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WAVELET MRA ON THE INTERVAL WITH DILATION FACTOR M

Abstract. In this paper we consider a class of M-channel subband coding schemes with perfect reconstruction property (see [14]). Along the lines of [12] we construct compactly supported biorthogonal wavelet bases of the interval, with dilation factor M, related to these schemes. In particular, we study the case of spline filters. It turns out a new MRA for the Interval.

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Introduction

Starting from the case $M = 2$ (see, e.g. [9]), the theory of scale factor $M > 2$ for spaces of functions defined on $\mathbb{R}^n$ has been developed by many authors (see, e.g., [14] and references cited therein). Let us recall that a multiresolution analysis of $L^2(\mathbb{R})$ with integer dilation factor $M > 2$ is defined exactly in the same way as for dilation $M = 2$, except that $f \in V_0$ if and only if $f(M\cdot) \in V_1$. The motivation for M larger than 2 comes from M-channel filter bank theory (see, e.g., [14],[13],[10] and references cited therein) and from the attempt to obtain a more flexible tiling of the time-scale space than in the case $M = 2$. For non-experts we refer to [15], where most of the basics of the subject are covered in tutorial fashion. Our new result is to provide a biorthogonal decomposition with dilation factor $M > 2$ for the unit Interval.

This decomposition is also useful for applications in various other fields: approximation of multivariate functions, parabolic types of evolution equations, numerical resolution of elliptic and semielliptic PDE's, etc (see, e.g., [11],[4],[2]). It is in fact the starting point to get a characterization of anisotropic function spaces defined on the unit square by means of the tensor product of 1-dimensional M-wavelets bases (see [7]).

In order to construct biorthogonal wavelet bases with dilation factor $M \geq 2$ on the interval, while making this paper self-contained and putting our own results in prospective, in Sec. 1 we have described in some detail related earlier work appearing in the literature for the case of the real line (see, e.g. [14]). In Sec. 2 we outline the construction of border scaling functions also for the (hitherto unknown) case $M > 2$ and we reduce the previous problem to check the non-singularity of the Gramian matrix X of the $M-$scaling function bases. Next, we turn our attention to the building of biorthogonal wavelet bases on the half line $(0, +\infty)$. In particular, we ought to underline the relevance of Propositions 6-8 (Sec. 3).

An important class of biorthogonal bases, very useful for applications, is given by the B-spline functions. In Sec. 4, using the expression of the spline filters derived in [14] for the real line, we compute the dimension of the border wavelets (Proposition

9) and we give a direct example (case $M = 3$, $L = \tilde{L} = 2$) of how to build border wavelets, showing the non-singularity of the Gramian matrix X introduced in Section 2.

This consideration of the half-line topic turns out to play a key role in the whole construction: once obtained, gluing together two such constructions (on $(0, +\infty)$ and $(-\infty, 1)$), in Sec. 5 we finally get the multilevel decomposition of the unit interval .

Throughout this paper, by abuse of notation, we use the symbol $\bigoplus$ (normally reserved for orthogonal sum) for a direct sum of spaces.

1. Biorthogonal decomposition in $\mathbb{R}$

The starting point of our construction is to give the conditions that must be satisfied by the filters. In this paper we shall consider only the case in which the scaling functions φ and $\widetilde{\varphi}$ are compactly supported i.e., the associated low-pass filters m_0 and $\widetilde{m}_0$ are trigonometric polynomials:

$$m_0(\xi) = \frac{1}{\sqrt{M}} \sum_{n=n_0}^{n_1} h_n e^{-in\xi}, \qquad \widetilde{m}_0(\xi) = \frac{1}{\sqrt{M}} \sum_{n=\tilde{n}_0}^{\tilde{n}_1} \widetilde{h}_n e^{-in\xi}. \tag{1}$$

An $M-$channel subband coding scheme (see [14]) is determined by M analyzing filters $\widetilde{m}_0, \dots, \widetilde{m}_{M-1}$ and M synthesizing filters $m_0, \dots, m_{M-1}$. A family of FIR filters

$$\widetilde{m}_0, \dots, \widetilde{m}_{M-1}, m_0, \dots, m_{M-1}$$

is said to have *perfect reconstruction* if the reconstructed signal, normalized by a factor of M, coincides with the input signal in the $M-$channel filter bank.

As for the low-pass filters, we can write the high-pass trigonometric polynomial, i.e., for $l = 1, \dots, M-1$, we have

$$m_l(\xi) = \frac{1}{\sqrt{M}} \sum_{n=n_0^{(l)}}^{n_1^{(l)}} h_n^{(l)} e^{-in\xi}, \qquad \widetilde{m}_l(\xi) = \frac{1}{\sqrt{M}} \sum_{n=n_0^{(l)}}^{n_1^{(l)}} \widetilde{h}_n^{(l)} e^{-in\xi}. \tag{2}$$

Now we are ready to show the new relations which we obtained between the filter coefficients . (Since the proofs are long and require only straightforward computations, we refer to [6]).

PROPOSITION 1. *We have*

$$\sum_{n\in\mathbb{Z}} h_n \widetilde{h}_{n-Mk} = \delta_{k,0}\,, \qquad \forall k \in \mathbb{Z}, \tag{3}$$

$$\sum_{n\in\mathbb{Z}} h_n = \sum_{n\in\mathbb{Z}} \widetilde{h}_n = \sqrt{M}\,, \tag{4}$$

$$\sum_{n\in\mathbb{Z}} h_n^{(l)} = \sum_{n\in\mathbb{Z}} \widetilde{h}_n^{(l)} = 0. \tag{5}$$

PROPOSITION 2. *The following relation holds*

$$\sum_{n\in\mathbb{Z}} h_n^{(l)}\widetilde{h}_{n-Mk}^{(l')} = \delta_{k,0}\delta_{l,l'}, \qquad \forall l, l' = 0,\dots,M-1. \tag{6}$$

In order to understand what we shall do in the sequel, let us recall the notion of a critical exponent for $M \geq 2$ (see, e.g., [3],[1],[13]). For $k = 0,\dots,M-1$, let $\theta_k = 2\pi k/M$ and $p(\xi)$ be a trigonometric polynomial such that $p(\theta_k) = \delta_{k,0}$, then there exists a positive integer L such that

$$p(\xi) = \left(\frac{1+e^{-i\xi}+e^{-i2\xi}+\dots+e^{-i(M-1)\xi}}{M}\right)^L P(\xi) \tag{7}$$

where $P(\xi)$ is a trigonometric polynomial such that $P(0) = 1$ and $P(\theta_k) \neq 0$. The critical exponent of P is the number τ defined as

$$\tau = \inf_{n>0}\left[\frac{1}{n}\log_M\left(\sup_{\xi\in\mathbb{R}}\prod_{j=1}^{n}\left|P(M^{-j}\xi)\right|\right)\right]. \tag{8}$$

The compactly supported distribution φ defined by means of its Fourier transform $\widehat{\varphi}(\xi) = \prod_{h=1}^{\infty} p(M^{-h}\xi)$, as in the case $M = 2$ (see, e.g.,[12]), satisfies $|\widehat{\varphi}(\xi)| \leq C(1+|\xi|)^{-L-\tau}$. Therefore, depending on the values of L and τ, φ belongs to various function spaces. If the family of FIR filters $\widetilde{m}_0,\dots,\widetilde{m}_{M-1}, m_0,\dots,m_{M-1}$ has perfect reconstruction and if m_0 and $\widetilde{m}_0$ are low-pass, that is:

$$m_0(\theta_k) = \widetilde{m}_0(\theta_k) = \delta_{k,0}, \tag{9}$$

then we can write

$$m_0(\xi) = \frac{1}{\sqrt{M}}\sum_{n=n_0}^{n_1} h_n e^{-in\xi} = \left(\frac{1+e^{-i\xi}+e^{-i2\xi}+\dots+e^{-i(M-1)\xi}}{M}\right)^L P(\xi) \tag{10}$$

with

$$P(0) = 1 \quad \text{and} \quad P(\theta_k) \neq 0, \quad k = 1,\dots,M-1. \tag{11}$$

Let us notice that, writing $m_0(\xi)$ as a polynomial in $z = e^{-i\xi}$ and using (10) and (11), we obtain the condition:

$$L \leq \frac{n_1 - n_0}{M-1}. \tag{12}$$

If τ is the critical exponent of m_0 and

$$\sigma := L - \tau - 1/2 > 0, \tag{13}$$

setting

$$\widehat{\varphi}(\xi)=\prod_{h=1}^{\infty} m_0(M^{-h}\xi) \tag{14}$$

we have $|\widehat{\varphi}(\xi)| \leq C(1+|\xi|)^{-1/2-\sigma}$, for some constant $C>0$, so that φ belongs to $L^2(\mathbb{R})$. From the definition (14) we derive the following relations

$$\begin{cases} \widehat{\varphi}(M\xi) & = & \widehat{\varphi}(\xi)m_0(\xi) \\ \widehat{\varphi}(0) & = & 1 \end{cases}$$

which we translate into a “refinement equation”, which will be useful in the sequel:

$$\varphi(x)=\sqrt{M}\sum_{n=n_0}^{n_1} h_n\varphi(Mx-n). \tag{15}$$

Wavelets are defined in the same way. For all $l=1,\dots,M-1$, we set

$$\widehat{\psi_l}(M\xi)=m_l(\xi)\widehat{\varphi}(\xi), \tag{16}$$

then ψ_l are compactly supported functions belonging to $L^2(\mathbb{R})$ and verifying

$$\psi_l(x)=\sqrt{M}\sum_{n=n_0^{(l)}}^{n_1^{(l)}} h_n^{(l)}\varphi(Mx-n). \tag{17}$$

From (16) we get $\widehat{\psi_l}(0)=0$, for $l=1,\dots,M-1$.

Let us briefly describe a multilevel decomposition of $L^2(\mathbb{R})$; for more details, see, e.g., [14] and [6]. For $j,k\in\mathbb{Z}$, we set

$$\varphi_{jk}(x)=M^{j/2}\varphi(M^jx-k) \qquad \text{and} \qquad \psi_{jkl}(x)=M^{j/2}\psi_l(M^jx-k) \tag{18}$$

(analogously their dual functions); they satisfy

$$\langle\varphi_{jk},\widetilde{\varphi}_{jh}\rangle=\delta_{kh}, \qquad \forall j,k,h\in\mathbb{Z}. \tag{19}$$

$$\langle\psi_{jkl},\widetilde{\psi}_{j_1k_1l_1}\rangle=\delta_{jj_1}\delta_{kk_1}\delta_{ll_1}, \qquad \forall j,j_1,k,k_1,l,l_1\in\mathbb{Z}. \tag{20}$$

Moreover, for every $j\in\mathbb{Z}$, $\Phi_j=\{\varphi_{jk}:k\in\mathbb{Z}\}$ are uniformly 2-stable bases for the spaces

$$V_j=\operatorname{span}_{L^2(\mathbb{R})}\{\varphi_{jk}:k\in\mathbb{Z}\}=\left\{\sum_{k\in\mathbb{Z}}\alpha_k\varphi_{jk}:\{\alpha_k\}\in l^2(\mathbb{Z})\right\}. \tag{21}$$

REMARK 1. Let us consider the isometries:

$$T_j \,:\, L^2(\mathbb{R}) \to L^2(\mathbb{R})\,, \qquad T_j v(x) = M^{j/2} v(M^j x), \quad j \in \mathbb{Z}, \tag{22}$$

then $\varphi_{jk} = T_j \varphi_{0k}$ and we can also define the spaces V_j from V_0:

$$V_j = T_j V_0\,. \tag{23}$$

obtaining an equivalent definition of (21).

The operators

$$P_j v = \sum_{k\in\mathbb{Z}} \langle v, \widetilde{\varphi}_{jk}\rangle \varphi_{jk}, \qquad Q_{jl} v = \sum_{k\in\mathbb{Z}} \langle v, \widetilde{\psi}_{jkl}\rangle \psi_{jkl}, \tag{24}$$

are uniformly bounded on $L^2(\mathbb{R})$ and satisfy the relations

$$P_{j+1} = P_j + \sum_{l=1}^{M-1} Q_{jl}. \tag{25}$$

We define the wavelet spaces

$$W_j := \bigoplus_{l=1}^{M-1} W_{jl} \qquad \text{with} \qquad W_{jl} := \operatorname{Im} Q_{jl}.$$

PROPOSITION 3. *We have* $W_j = \{v \in V_{j+1} \,:\, \langle v, \tilde{v}\rangle = 0, \quad \forall \tilde{v} \in \widetilde{V}_j\}$.

The proof is the same as the case $M = 2$ (see [6]).
It follows that $W_j \perp \widetilde{V}_j$ and $\widetilde{W}_j \perp V_j$, for all $j \in \mathbb{Z}$, hence

$$\langle \psi_l(\cdot - k), \widetilde{\varphi}(\cdot - h)\rangle = 0 \qquad \forall k, h \in \mathbb{Z},\ l = 1, \dots, M-1. \tag{26}$$

Taking $h = 0$, we can translate (26) in terms of filters' coefficients as

$$\sum_{n\in\mathbb{Z}} h_n^{(l)} \widetilde{h}_{n-Mk} = 0, \quad \forall k \in \mathbb{Z},\ l = 1, \dots, M-1.$$

From (20) we get that $\Psi_j = \{\psi_{jkl} \,:\, k \in \mathbb{Z},\ l = 1, \dots, M-1\}$ are uniformly 2-stable bases of W_j. Let us recall [14, Thm. 5]

THEOREM 1. *Consider the FIR filters* $\widetilde{m}_0, \dots, \widetilde{m}_{M-1}, m_0, \dots, m_{M-1}$ *defined previously, such that perfect reconstruction holds. Then the families* $\{M^{j/2}\psi_l(M^j x - k)\}$ *and* $\{M^{j/2}\widetilde{\psi}_l(M^j x - k)\}$, *where* $j, k \in \mathbb{Z}$ *and* $l = 1, \dots, M-1$, *are dual Riesz bases of* $L^2(\mathbb{R})$. *In particular, for every* $f \in L^2(\mathbb{R})$ *we have that*

$$f = \sum_{j,k\in\mathbb{Z}} \sum_{l=1}^{M-1} \langle f, \widetilde{\psi}_{jkl}\rangle \psi_{jkl} = \sum_{j,k\in\mathbb{Z}} \sum_{l=1}^{M-1} \langle f, \psi_{jkl}\rangle \widetilde{\psi}_{jkl}\,. \tag{27}$$

In conclusion, one can show

THEOREM 2. *The following three conditions are equivalent:*

a) $\frac{d^r m_0}{d\xi^r}(\theta_k) = 0, \qquad 0 \le r \le L-1,\ \forall k = 1, \dots, M-1;$

b) the span $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ *contains locally the polynomials* $\mathbb{P}_{L-1}$ *of degree* $\le L-1$*;*

c) $\displaystyle\int_{\mathbb{R}} x^r \widetilde{\psi}_l(x)dx = 0, \qquad 0 \le r \le L-1,\ l = 0, \dots, M-1.$

REMARK 2. (i) The equivalence $a) \Leftrightarrow b)$ is known to hold in bigger generality, see [5, Prop. 2.1] and references therein.
(ii) A detailed proof is contained in [6].

2. Scaling function spaces for the half-line

Starting from a biorthogonal decomposition on $\mathbb{R}$ as described before, we aim at constructing dual scaling function spaces $V_j(\mathbb{R}^+)$ and $\widetilde{V}_j(\mathbb{R}^+)$ which will form a multilevel decomposition of $L^2(\mathbb{R}^+)$. Making similar computations as developed for the case $M = 2$ (see, e.g. [12]) we get

$$\begin{aligned} \operatorname{supp}\varphi &= \left[\frac{n_0}{M-1}, \frac{n_1}{M-1}\right] \\ \operatorname{supp}\psi_l &= \left[\frac{n_0^{(l)}}{M} + \frac{n_0}{M(M-1)}, \frac{n_1^{(l)}}{M} + \frac{n_1}{M(M-1)}\right], \quad l = 1, \dots, M-1. \end{aligned} \tag{28}$$

Without loss of generality, we shall suppose $L \le \widetilde{L}$ and $\tilde{n}_0 \le n_0 \le 0 \le n_1 \le \tilde{n}_1$, so that $\operatorname{supp}\varphi \subseteq \operatorname{supp}\widetilde{\varphi}$. From now on, we will append a superscript $^{\mathbb{R}}$ to all the functions defined on the real line. Note that if $k \ge -\frac{n_0}{M-1}$, $\varphi_{0k}^{\mathbb{R}}$ has support contained in $[0, +\infty)$. More precisely, $\operatorname{supp}\varphi_{0k}^{\mathbb{R}} = \left[\frac{n_0}{M-1} + k, \frac{n_1}{M-1} + k\right]$. Let us fix a nonnegative integer δ and set $k_0^* := \lceil -\frac{n_0}{M-1} \rceil + \delta$; observe that $k_0^* = \min\{k \in \mathbb{Z} : \operatorname{supp}\varphi_{0k}^{\mathbb{R}} \subseteq [\delta, +\infty)\}$. Define

$$V^{(+)} = \operatorname{span}\{\varphi_{0k|[0,+\infty)}^{\mathbb{R}} : k \ge k_0^*\}; \tag{29}$$

this space will be identified in a natural way with a subspace of $V_0(\mathbb{R})$ and will not be modified by the subsequent construction. To obtain the right scaling space $V_0(\mathbb{R}^+)$ for the half-line, we will add to the basis $\{\varphi_{0k|[0,+\infty)}^{\mathbb{R}} : k \ge k_0^*\}$ of $V^{(+)}$ a finite number of new functions. As in [12], these functions will be constructed so that the reproducing property of polynomials is maintained. For any polynomial $p \in \mathbb{P}_{L-1}$ and every fixed $x \in \mathbb{R}$, $p(x) = \sum_{k\in\mathbb{Z}} \breve{p}_{0k}\, \varphi_{0k}^{\mathbb{R}}(x)$. Therefore, if $\{p_\alpha : \alpha = 0, \dots, L-1\}$ is a basis for

$\mathbb{P}_{L-1}$, for every $x \geq 0$, we have

$$p_\alpha(x) = \sum_{k=-\lceil n_1/(M-1)\rceil+1}^{k_0^*-1} c_{\alpha k}\varphi_{0k}^{\mathbb{R}}(x) + \sum_{k\geq k_0^*} c_{\alpha k}\varphi_{0k}^{\mathbb{R}}(x), \tag{30}$$

where

$$c_{\alpha k} := (\check{p}_\alpha)_{0k} = \langle p_\alpha, \widetilde{\varphi}_{0k}\rangle, \quad \alpha = 0, \ldots, L-1. \tag{31}$$

Since the second sum in (30) is a linear combination of elements of $V^{(+)}$, in order to locally generate all polynomials of degree $\leq L-1$ on the half-line, we will add to this space all linear combinations of the functions

$$\phi_\alpha(x) = \sum_{k=-\lceil n_1/(M-1)\rceil+1}^{k_0^*-1} c_{\alpha k}\varphi_{0k}^{\mathbb{R}}(x)\,, \qquad \alpha = 0, \ldots, L-1. \tag{32}$$

As in the case $M = 2$, the following proposition holds (see [12, Prop. 3.1]).

PROPOSITION 4. *The functions* $\phi_\alpha,\ \alpha = 0, \ldots, L-1$ *and* $\varphi^{\mathbb{R}}_{0k|[0,+\infty)},\ k \geq k_0^*$, *are linearly independent.*

As a consequence, it is natural to define

$$V_0(\mathbb{R}^+) = \text{span}\,\{\phi_\alpha\,:\,\alpha = 0, \ldots, L-1\} \oplus V^{(+)}. \tag{33}$$

We rename the functions in the following way

$$\varphi_{0k} = \begin{cases} \phi_k & \text{if } k = 0, \ldots, L-1, \\ \varphi^{\mathbb{R}}_{0,k_0^*+k-L} & \text{if } k \geq L. \end{cases} \tag{34}$$

We study now the biorthogonality of the dual generators of $V_0(\mathbb{R}^+)$ and $\widetilde{V}_0(\mathbb{R}^+)$. Setting $k^* := \{k_0^*, \tilde{k}_0^*\} = \max\left\{\left\lceil -\frac{n_0}{M-1}\right\rceil + \delta,\ \left\lceil -\frac{\tilde{n}_0}{M-1}\right\rceil + \tilde{\delta}\right\}$, let us observe that $\{\varphi_{0k}\,:\,k \geq k^*\}$ and $\{\widetilde{\varphi}_{0k}\,:\,k \geq k^*\}$ are already biorthogonal. In order to get a pair of dual systems using our "blocks" we have therefore to match the dimensions of the spaces spanned by $\{\phi_\alpha\,:\,\alpha = 0, \ldots, L-1\} \cup \{\varphi_{0k}^{\mathbb{R}}\,:\,k = k_0^*, \ldots, k^*-1\}$ and by $\{\widetilde{\phi}_\beta\,:\,\beta = 0, \ldots, \widetilde{L}-1\} \cup \{\widetilde{\varphi}_{0k}^{\mathbb{R}}\,:\,k = \tilde{k}_0^*, \ldots, k^*-1\}$. This requirement can be translated into an explicit relation between δ and $\tilde{\delta}$; indeed, we must have $L - k_0^* = \widetilde{L} - \tilde{k}_0^*$, i.e.,

$$\tilde{\delta} - \delta = \widetilde{L} - L + \left\lceil -\frac{n_0}{M-1}\right\rceil - \left\lceil -\frac{\tilde{n}_0}{M-1}\right\rceil. \tag{35}$$

Since $\widetilde{L} \geq L$, we get $k^* = \lceil -\frac{\tilde{n}_0}{M-1}\rceil + \tilde{\delta}$.

REMARK 3. The two parameters δ and $\tilde{\delta}$ have been introduced exactly because we want the equality of the cardinality of the sets previously indicated. On the other hand, we want to choose them as small as possible in order to minimize the perturbation due to the boundary.

Let us define the spaces V_0^B and $\widetilde{V}_0^B$ spanned by the so called *boundary scaling functions* as

$$V_0^B := \text{span}\,\{\varphi_{0k} \,:\, k = 0, \dots, \widetilde{L} - 1\},\ \widetilde{V}_0^B = \text{span}\,\{\widetilde{\varphi}_{0k} \,:\, k = 0, \dots, \widetilde{L} - 1\}, \tag{36}$$

and the spaces V_0^I and $\widetilde{V}_0^I$ spanned by the *interior scaling functions* as

$$V_0^I := \text{span}\,\{\varphi_{0k} :\ k \geq \widetilde{L}\}, \qquad \widetilde{V}_0^I := \text{span}\,\{\widetilde{\varphi}_{0k} :\ k \geq \widetilde{L}\}. \tag{37}$$

Thus we have

$$V_0(\mathbb{R}^+) = V_0^B \oplus V_0^I\,, \qquad \widetilde{V}_0(\mathbb{R}^+) = \widetilde{V}_0^B \oplus \widetilde{V}_0^I\,. \tag{38}$$

The only functions that we have to modify in order to obtain biorthogonal systems are the border ones. The problem is to find a basis of V_0^B, say $\{\eta_{0k} \,:\, k = 0, \dots, \widetilde{L} - 1\}$, and one of $\widetilde{V}_0^B$, say $\{\tilde{\eta}_{0l} \,:\, l = 0, \dots, \widetilde{L} - 1\}$, such that $\langle \eta_{0k}, \tilde{\eta}_{0l} \rangle = \delta_{kl}$, for $k, l = 0, \dots, \widetilde{L} - 1$. Setting $\eta_{0k} = \sum_{m=0}^{\widetilde{L}-1} a_{km}\varphi_{0m}$ and $\tilde{\eta}_{0k} = \sum_{m=0}^{\widetilde{L}-1} \tilde{a}_{km}\widetilde{\varphi}_{0m}$, and calling X the Gramian matrix of components

$$X_{kl} = \langle \varphi_{0k}, \widetilde{\varphi}_{0l} \rangle, \qquad k, l = 0, \dots, \widetilde{L} - 1, \tag{39}$$

this is equivalent to the problem of finding two $\widetilde{L} \times \widetilde{L}$ real matrices, say $A = (a_{km})$ and $\widetilde{A} = (\tilde{a}_{km})$, satisfying

$$AX\widetilde{A}^T = I. \tag{40}$$

A necessary and sufficient condition for (40) to have solutions is clearly the non-singularity of X, or equivalently $V_0^B \cap (\widetilde{V}_0^B)^\perp = \{0\}$. We know at present no general result establishing the invertibility of X. It can be proved, e.g., for orthogonal systems and for systems arising from B-spline functions in the case $M = 2$ (see [12] and also [8]). We conjecture that this holds for B-spline functions for $M \geq 2$; as an example, in the next paragraph we show the case $M = 3$ and $L = \widetilde{L} = 2$. We also have results for $M > 3$, which we intend to publish elsewhere. From now on we will assume that this condition is verified and we suppose, renaming if necessary (similarly to what we did in (34)), that $\{\varphi_{0k}\}_{k\geq 0}$ and $\{\widetilde{\varphi}_{0l}\}_{l\geq 0}$ are dual biorthogonal bases.

PROPOSITION 5. *For any $j \in \mathbb{N}$, one has the inclusions*

$$V_j(\mathbb{R}^+) \subset V_{j+1}(\mathbb{R}^+).$$

The refinement equation for the modified border functions are

$$\varphi_{0k} = M^{-(k+1/2)}\varphi_{1k} + \sum_{m\geq L} H_{km}\varphi_{1m}^{\mathbb{R}} \tag{41}$$

where

$$H_{km} = c_{k,k_0^*+m-L}\, M^{-(k+1/2)} - \sum_{l \geq k_0^*} c_{kl}\, h_{k_0^*+m-L-Ml}, \tag{42}$$

for all $k = 0, \dots, L-1$ and for $m \geq L$.

3. Wavelet function spaces for the half-line

We now have all the tools to build the detail spaces and wavelets on the half-line. We start from level $j = 0$ and look for a complement space $W_0(\mathbb{R}^+)$ such that $V_1(\mathbb{R}^+) = V_0(\mathbb{R}^+) \oplus W_0(\mathbb{R}^+)$ and $W_0(\mathbb{R}^+) \perp \widetilde{V}_0(\mathbb{R}^+)$. To this end, let us consider the basis functions of $V_1(\mathbb{R}^+)$ and let us write them as a sum of a function of $V_0(\mathbb{R}^+)$ and a function which will be an element of $W_0(\mathbb{R}^+)$. Since we have based our construction on the existence of a multilevel decomposition on the real line, we start with two equations that will be extensively used in the sequel. Defining $\varphi^{\mathbb{R}}_{jm}(x) = M^{j/2}\varphi^{\mathbb{R}}(M^j x - m)$, $\psi^{\mathbb{R}}_{ljm}(x) = M^{j/2}\psi^{\mathbb{R}}_l(M^j x - m)$, and using the refinement equations (15):

$$\varphi^{\mathbb{R}}_{jm} = \sum_{k=n_0+Mm}^{n_1+Mm} h_{k-Mm}\varphi^{\mathbb{R}}_{j+1\,k} \qquad \psi^{\mathbb{R}}_{ljm} = \sum_{k=n_0^{(l)}+Mm}^{n_1^{(l)}+Mm} h^{(l)}_{k-Mm}\varphi^{\mathbb{R}}_{j+1,k} \tag{43}$$

$$\widetilde{\varphi}^{\mathbb{R}}_{jm} = \sum_{k=\tilde{n}_0+Mm}^{\tilde{n}_1+Mm} \tilde{h}_{k-Mm}\widetilde{\varphi}^{\mathbb{R}}_{j+1\,k} \qquad \widetilde{\psi}^{\mathbb{R}}_{ljm} = \sum_{k=\tilde{n}_0^{(l)}+Mm}^{\tilde{n}_1^{(l)}+Mm} \tilde{h}^{(l)}_{k-Mm}\widetilde{\varphi}^{\mathbb{R}}_{j+1,k}. \tag{44}$$

In view of the multilevel decomposition of $L^2(\mathbb{R})$ we mentioned in (27) we have:

$$\varphi^{\mathbb{R}}_{j+1,k} = \sum_m \langle \varphi^{\mathbb{R}}_{j+1,k}, \widetilde{\varphi}^{\mathbb{R}}_{jm}\rangle \varphi^{\mathbb{R}}_{jm} + \sum_{l=1}^{M-1}\sum_m \langle \varphi^{\mathbb{R}}_{j+1,k}, \widetilde{\psi}^{\mathbb{R}}_{ljm}\rangle \psi^{\mathbb{R}}_{ljm}, \tag{45}$$

and using (44), Equation (45) becomes

$$\varphi^{\mathbb{R}}_{j+1,k} = \sum_{Mm=k-\tilde{n}_1}^{Mm=k-\tilde{n}_0} \tilde{h}_{k-Mm}\varphi^{\mathbb{R}}_{jm} + \sum_{l=1}^{M-1}\sum_{Mm=k-\tilde{n}_1^{(l)}}^{Mm=k-\tilde{n}_0^{(l)}} \tilde{h}^{(l)}_{k-Mm}\psi^{\mathbb{R}}_{ljm}. \tag{46}$$

Interior wavelets. Since $V_0(\mathbb{R}^+)$ contains the subspace $V^{(+)}$ defined in (29), $V_1(\mathbb{R}^+)$ contains the subspace $T_1 V^{(+)} = \{\varphi^{\mathbb{R}}_{1k|[0,+\infty)} \,:\, k \geq k_0^*\}$. Considering the right-hand side of (43) in the case $j = 0$, let us determine the integer m such that all the indices k in the sum are greater than or equal to k_0^*. This is equivalent to $Mm \geq k_0^* - n_0^{(l)}$. For $l = 1, \dots, M-1$, we set

$$m^*_{0l} := \left\lceil \frac{1}{M}\left(\left\lceil -\frac{n_0}{M-1}\right\rceil + \delta - n_0^{(l)}\right)\right\rceil \leq m; \tag{47}$$

for every $m \geq m^*_{0l}$, we define $\psi_{l0m} := \psi^{\mathbb{R}}_{l0m\,|[0,+\infty)}$, $\quad \forall m \geq m^*_{0l}$ and

$$W^{(+)} := \bigoplus_{l=1}^{M-1} W^{(+)}_l, \quad \text{with } W^{(+)}_l := \text{span}\,\{\psi^{\mathbb{R}}_{l0m\,|[0,+\infty)} \,:\, m \geq m^*_{0l}\}. \tag{48}$$

Let us observe that $W^{(+)} \subset V_1(\mathbb{R}^+)$ and $W^{(+)} \perp \widetilde{V}_0(\mathbb{R}^+)$, hence $W^{(+)} \subseteq W_0(\mathbb{R}^+)$. Analogously, we build $\widetilde{m}^*_{0l}$ and $\widetilde{W}^{(+)}$. Setting

$$m^*_l := \max(m^*_{0l}, \widetilde{m}^*_{0l}), \quad l = 1, \ldots, M-1,$$

we define the so-called *interior wavelets* by

$$W^I_0 := \bigoplus_{l=1}^{M-1} W^I_{0l}, \quad \text{with } W^I_{0l} := \text{span}\,\{\psi^{\mathbb{R}}_{l0m\,|[0,+\infty)} \,:\, m \geq m^*_l\}. \tag{49}$$

Border wavelets. Let us now call W^S_l a generic supplementary space of $W^{(+)}_l$ in $W_0(\mathbb{R}^+)$ and set $W^S := \bigoplus_{l=1}^{M-1} W^S_l$.

PROPOSITION 6. *We have*

$$\dim\, W^S_l = m^*_{0l} - \left\lceil \frac{1}{M}\left(\left\lceil -\frac{n_0}{M-1}\right\rceil - \tilde{n}^{(l)}_1\right)\right\rceil \tag{50}$$

and

$$\dim\, W^S = \sum_{l=1}^{M-1}\left\{m^*_{0l} - \left\lceil \frac{1}{M}\left(\left\lceil -\frac{n_0}{M-1}\right\rceil - \tilde{n}^{(l)}_1\right)\right\rceil\right\} \tag{51}$$

$$= \sum_{l=1}^{M-1}\left\lfloor \frac{1}{M}(\delta + (\tilde{n}^{(l)}_1 - \tilde{n}^{(l)}_0)\right\rfloor. \tag{52}$$

Proof. Let $K > 0$ be an integer such that all wavelets and scaling functions are interior ones on $[K, +\infty)$. Then, if $V^B_1 = T_1 V^B_0$,

$$\begin{aligned} &V^B_1 \oplus \text{span}\,\{\varphi^{\mathbb{R}}_{1k} \,|\, k^*_0 \leq k < MK - \frac{n_0}{M-1}\} = \\ &\left[V^B_0 \oplus \text{span}\left\{\varphi^{\mathbb{R}}_{0k} \,|\, k^*_0 \leq k < K - \frac{n_0}{M-1}\right\}\right] \oplus \\ &\left[W^S \bigoplus_{l=1}^{M-1} \text{span}\left\{\psi^{\mathbb{R}}_{l0m} \,|\, m^*_{0l} \leq m < K - \frac{n_0}{M(M-1)} - \frac{\tilde{n}^{(l)}_1}{M}\right\}\right]. \end{aligned}$$

Since supp $\varphi^{\mathbb{R}}_{0k} = [\frac{n_0}{M-1} + k, \frac{n_1}{M-1} + k]$, then supp $\varphi^{\mathbb{R}}_{0k} \subseteq [K, +\infty)$ if $k \geq K - \frac{n_0}{M-1}$. Similarly, since supp $\varphi^{\mathbb{R}}_{1k} = [\frac{n_0}{M(M-1)} + \frac{k}{M}, \frac{n_1}{M(M-1)} + \frac{k}{M}]$, then supp $\varphi^{\mathbb{R}}_{1k} \subseteq [K, +\infty)$ if $k \geq MK - \frac{n_0}{M-1}$.

Moreover, for every $l = 1, \dots, M-1$, the first interior wavelet used in (46) to generate $\varphi^{\mathbb{R}}_{1,\lceil MK-\frac{n_0}{M-1}\rceil}$ is $\psi^{\mathbb{R}}_{l0m_l}$ with $m_l \geq K - \frac{n_0}{M(M-1)} - \frac{\tilde{n}_1^{(l)}}{M}$. In terms of the dimensions of the spaces, the previous space equality becomes:

$$\widetilde{L} + MK + \left\lceil -\frac{n_0}{M-1} \right\rceil - k_0^* = \widetilde{L} + K + \left\lceil -\frac{n_0}{M-1} \right\rceil - k_0^* + \dim W^S$$
$$+ \sum_{l=1}^{M-1} \left(K + \left\lceil \frac{1}{M} (\left\lceil -\frac{n_0}{M-1} \right\rceil - \tilde{n}_1^{(l)}) \right\rceil - m_{0l}^* \right),$$

from which (51) follows. Finally, since

$$\lceil \frac{1}{M} (\lceil -\frac{n_0}{M-1} \rceil - \tilde{n}_1^{(l)}) \rceil - m_{0l}^* = \lceil \frac{1}{M} (n_0^{(l)} - \tilde{n}_1^{(l)}) - \frac{\delta}{M} \rceil,$$

we obtain (52). ☐

Remark 4. If $M = 2$, we have $l = 1$, $m_{0l}^* = m_0^*$, $\tilde{n}_1^{(1)} = 1 - n_0$, so that $\dim W^S = m_0^*$, as shown in [12].

To build W^S we need some functions that, added to $W^{(+)}$, will generate both V_1^B and the interior scaling functions that cannot be obtained in (46) using $W^{(+)}$ and V_0^I. In view of (41), we only have to consider the problem of generating interior scaling functions. Let us now look for the functions $\varphi^{\mathbb{R}}_{1k}$ generated by $\varphi^{\mathbb{R}}_{0m}$, for $m \geq k_0^*$, and by $\psi^{\mathbb{R}}_{l0m}$, with $m \geq m_{0l}^*$, for $l = 1, \dots, M-1$. Working separately on the two sums of (46):

(a) we must have $m \geq (k - \tilde{n}_1)/M$. Imposing $m \geq k_0^*$ and seeking for integer solutions, we get $\left\lceil \frac{k-\tilde{n}_1}{M} \right\rceil \geq k_0^*$, from which we obtain

(53) $$k \geq Mk_0^* + \tilde{n}_1 - (M-1) := \bar{k}_0.$$

(b) We want $m \geq m_{0l}^*$, therefore we must have $\left\lceil \frac{k-\tilde{n}_1^{(l)}}{M} \right\rceil \geq m_{0l}^*$; and consequently

(54) $$k \geq Mm_{0l}^* + \tilde{n}_1^{(l)} - (M-1) := \bar{k}_l, \quad \forall l = 1, \dots, M-1.$$

If we set

(55) $$\bar{k} = \max_{0 \leq r \leq M-1} (\bar{k}_r),$$

then, span $\{\varphi^{\mathbb{R}}_{1k|[0,+\infty)} \ : \ k \geq \bar{k}\} \subseteq V_0^I \oplus W^{(+)}$. We are left with the problem of generating some functions of $V_1(\mathbb{R}^+)$, precisely $\varphi^{\mathbb{R}}_{1k}$ with $k_0^* \leq k < \bar{k}$. These $\bar{k} - k_0^*$ functions are not all linearly independent.

PROPOSITION 7. *We have*

$$\dim\, W^S < \overline{k}_0 - k_0^*. \tag{56}$$

Proof. Let us observe that $\bar{k}_l \geq k_0^* - n_0^{(l)} + \tilde{n}_1^{(l)}$. At first, suppose that there is an $l^* > 0$ such that $\tilde{n}_1^{(l^*)} - n_0^{(l^*)} \geq Mk_0^* + \tilde{n}_1$, and $\tilde{n}_1^{(l^*)} - n_0^{(l^*)} \geq \tilde{n}_1^{(l)} - n_0^{(l)}$, for $l = 1, \dots, M-1$; since $Mk_0^* + \tilde{n}_1 - (M-1) > (M-1)\delta$, then

$$(M-1)\delta + \sum_{l=1}^{M-1} (\tilde{n}_1^{(l)} - n_0^{(l)}) < M(\tilde{n}_1^{(l^*)} - n_0^{(l^*)})$$

and, dividing by M, we obtain the result. It remains only to consider the case $Mk_0^* + \tilde{n}_1 \geq \tilde{n}_1^{(l)} - n_0^{(l)}$, for all $l = 1, \dots, M-1$:

$$(M-1)\delta + \sum_{l=1}^{M-1} (\tilde{n}_1^{(l)} - n_0^{(l)}) < M(Mk_0^* + \tilde{n}_1^{(l)}) - (M-1)$$

and dividing by M we again obtain the assertion. ☐

In the case $M = 2$, it has been shown that, for $k = k_0^*, \dots, \overline{k} - 1$, one out of two $\varphi^{\mathbb{R}}_{1k}$ depends on the previous one through elements of level zero (see [12]). Here, under the assumption

$$\left\lceil \frac{1}{M} \left(\left\lceil -\frac{n_0}{M-1} \right\rceil - \tilde{n}_1^{(l)} \right) \right\rceil = 0 \tag{57}$$

we get a similar relation. We shall prove later that this condition is satisfied, e.g., for B-spline functions. Observe that (57) implies $\dim\, W_l^S = m_{0l}^*$, for all $l = 1, \dots, M-1$.

PROPOSITION 8. *If condition (57) holds, we have*

$$\varphi^{\mathbb{R}}_{1,\overline{k}_l - Mm + r\,|[0,+\infty)} \in S_{lm} \oplus V_0^I \oplus W_{0l}^{(+)} \oplus_{a \neq l} W_a^S,$$

with

$$S_{lm} := \text{span}\,\{\varphi^{\mathbb{R}}_{1,\overline{k}_l - Mt + r_l\,|[0,+\infty)} \;:\; 1 \leq t \leq m_{0l}^*\},$$

where $r_l \in \{\tilde{n}_1^{(l)} - n_0\}_{mod M}$, $0 < r_l < M$, $0 \leq r < M$.

Proof. It can be obtained using the same technique developed in [12, Prop. 5.2]. Therefore, we give only a sketch of the proof, in order to point out the main differences. Let $r = 0$, set $m = 1$ and consider the linear combination

$$h_{n_0} \varphi^{\mathbb{R}}_{1,\overline{k}_l - M} + h_{n_0 + r_l} \varphi^{\mathbb{R}}_{1,\overline{k}_l - M + r_l} \tag{58}$$

which gives, for $\psi^{\mathbb{R}}_{l0n}$:

$$\cdots + \left(h_{n_0}\tilde{h}_{\tilde{n}_1^{(l)}-r_l} + h_{n_0}\tilde{h}_{\tilde{n}_1^{(l)}}\right)\psi^{\mathbb{R}}_{l0,m^*_{0,l}-1} + \sum_{n\geq m^*_{0,l}} d_{1n}\psi^{\mathbb{R}}_{l0n}.$$

Equation (3) with $k = n_0 - \tilde{n}_1^{(l)} + \frac{r_l}{M} \neq 0$ becomes

$$h_{n_0}\tilde{h}_{\tilde{n}_1^{(l)}-r_l} + h_{n_0+r_l}\tilde{h}_{\tilde{n}_1^{(l)}} = 0$$

and the thesis follows. If $0 < r < r_l$, we can consider (58) again, and by means of (57) we obtain the same result. If $r > r_l$, using (47) and (57), the wavelets contributing to generate the scaling function are $\psi^{\mathbb{R}}_{l0n}$, with $n \geq m^*_{0l}$, hence we get the result again. For $m > 1$, by induction, one can get the assertion. $\square$

We set $Q^I_{0l}v = \sum_{m\geq m^*_l}\langle v, \widetilde{\psi}^{\mathbb{R}}_{l0m}\rangle\psi^{\mathbb{R}}_{l0m}$. Then, for $i = 1, \ldots, m^*_{0l}$ we define

$$\psi_{l0m^*_{0l}-i} := (\,Id - P_0 - \sum_{k=1,k\neq l}^{M-1} Q_{0k} - Q^I_{0l}\,)\varphi^{\mathbb{R}}_{1,\overline{k}_l-Mi+r_l}, \tag{59}$$

and

$$W^B_{0l} := \operatorname{span}\{\psi_{l0m}\,:\, m = 0, \ldots, m^*_{0l}-1\}, \qquad W^B_0 := \bigoplus_{l=1}^{M-1} W^B_{0l};$$

it follows $W_0(\mathbb{R}^+) = W^B_0 \oplus W^I_0$.

As for the scaling functions, we must find a couple of biorthogonal bases for the spaces $W_0(\mathbb{R}^+)$ and $\widetilde{W}_0(\mathbb{R}^+)$. Similarly to the case $M = 2$, setting $m^*_l = \max\{\widetilde{m}^*_{0l}, m^*_{0l}\}$, for all $l = 1, \ldots, M-1$, we have to find two $m^*_l \times m^*_l$ matrices $F_l = (\,f_{lmr})$ and $\widetilde{F}_l = (\,\tilde{f}_{lns})$ such that

$$\langle \sum_{r=0}^{m^*_l-1} f_{lmr}\psi_{l0r}\,,\ \sum_{s=0}^{m^*_l-1}\tilde{f}_{lns}\widetilde{\psi}_{0s}\rangle = \delta_{mn}\,, \qquad \forall m, n = 0, \ldots, m^*_l - 1.$$

Calling Z_l the $m^*_l \times m^*_l$ matrix of components $Z_{lmn} = \langle\psi_{l0m}, \widetilde{\psi}_{l0n}\rangle$, this condition is equivalent to $F_l Z_l \widetilde{F_l}^T = I$. Again, it is enough to prove that the matrices Z_l are non-singular and this follows from the assumed invertibility of the matrix X (defined in (39)). Finally, for any $j \in \mathbb{N}$, we set

$$W_j(\mathbb{R}^+) = T_j W_0(\mathbb{R}^+) \qquad \text{and} \qquad \widetilde{W}_j(\mathbb{R}^+) = \widetilde{T}_j\widetilde{W}_0(\mathbb{R}^+); \tag{60}$$

they satisfy the same properties as in the case of the real line (for more details see [6]).

4. The B-spline case

Let us recall that the basic box spline N_L of order L and degree $L-1$ can be defined as the $(L-1)-$fold convolution of the characteristic function of [0, 1]. The space V_j^L is defined as the space of all square integrable splines of order L having $M^{-j}\mathbb{Z}$ as set of knots. Exactly as in the $M=2$ case, the family $\{V_j\}_{j\in\mathbb{Z}}$ is a multiresulution analysis with dilation factor M and scaling function N_L. In [14], P. Soardi found the expression for the FIR filters m_l and $\widetilde{m}_l$, for $l=0,\dots,M-1$. More precisely, the low pass synthesis filter is

$$m_0(\xi)=\begin{cases}\left(\frac{sin(M\xi/2)}{Msin(\xi/2)}\right)^L & \text{if} \quad L \text{ is even}\\ e^{-i(M-1)\xi/2}\left(\frac{sin(M\xi/2)}{Msin(\xi/2)}\right)^L & \text{if} \quad L \text{ is odd.}\end{cases} \tag{61}$$

For the analysis of high pass filters the general expression is

$$\widetilde{m}_l(\xi)=\begin{cases}\left(\frac{2}{M^{1/(M-1)}}\right)^L e^{-il\xi}cos^L\left(\frac{\xi+\pi}{2}\right) & \text{if} \quad L \text{ is even}\\ \left(\frac{2}{M^{1/(M-1)}}\right)^L e^{-il\xi+i(\xi+\pi)/2}cos^L\left(\frac{\xi+\pi}{2}\right) & \text{if} \quad L \text{ is odd.}\end{cases} \tag{62}$$

The analysis of low pass filters $\widetilde{m}_0$ is obtained by means of Bernstein polynomials (for details see [14] and [6]). If we set $2N=L+\tilde{L}$, then

$$\widetilde{m}_0(\xi)=\begin{cases}\left(\frac{sin(M\xi/2)}{Msin(\xi/2)}\right)^{\tilde{L}} P_N(\xi) & \text{if} \quad L \text{ is even}\\ e^{-i(M-1)\xi/2}\left(\frac{sin(M\xi/2)}{Msin(\xi/2)}\right)^{\tilde{L}} P_N(\xi) & \text{if} \quad L \text{ is odd.}\end{cases} \tag{63}$$

Let us point out the important dimensional equality we mentioned previously.

PROPOSITION 9. *Considering the B-spline functions we have*

$$\dim\, W_l^S = m_{0l}^*.$$

Proof. As a consequence of Definitions (61), (62), (63), we get

$$n_0=-\left\lceil\tfrac{L}{2}\right\rceil(M-1) \qquad \text{and} \qquad \tilde{n}_1^{(l)}=\left\lceil\tfrac{L}{2}\right\rceil+l,$$

for $l=1,\dots,M-1$. Since $\frac{1}{M}\left(\left\lceil-\frac{n_0}{M-1}\right\rceil-\tilde{n}_1^{(l)}\right)=-\frac{l}{M}$ and $\left\lceil-\frac{l}{M}\right\rceil=0$, for every $l=1,\dots,M-1$, from (51) we obtain the result. □

4.1. MRA in the case $M=3$, $L=\tilde{L}=2$

We will detail here, as an example, the construction of biorthogonal systems on the half line and we shall show the non-singularity of the matrix X (see (39)).

The low-pass synthesizing filter (61) in the case $M = 3$, $L = \tilde{L} = 2$, becomes

$$m_0(\xi) = e^{2i\xi}\left(\frac{1 + e^{-i\xi} + e^{-2i\xi}}{3}\right)^2 = \frac{1}{3}\left(\frac{1}{3}e^{2i\xi} + \frac{2}{3}e^{i\xi} + 1 + \frac{2}{3}e^{-i\xi} + \frac{1}{3}e^{-2i\xi}\right)$$

For $l = 1, 2$, the expression (62) gives the analysis high pass filters

$$\widetilde{m}_1(\xi) = \frac{1}{3}\left(-1 + 2e^{-i\xi} - e^{-2i\xi}\right),\ \widetilde{m}_2(\xi) = \frac{1}{3}\left(-e^{-i\xi} + 2e^{-2i\xi} - e^{-3i\xi}\right).$$

The analyzing low pass filter $\widetilde{m}_0$ defined in (63) is the following

$$\begin{aligned}\widetilde{m}_0(\xi) &= \frac{1}{81}(\frac{2}{3}e^{6i\xi} - 2e^{4i\xi} - \frac{32}{3}e^{3i\xi} + 7e^{2i\xi} + 22e^{i\xi} + 47 \\ &+ 22e^{-i\xi} + 7e^{-2i\xi} - \frac{32}{3}e^{-3i\xi} - 2e^{-4i\xi} + \frac{2}{3}e^{-6i\xi}).\end{aligned}$$

The high pass synthesis filters are

$$\begin{aligned}m_1(\xi) &= \frac{1}{27}(\frac{2}{27}e^{8i\xi} + \frac{4}{27}e^{7i\xi} + \frac{2}{9}e^{6i\xi} + \frac{32}{27}e^{5i\xi} - \frac{70}{27}e^{4i\xi} - 4e^{3i\xi} + \frac{40}{9}e^{2i\xi} \\ &+ \frac{116}{9}e^{i\xi} - \frac{98}{27}e^{-i\xi} - \frac{43}{27}e^{-2i\xi} + \frac{4}{9}e^{-3i\xi} + \frac{8}{27}e^{-4i\xi} + \frac{4}{27}e^{-5i\xi}).\end{aligned}$$

$$\begin{aligned}m_2(\xi) &= \frac{1}{27}(\frac{4}{27}e^{8i\xi} + \frac{8}{27}e^{7i\xi} + \frac{4}{9}e^{6i\xi} - \frac{43}{27}e^{5i\xi} - \frac{98}{27}e^{4i\xi} - \frac{17}{3}e^{3i\xi} + \frac{116}{9}e^{2i\xi} \\ &+ \frac{40}{9}e^{i\xi} - 4 - \frac{70}{27}e^{-i\xi} - \frac{32}{27}e^{-2i\xi} + \frac{2}{9}e^{-3i\xi} + \frac{4}{27}e^{-4i\xi} + \frac{2}{27}e^{-5i\xi}).\end{aligned}$$

To sum up, we get: $n_0 = -2$, $\tilde{n}_0 = -6$, $n_1 = 2$, $\tilde{n}_1 = 6$, $n_0^{(1)} = -8$, $\tilde{n}_0^{(1)} = 0$, $n_1^{(1)} = 5$, $\tilde{n}_1^{(1)} = 2$, $n_0^{(2)} = -8$, $\tilde{n}_0^{(2)} = 1$, $n_1^{(2)} = 5$, $\tilde{n}_1^{(2)} = 3$.

In order to satisfy condition (35), we must have $\tilde{\delta} - \delta = -2$; therefore we set $\tilde{\delta} = 0$ and $\delta = 2$. Let us observe that, with this choice, condition (12) (that holds also for the dual case), is fulfilled. As a result $k_0^* = k^* = 3$, then

$$V_0^I = \text{span}\,\{\varphi_{0k} := \varphi^{\mathbb{R}}_{0k|[0,+\infty)}\,|\,k \geq 3\},\quad \widetilde{V}_0^I = \text{span}\,\{\widetilde{\varphi}_{0k} := \widetilde{\varphi}^{\mathbb{R}}_{0k|[0,+\infty)}\,|\,k \geq 3\}.$$

We have to build the border scaling functions, by means of (32). We can choose $p_\alpha = x^\alpha, \alpha = 0, 1$ as a basis for $\mathbb{P}_1$, since $\varphi^{\mathbb{R}}_{0k}(l) = \delta_{kl}$, we get $l^\alpha = \sum_{k\in\mathbb{Z}} c_{\alpha k}\delta_{kl} = c_{\alpha l}$, $\forall k \in \mathbb{Z},\ \alpha = 0, 1$. Therefore

$$\phi_0(x) = c_{00}\varphi(x) + c_{01}\varphi(x-1) + c_{02}\varphi(x-2) = \begin{cases} 1 & \text{if } x \in [0,2], \\ 3 - x & \text{if } x \in [2,3], \\ 0 & \text{if } x \geq 3 \end{cases}$$

and

$$\phi_1(x) = c_{10}\varphi(x) + c_{11}\varphi(x-1) + c_{12}\varphi(x-2) = \begin{cases} x & \text{if } x \in [0,2], \\ 6-2x & \text{if } x \in [2,3], \\ 0 & \text{if } x \geq 3. \end{cases}$$

By means of the moments of φ we get $\tilde{c}_{0k} = 1$, $\tilde{c}_{1k} = k$. As in (34) we set $\varphi_{00} = \phi_0$, $\varphi_{01} = \phi_1$, $\widetilde{\varphi}_{00} = \widetilde{\phi}_0$, $\widetilde{\varphi}_{01} = \widetilde{\phi}_1$. Let us observe that $\widetilde{\phi}_l(x) = x^l - \sum_{m\geq 2} \tilde{c}_{lm}\widetilde{\varphi}_{0m}(x)$ for $l = 0, 1$, then $\langle \phi_k, \widetilde{\phi}_l \rangle = \langle \phi_k, x^l \rangle$ for $k, l = 0, 1$, and the non-singularity of the matrix X (defined in (39)) follows.

Let us build the space $W_0(\mathbb{R}^+) = W_{01}(\mathbb{R}^+) \oplus W_{02}(\mathbb{R}^+)$ and the dual one. From (47) we get $m^*_{0l} = 4$ and $\widetilde{m}^*_{0l} = 1$, for $l = 1, 2$. Then $m^*_l = 4$ for $l = 1, 2$ and $m^* = 4$, so that $W^I_{0l} = \text{span}\,\{\psi_{l0m} = \psi^{\mathbb{R}}_{l0m\,|[0,+\infty)} \,|\, m \geq 4\}$, for $l = 1, 2$, and $W^I_0 = W^I_{01} \oplus W^I_{02}$ (the same result holds for the dual space). Using (54), we have $\overline{k}_1 = 12$, $\overline{k}_2 = 13$, $\overline{\tilde{k}}_1 = 6$, $\overline{\tilde{k}}_2 = 6$, which allow us to build four modified wavelets for each W^S_1 and W^S_2 and one for each $\widetilde{W}^S_1$ and $\widetilde{W}^S_2$, so that, by (59) with $r_1 = 1, r_2 = 2, \tilde{r}_1 = 2, \tilde{r}_2 = 2$, we obtain, respectively: $\psi_{100}, \psi_{101}, \psi_{102}, \psi_{103}$ by means of the projections of $\varphi^{\mathbb{R}}_{1,10}, \varphi^{\mathbb{R}}_{1,7}, \varphi^{\mathbb{R}}_{1,4}, \varphi^{\mathbb{R}}_{1,1}$; $\psi_{200}, \psi_{201}, \psi_{202}, \psi_{203}$ by means of the projections of $\varphi^{\mathbb{R}}_{1,11}, \varphi^{\mathbb{R}}_{1,8}, \varphi^{\mathbb{R}}_{1,5}, \varphi^{\mathbb{R}}_{1,2}$.

5. Biorthogonal decomposition of the unit interval

This section describes how the multilevel decomposition of the half-line derived in the previous sections can be adapted to build a multilevel decomposition of the unit interval $I = (0, 1)$. Let us notice that, provided the scale is fine enough, the presence of the left boundary point does not influence the construction of the right one. More precisely, if we call $\varphi^{(0)}_{jl}$ and $\varphi^{(1)}_{jr}$ the modified boundary scaling functions (here, and from now on, the superscript $^{(0)}$ or $^{(1)}$ refers to the boundary point 0 or 1, respectively), we will choose a level j_0 such that, for all $j \geq j_0$,

$$V_j(I) = \text{span}\,\{\varphi^{(0)}_{jl} \,:\, l \in \mathcal{I}_L\} \oplus \text{span}\,\{\varphi_{jk} \,:\, k \in \mathcal{I}_I\} \oplus \text{span}\,\{\varphi^{(1)}_{jr} \,:\, r \in \mathcal{I}_R\},$$

with $\mathcal{I}_I \neq \emptyset$ and where the boundary functions $\varphi^{(0)}_{jl}$ and $\varphi^{(1)}_{jr}$ are constructed independently.

The basic idea is to start from two multiresolution analyses on the half-lines $I_0 = (0, +\infty)$ and $I_1 = (-\infty, 1)$, and paste them together in a suitable way to get the spaces $V_j(I)$. In turn, to obtain a decomposition on I_1, we first consider a decomposition on $\mathbb{R}^- = (-\infty, 0)$ and then we translate it by a unit.

We therefore fix two nonnegative integers δ_1 and $\tilde{\delta}_1$ and we define the boundary functions as in (32); matching the dimensions of $V_0(\mathbb{R}^-)$ and $\widetilde{V}_0(\mathbb{R}^-)$, we obtain a relation similar to (35),

$$\tilde{\delta}_1 - \delta_1 = \widetilde{L} - L - \left\lceil \frac{\tilde{n}_1}{M-1} \right\rceil + \left\lceil \frac{n_1}{M-1} \right\rceil. \tag{64}$$

Using the operator $\tau : x \mapsto x - 1$, we translate the origin into the right edge of our interval. It is easy to see that, calling

$$\phi^{(1)}_{j\alpha}(x) = \sum_{k=M^j+1-\delta_1-\lceil n_1/(M-1)\rceil}^{M^j+\lceil -n_0/(M-1)\rceil-1} c^{(1)}_{\alpha,k-M^j}\, \varphi^{\mathbb{R}}_{jk}(x)\,, \qquad x \le 1,$$

we have

$$V_j(-\infty,1)=\mathrm{span}\{\phi^{(1)}_{j\alpha}\colon\ \alpha = 0,\ldots,L-1\} \oplus \mathrm{span}\left\{\varphi^{\mathbb{R}}_{jk}\colon\ k \le M^j - \delta_1 - \left\lceil \frac{n_1}{M-1} \right\rceil\right\}.$$

As mentioned before, we wish to maintain the situation at the two boundary points decoupled. This means we want to have at least one interior function in $V_j(I)$. This requirement yields the condition $\left\lceil -\frac{n_0}{M-1} \right\rceil + \delta_0 \le M^j - \delta_1 - \left\lceil \frac{n_1}{M-1} \right\rceil$. Taking also into account the dual relation, we have to set a coarsest level j_0 such that for all $j \ge j_0$,

$$\text{(65)}\quad M^j \ge \max\left\{ \left\lceil \frac{n_1}{M-1} \right\rceil + \left\lceil -\frac{n_0}{M-1} \right\rceil + \delta_0 + \delta_1,\right.$$
$$\left. \left\lceil \frac{\tilde{n}_1}{M-1} \right\rceil + \left\lceil \lceil -\frac{\tilde{n}_0}{M-1} \right\rceil + \tilde{\delta}_0 + \tilde{\delta}_1 \right\}.$$

By (35) and (64), we have

$$\left(\left\lceil \frac{\tilde{n}_1}{M-1} \right\rceil + \left\lceil -\frac{\tilde{n}_0}{M-1} \right\rceil + \tilde{\delta}_0 + \tilde{\delta}_1 \right) - \left(\left\lceil \frac{n_1}{M-1} \right\rceil + \left\lceil -\frac{n_0}{M-1} \right\rceil + \delta_0 + \delta_1 \right)$$
$$= 2(\widetilde{L} - L) \ge 0,$$

therefore we can fix

$$\text{(66)}\qquad j_0 \ge \left\lceil \log_M \left(\left\lceil \frac{\tilde{n}_1}{M-1} \right\rceil - + \left\lceil -\frac{\tilde{n}_0}{M-1} \right\rceil + \tilde{\delta}_0 + \tilde{\delta}_1 \right) \right\rceil.$$

Thus, we have, for $j \ge j_0$,

$$\begin{aligned} V_j(I) = \quad & \mathrm{span}\,\{\phi^{(0)}_{jk}\ :\ k = 0,\ldots,L-1\} \\ \oplus \quad & \mathrm{span}\,\{\varphi^{\mathbb{R}}_{jk}\ :\ k = \lceil -n_0/(M-1)\rceil + \delta_0,\ldots,M^j - \delta_1 - \lceil n_1/(M-1)\rceil\} \\ \oplus \quad & \mathrm{span}\,\{\phi^{(1)}_{jk}\ :\ k = 0,\ldots,L-1\}, \end{aligned}$$

and similarly for the dual spaces $\widetilde{V}_j(0,1)$. By construction, and thanks to the choice of j_0, all the biorthogonality properties are maintained. Finally, we observe that

$$\dim V_j(I) = M^j + 2L + 1 - \delta_0 - \delta_1 - \left\lceil \frac{n_1}{M-1} \right\rceil - \left\lceil -\frac{n_0}{M-1} \right\rceil, \qquad \forall j \ge j_0.$$

Since $V_{j+1}(I) = V_j(I) \oplus W_j(I)$, this implies

$$\dim W_j(I) = M^{j+1} - M^j = M^j(M-1).$$

REMARK 5. If $M = 2$, we get $\dim W_j(I) = 2^j$, which is the same as in [12].

REMARK 6. We have built a multilevel decomposition $\{V_j(I), P_j\}_{j\geq 0}$ of $L^2(I)$ and an **MRA** of $L^2(I)$, in the following sense:

(i) For $j \geq 0$

$$V_j = \operatorname{span}\{\varphi_{jk} : k \in \breve{\mathcal{K}}_j\} = \left\{ \sum_{k\in\breve{\mathcal{K}}_j} \alpha_k \varphi_{jk} : \{\alpha_k\} \in l^2(\breve{\mathcal{K}}_j) \right\},$$

with

$$\left\| \sum_{k\in\breve{\mathcal{K}}_j} \alpha_k \varphi_{jk} \right\| \asymp \left(\sum_{k\in\breve{\mathcal{K}}_j} |\alpha_k|^2 \right)^{1/2};$$

(ii) $V_j \subset V_{j+1}$, $\bigcap_{j\geq 0} V_j = V_0$, $\overline{\bigcup_{j\geq 0} V_j} = L^2(I)$;

(iii) $v(x) \in V_j \Rightarrow v(Mx) \in V_{j+1}$.

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