstein set of points not Complete Intersections.

The answer to the question is positive. To proof, we need of following result by Davis, Geramita and Orecchia [2]:

THEOREM 2. *Let I be the ideal of a set X of s distinct points in $\mathbb{P}^n$ and suppose that the Hilbert function of X has the first difference which is symmetric and that every subset of X having cardinality $s - 1$ has the same Hilbert function. Then the homogeneous coordinate ring of X is a Gorenstein ring.*

THEOREM 3. *Let $X \subset \mathbb{P}^3$ be a reduced Complete Intersections of the form $(1, a, b)$ and let $P \in X$ a point. Take a line L through P, not in the plane that contains X and fix a set Y of $a + b - 1$ distinct points on L, containing P. Define*

$$W := (X \cup Y) \setminus \{P\}.$$

Then W is an arithmetically Gorenstein zeroscheme.

Proof. Suppose $a \leq b$. Let $I_X = (F_1, F_2, F_3)$, where $\deg(F_1) = 1$, $\deg(F_2) = a$ and $\deg(F_3) = b$. The h-vector of the Complete Intersection X is

$$h_X = (1, 2, 3, \dots, a-1, a, a, \dots, a, a, a-1, \dots, 3, 2, 1),$$

where the two "$a - 1$" entries correspond to the forms of degrees $a - 2$ and b. So the length of h_X is $a + b - 1$. Let Y be the set of $a + b - 1$ points on L; I_Y will be (L_1, L_2, L_3), where $I_L = (L_1, L_2)$ and $\deg(L_3) = a + b - 1$. Since $P = X \cap Y$, we have $I_X + I_Y \subset I_P$. But $I_X + I_Y$ is $(F_1, F_2, F_3, L_1, L_2, L_3)$ and the $I_P = (L_1, L_2, F_1)$, so we have that $I_X + I_Y$ is the satured ideal I_P. Obviously, the h-vector of I_Y is $h_Y = (1, 1, \dots, 1, 1)$, because we have $a + b - 1$ points on a line. From the next exact sequence we can calculate the h-vector of $X \cup Y$:

$$0 \to I_X \cap I_Y \to \quad I_X \quad \oplus \quad I_Y \quad \to I_X + I_Y \to 0$$

I_X	I_Y	$I_X + I_Y$
1	1	1
2	1	0
$\vdots$	$\vdots$	$\vdots$
$a-1$	1	0
a	1	0
a	1	0
$\vdots$	$\vdots$	$\vdots$
a	1	0
a	1	0
$a-1$	1	0
$\vdots$	$\vdots$	$\vdots$
2	1	0
1	1	0

So, we obtain $h_{X \cup Y} = (1, 3, 4, 5, \dots, a, a+1, a+1, \dots, a+1, a+1, a, \dots, 5, 4, 3, 2)$. If we consider $X \cup Y \setminus \{P\} = W$, it has h-vector

$$(1, 3, 4, 5, \dots, a, a+1, a+1, \dots, a+1, a+1, a, \dots, 5, 4, 3, 1)$$

which is symmetric. In fact, suppose that the h-vector does not decrease at the last position. Then there is a form F of degree less than $a + b - 2$ which is zero on W but not on P. So, if we consider the curve given by $F = 0$ in the plane $F_1 = 0$, we have a form of degree less than $a + b - 2$ which is zero on all but one the points of a Complete Intersection (a, b), but this is not possible by Remark 1.

Now, we use Theorem 2 to prove that this set of points is Gorenstein. Cut a point off this set to obtain a set W': it is sufficient to prove that $h_{W'}$ is the same for any point we cut off. There are two possible cases:

1) the point is on the line $L_1 = 0$, $L_2 = 0$,
2) the point is on the plane $F_1 = 0$.

Case 1. Let $W' = W \setminus \{Q\}$, where $Q \in L \cap W$. The only possible h-vector for W' is

$$(1, 3, 4, 5, \ldots, a, a+1, a+1, \ldots, a+1, a+1, a, \ldots, 5, 4, 3).$$

In fact, it cannot decrease in any other point, because in this case there would be a form F of degree less than or equal to $a + b - 3$ that is zero on all the points of W' and not on Q. So, $F = 0$ on $ab - 1$ points of the Complete Intersection X, then, for Remark 1, we know that F is also zero on the other point of X, that is P. So $a + b - 2$ points of L are zeros of F, then F is zero on L and so $F(Q) = 0$. This is a contradiction.

Case 2. Let $Q \in X \setminus \{P\}$, for the same reasons of the case 1, we cannot have a form of degree less than or equal to $a + b - 3$ that is zero on W' and not on Q. If F exists, it is zero on $a+b-2$ points of L, so L is contained in $F = 0$ and so $F(P) = 0$. Then F is zero on $a+b-1$ points of X and, for Remark 1, $F(Q) = 0$.

Then, the only possible h-vector for W' is

$$(1, 3, 4, 5, \ldots, a, a+1, a+1, \ldots, a+1, a+1, a, \ldots, 5, 4, 3).$$

□

REMARK 2. If $a \neq 1$ and $b \neq 1$, the Gorenstein set of points which we found, W, is not a Complete Intersection. In fact, in this case W is not contained in any hyperplane, but we have two independent forms of degree two which are zero on W. With the above notation, those forms are F_1L_1 and F_1L_2. Moreover, every form of degree two in I_W must contain F as factor by Bezout's Theorem. So, in every set of minimal generators of I_W we have two forms of degree 2 which are not a regular sequence.

4. Conclusion

In the previous section we showed a new method to construct aG zerodimensional schemes not complete intersection. By this way, we can easily visualize the position of these points and obtain more informations about the "geometry" of the scheme, as the next example shows.

EXAMPLE 5. We know that the coordinate ring of a set of five general points in $\mathbb{P}^3$ is Gorenstein, where general means that not four are on a plane. We want give a proof using Theorem 3.

In fact let P_1, P_2, P_3, P_4, P_5 be five general points in $\mathbb{P}^3$. Let $L_1 = 0$ be the plane containing P_1, P_2, P_3 and $L_2 = 0$, $L_3 = 0$ the line through P_4 and P_5. So we have a new point P_6, i.e. the intersection between this plane and this line. The four points in the plane are complete intersection of L_1 and two forms of degree two, because no three of them are collinear. In fact, if P_6 and two points on the plane are collinear, then P_4, P_5 and those points are on a plane, and this is a contradiction. So, by Theorem 3, P_1, P_2, P_3 and $2 + 2 - 2$ points on a line through P_6 but not in the plane form an arithmetically Gorenstein zeroscheme. If we choose L the line through P_6 and P_4 and P_5 the points on L, we have the conclusion.

REMARK 3. Unfortunately in this way we can obtain very particular schemes: all these schemes have h-vector

$$(1, 3, 4, 5, \dots, a, a+1, a+1, \dots, a+1, a+1, a, \dots, 5, 4, 3, 1);$$

so, we cannot build the scheme of the Example 2. But, this scheme too, can be obtained from the union of a residual scheme and a "suitable" complete intersection.

Recently, in a joint work with R. Notari and M.L. Spreafico, we generalized this construction obtaining a bigger family of Gorenstein schemes of codimension three.

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AMS Subject Classification: 14M06, 13C40, 14M05.

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