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C_0-SEMIGROUP AND OPERATOR IDEALS

Abstract. Let $T(t), 0 \leq t < \infty$, be a one parameter c_0-semigroup of bounded linear operators on a Banach space X with infinitesimal generator A and $R(\lambda, A)$ be the resolvent operator of A. The Hille-Yosida Theorem for c_0-semigroups asserts that the resolvent operator of the infinitesimal generator A satisfies $\|R(\lambda, A)\| \leq \frac{M}{\lambda-\omega}$ for some constants $M > 0$ and $\lambda \in R$ (the set of real numbers), $\lambda > \omega$. The object of this paper is to investigate when a c_0-semigroup can be in some operator ideals and to show that the previous inequality may not hold for certain ideal norms, including the p-summing, nuclear and the Schatten norms.

Introduction

Let X be a Banach space. A one parameter family $T(t)$, $0 \leq t < \infty$, of bounded linear operators from X into X is called a one parameter semigroup of bounded linear operators on X if : (i) $T(0) = I$ (the identity operator) and (ii) $T(s+t) = T(s)T(t)$ for all s, t in $[0, \infty)$. A semigroup, $T(t)$, is called strongly continuous if $\lim_{t\to 0^+} T(t)x = x$ for every $x \in X$. A strongly continuous semigroup of bounded linear operators is called a c_0-semigroup. If $T(t)$ is a c_0-semigroup, then there exist two constants $M > 0$ and $\omega \in (-\infty, \infty)$, such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \in [0, \infty)$. The linear operator A defined by :

$$D(A) = \{x \in X : \lim_{t\to 0^+} \tfrac{T(t)x-x}{t} \text{ exists}\} \text{ and } Ax = \lim_{t\to 0^+} \tfrac{T(t)x-x}{t} \text{ for } x \in D(A),$$

is called the infinitesimal generator of the semigroup $T(t)$ and $D(A)$ is the domain of A. The resolvent set of A is denoted by $\rho(A)$ and for $\lambda \in \rho(A)$, the operator $R(\lambda, A) = (\lambda - A)^{-1}$ is the resolvent operator of A. It is known, for c_0-semigroups, that $D(A)$ is dense in X, A is a closed operator and the resolvent operator $R(\lambda, A)$ is a bounded operator for all $\lambda \in \rho(A)$. We refer to [6] and [12] for excellent monographs on semigroups.

The Hille-Yosida Theorem (see[12]) asserts that, the resolvent operator $R(\lambda, A)$ of the infinitesimal generator A of a c_0-semigroup $T(t)$ satisfies the inequality :

$$\|R(\lambda, A)\| \leq \frac{M}{\lambda - \omega} \tag{1}$$

for ω and M as above and for $\lambda > \omega$. The norm of the resolvent operator in (1) is the usual operator norm. However, there are many important norms on different classes of bounded linear operators on X. It is natural to ask : **Does inequality** (1) **hold true for norms other than the operator norm?**

In this paper we address two questions :

(i) If $T(t) \in J(X) \subseteq L(X)$ for some ideal $J(X)$ with ideal norm $\|.\|_J$, can we prove a similar type of inequality of the form:

$$\|R(\lambda, A)\|_J \leq \frac{M}{\lambda - \omega} \tag{2}$$

for ω and M as above and for $\lambda > \omega$ with $\lambda \in \rho(A)$?

(ii) When can a c_0-semigroup be in some ideal $J(X) \subseteq L(X)$?

Problem (ii) was studied by Pazy [10] for compact operators $K(X) \subseteq L(X)$ on any Banach space X. For the ideal of Hilbert-Schmidt operators on a Hilbert space H, $C_2(H)$, the problem was discussed by Pazy [11] . Khalil and Deeb [8] studied the problem for the ideal of Schatten Classes $C_p(H) \subseteq L(H)$.

Throughout this paper, the dual of a Banach X is denoted by X^*, and $B_1(X)$ is the open unit ball of X. For $x^* \in X^*$ and $x \in X$ the value of x^* at x is denoted by $< x^*, x >$ and $x^* \otimes y$ denotes the operator $(x^* \otimes y)(x) =< x^*, x > y$. The set of real numbers will be denoted by R, and the set of positive integers by N.

1. Basic properties and examples of operator ideals

Let X be a Banach space and $L(X)$ be the space of all bounded linear operators from X into X. For $T \in L(X)$, $\|T\|$ denotes the operator norm of T.

DEFINITION 1. *Let $J(X)$ be a subset of $L(X)$. The set $J(X)$ is called an **ideal** of operators in $L(X)$ if :*

(i) $J(X)$ is a subspace of $L(X)$.

(ii) $PTQ \in J(X)$ for all $T \in J(X)$ and $P, Q \in L(X)$.

(iii) $J(X)$ contains all finite rank operators in $L(X)$.

*A function $\|.\|_J : J(X) \longrightarrow [0, \infty)$ is called an **ideal norm** on $J(X)$ if the followings are satisfied :*

(i) $\|.\|_J$ is a norm on $J(X)$.

(ii) $\|x^* \otimes y\|_J = \|x^*\| \, \|y\|$ for all one rank operators $x^* \otimes y$.

(iii) $\|PTQ\|_J \leq \|P\| \, \|Q\| \, \|T\|_J$ for all P, Q in $L(X)$ and $T \in J(X)$.

We refer to [7] and [13] for an excellent treatment of operator ideals. Now we present the basic examples of ideals of operators in $L(X)$ and the associated ideal norms. Let $p, q \in [1, \infty)$.

(**a**) An operator $T \in L(X)$ is called (p, q)-**summing** if there exists $\lambda > 0$ such that

$$\left(\sum_{i=1}^{n} \|T(x_i)\|^p \right)^{\frac{1}{p}} \leq \lambda \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^{n} \left| \langle x_i, x^* \rangle \right|^q \right)^{\frac{1}{q}} \tag{3}$$

for every finite set $\{x_1, x_2, ...x_n\} \subseteq X$. Let $\Pi_{p,q}(X)$ denote the set of all (p, q)-summing operators in $L(X)$. For $T \in \Pi_{p,q}(X)$, the (p, q)-summing norm of T is $\|T\|_{\Pi(p,q)} = \inf\{\lambda :$ (3) holds$\}$. It is known that, $\Pi_{p,q}(X)$ is an ideal of operators in $L(X)$ and $\|.\|_{\Pi(p,q)}$ is an ideal norm on $\Pi_{p,q}(X)$.

(**b**) An operator $T \in L(X)$ is called **strongly** (p,q)**-summing** if there exists $\lambda > 0$ such that :

$$\sup \left| \sum_{i=1}^{n} \left\langle Tx_i, x_i^* \right\rangle \right| \leq \lambda \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}} \tag{4}$$

for all finite sets $\{x_1, x_2, ...x_n\} \subseteq X$, where the supremum is taken over all finite sets $\{x_1^*, ...x_n^*\} \subseteq X^*$ for which $\sum \left|< x_i^*, x >\right|^q \leq 1$ for all $x \in B_1(X)$. Let $D_{p,q}(X)$ denote the set of all strongly (p,q)-summing operators in $L(X)$. For $T \in D_{p,q}(X)$, the strongly (p,q)-summing norm of T is $\|T\|_{D(p,q)} = \inf\{\lambda : (4) \text{ holds}\}$. It is known that, $D_{p,q}(X)$ is an ideal of operators in $L(X)$ and $\|T\|_{D(p,q)}$ is an ideal norm on $D_{p,q}(X)$.

(**c**) Let H be a Hilbert space and T be a compact operator in $L(H)$. Then T has a representation $T = \sum_{i=1}^{\infty} \lambda_i e_i \otimes f_i$, where (e_i), (f_i) are sequences of orthonormal vectors in H and $\lim_{i \to \infty} \lambda_i = 0$. **The Schatten class of index p** on H, denoted by $C_p(H)$, is the set of compact operators $T = \sum_{i=1}^{\infty} \lambda_i e_i \otimes f_i$ in $L(X)$ for which $\sum |\lambda_i|^p < \infty$. For $T \in C_p(H)$, set $\|T\|_p = \sup(\sum |< T\theta_i, \delta_i >|^p)^{\frac{1}{p}}$, where the supremum is taken over all orthonormal sequences (θ_i), (δ_i) in H. It is known that, $C_p(H)$ is an ideal of operators in $L(H)$ and $\|.\|_p$ is an ideal norm on $C_p(H)$.

(**d**) An operator $T \in L(X)$ is called **Cohen** (p,q)**-nuclear operator** if there exists $\lambda > 0$ such that :

$$\sup \left| \sum_{i=1}^{n} \left\langle Tx_i, x_i^* \right\rangle \right| \leq \lambda \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^{n} \left| \left\langle x_i, x^* \right\rangle \right|^q \right)^{\frac{1}{q}} \tag{5}$$

for all finite sets $\left\{x_1, x_2, ...x_n\right\} \subseteq X$, where the supremum on the left hand side of the inequality (5) is taken over all (x_i^*) in X^* for which $\sum \left|< x_i^*, x >\right|^p \leq 1$ for all $x \in B_1(X)$. Let $N_{p,q}(X)$ denote the set of all Cohen (p,q)-nuclear operators in $L(X)$. For $T \in N_{p,q}(X)$, set $\|T\|_{N(p,q)} = \inf\{\lambda : (5) \text{ holds}\}$. It is known,[2], that $N_{p,q}(X)$ is an ideal of operators in $L(X)$ and $\|.\|_{N(p,q)}$ is an ideal norm on $N_{p,q}(X)$.

(**e**) An operator $T \in L(X)$ is called **integral operator** if T admits a factorization:

$$\begin{array}{cccccc} & X & \stackrel{T}{\rightarrow} & X & \stackrel{i}{\rightarrow} & X^{**} \\ P & \downarrow & & & & \uparrow \quad Q \\ & L^{\infty}(\Omega,\mu) & & \stackrel{j}{\rightarrow} & & L^1(\Omega,\mu), \end{array}$$

where μ is a finite regular Borel measure on some compact Hausdorff space Ω, $J : L^{\infty}(\Omega,\mu) \to L^1(\Omega,\mu)$ is the canonical inclusion of $L^{\infty}(\Omega,\mu)$ into $L^1(\Omega,\mu)$, $Q \in L(L^1(\Omega,\mu), X^{**})$, $P \in L(X, L^{\infty}(\Omega,\mu))$ and I is the canonical embedding of X into X^{**}. Let $I(X)$ denote the set of all integral operators in $L(X)$. For $T \in I(X)$, set

$\|T\|_{int} = |\mu|(\Omega)$. It is known that $I(X)$ is an ideal of operators in $L(X)$ and $\|.\|_{int}$ is an ideal norm on $I(X)$.

(**f**) An operator $T \in L(X)$ is called **nuclear operator** if T has a nuclear representation : $T = \sum_{i=1}^{\infty} x_i^* \otimes y_i$ with $\sum_{i=1}^{\infty} \|x_i^*\| \|y_i\| < \infty$. The set of nuclear operators on X is denoted by $N(X)$ and for $T \in N(X)$, set $\|T\|_n = \inf \sum_{i=1}^{\infty} \|x_i^*\| \|y_i\|$, where the infimum is taken over all nuclear representations of T. It is known that, $N(X)$ is an ideal of operators in $L(X)$ and $\|T\|_n$ is an ideal norm on $N(X)$. Further, if X is reflexive, then $N(X) = I(X)$.

Examples (a), (b), (c), (e) and (f) are discussed in details in [7] and [13].

2. Operator ideals and the s.d. Property

Let $J(X)$ be an ideal of operators in $L(X)$ with ideal norm $\|.\|_J$.

DEFINITION 2. *The ideal $J(X)$ is said to have the* ***strong domination property*** *if for every sequence (T_n) of operators in $J(X)$ for which $\lim_{n\to\infty} T_n x = Tx$ for all $x \in X$ and $\|T_n\|_J \le \xi$ for some $\xi > 0$ and all $n = 1, 2, 3...$, then $T \in J(X)$ and $\|T\|_J \le \xi$. We will write* ***s.d*** *property for the strong domination property.*

If X is infinite dimensional, then the ideal of compact operators $K(X) \subseteq L(X)$ does not have the s.d. property since I, the identity operator, is not compact.

Now, we present some examples of operator ideals that have the s.d. property.

LEMMA 1. *The following ideals of operators have the s.d. property :* $\Pi_{p,q}(X)$, $C_p(H)$, $p \ge 2$, $D_{p,q}(X)$, $N_{p,q}(X)$ *and* $I(X)$.

Proof. That $\Pi_{p,q}(X)$ satisfies the s.d. property follows from the definition. The case of $C_p(H)$ follows from $C_p(H) = \Pi_{p,2}(H)$, $p \ge 2$, [9]. The proof for $D_{p,q}(X)$ and $N_{p,q}(X)$ follows from the definition of such ideals. So we prove only the case of $I(X)$.

Let (T_n) be a sequence in $I(X)$ such that $\sup_n \|T_n\|_{in} \le \xi$ for some $\xi > 0$ and $\lim_{n\to\infty} T_n x = Tx$ for all $x \in X$. By Proposition 17.5.2 in [7] we have :

$$|tr(T_n S)| \le \|T_n\|_{int} \|S\|$$

for every finite rank operator S in $L(X)$. If $S = \sum_{i=1}^{m} x_i^* \otimes x_i$, then

$$|tr(T_n S)| = \left| \sum_{i=1}^{m} \langle T_n x_i \, , \, x_i^* \rangle \right| \le \|T_n\|_{int} \|S\| \le \xi \|S\| .$$

Since $\lim\limits_{n\to\infty} T_n x = Tx$ for all $x \in X$ we get :

$$|tr(TS)| = \left| \sum_{i=1}^{m} \left\langle Tx_i \, , x_i^* \right\rangle \right| \leq \xi \, \|S\| \, .$$

Another application of Proposition 17.5.2 in [7] implies T is an integral operator and $\|T\|_{in} \leq \xi$. □

COROLLARY 1. *If Y is reflexive, then the ideal of nuclear operators $N(X)$ satisfies the s.d. property.*

Proof. Since Y is reflexive, then by Theorem 17.6.4 in [7] we get $N(X) = I(X)$. Hence, $N(X)$ satisfi es the s.d. property. □

3. c_0-semigroups and operator ideals

Let $J(X)$ be an ideal of operators in $L(X)$ with ideal norm $\|.\|_J$ such that $(J(X), \|.\|_J)$ is a Banach space. We call $J(X)$ a Banach ideal. Let $T(t)$, $0 \leq t < \infty$, be a one parameter c_0-semigroup of operators in $L(X)$ with infi nitesimal generator A and resolvent operator $R(\lambda, A)$. The main result of this section is : Inequality (2) can't hold true for any ideal norm $\|.\|_J$ different from the operator norm, for which $J(X)$ satisfi es the s.d. property.

LEMMA 2. *Let $T(t)$ be a c_0-semigroup on a Banach space X and $J(X)$ be an ideal in $L(X)$. If $T(t_0) \in J(X)$ for some $t_0 > 0$, then $T(t) \in J(X)$ for all $t > t_0$.*

Proof. For $t > t_0$,

$$T(t) = T(t - t_0 + t_0) = T(t_0)T(t - t_0).$$

Since $T(t - t_0)$ is a bounded linear operator for all $t > t_0$ and $T(t_0) \in J(X)$, then $T(t) \in J(X)$ for all $t > t_0$. □

LEMMA 3. *Let $T(t)$ be a c_0-semigroup on a Banach space X with infinitesimal generator A and $J(X)$ be an ideal in $L(X)$ that has the s.d. property. If $T(t) \in J(X)$ for all $t > 0$ and $\|T(t)\|_J < \xi$ in $(0, \epsilon)$ for some $\epsilon > 0$, then for $0 < a < \epsilon$, the operator $G_a : X \to X$, $G_a x = \int_0^a e^{-\lambda s} T(s)x \, ds$, for any $x \in X$, belongs to $J(X)$ and $\|G_a\|_J < \xi \frac{1-e^{-\lambda a}}{\lambda}$.*

Proof. For all $n \in N$, since $T(t) \in J(X)$ for all $t > 0$ and $\|T(t)\|_J < \xi$ in $(0, \epsilon)$ for some $\epsilon > 0$, the operators G_{an} defi ned by $G_{an}x = \sum_{k=1}^{n} \frac{e^{-\lambda t_k} T(t_k)x}{n}$ is in $J(X)$ and

$$\|G_{an}\|_J = \left\| \sum_{k=1}^{n} \frac{e^{-\lambda t_k} T(t_k)}{n} \right\|_J \leq \sum_{k=1}^{n} \frac{e^{-\lambda t_k} \, \|T(t_k)\|_J}{n} < \xi \sum_{k=1}^{n} \frac{e^{-\lambda t_k}}{n} < \xi \frac{1 - e^{-\lambda a}}{\lambda},$$

where, $\frac{(k-1)a}{n} < t_k < \frac{ka}{n}$.

Since $T(s)$ is strongly continuous, the operators G_{na} converge strongly to the operator G_a. By the s.d. property the operator $G_a \in J(X)$ and $\|G_a\|_J < \xi \frac{1-e^{-\lambda a}}{\lambda}$. □

Now we prove one of the main results of this paper.

THEOREM 1. *Let $T(t)$ be a c_0-semigroup on a Banach space X with infinitesimal generator A and $J(X)$ be a Banach ideal in $L(X)$ that has the s.d. property. Then following are equivalent :*

(i) $R(\lambda, A) \in J(X)$ for all $\lambda \in \rho(A)$ and $\|R(\lambda, A)\|_J \leq \frac{\beta}{\lambda-\omega}$ for some $\beta > 0$ and $\lambda > \omega \geq 0$.

(ii) $T(t) \in J(X)$ for all $t > 0$ and $\|T(t)\|_J \leq \xi$ in $(0, \varepsilon)$ for some $\varepsilon > 0$.

Proof. $(ii) \longrightarrow (i)$. For $n \in N$, $\lambda \in R$, $\lambda > \omega$, define :

$$\begin{aligned} R_n(\lambda, A)x &= \int_{\frac{1}{n}}^{\infty} e^{-\lambda s} T(s)x\, ds \\ &= \int_{\frac{1}{n}}^{\infty} e^{-\lambda s} T(\frac{1}{n}) T(s - \frac{1}{n})x\, ds \\ &= T(\frac{1}{n}) \int_{\frac{1}{n}}^{\infty} e^{-\lambda s} T(s - \frac{1}{n})x\, ds. \end{aligned}$$

Since $J(X)$ is an ideal in $L(X)$, $T(\frac{1}{n}) \in J(X)$ for all $n \in N$, and the operator P, defined by $P(x) = \int_{\frac{1}{n}}^{\infty} e^{-\lambda s} T(s - \frac{1}{n})\, x ds$, is a bounded linear operator in $L(X)$ for $\lambda > \omega$ and all $n \in N$, the operators $R_n(\lambda, A) \in J(X)$ for all n and all $\lambda \in R$, $\lambda > \omega > 0$. Further more, since $T(t)$ is a c_0-semigroup, then $\left\| T(s - \frac{1}{n}) \right\| \leq M e^{\omega(s - \frac{1}{n})}$, [12]. So using (ii) for $n \in N$, $\frac{1}{n} < \epsilon$, we get:

$$\begin{aligned} \|R_n(\lambda, A)\|_J = \left\| T(\frac{1}{n}) P \right\|_J &\leq \left\| T(\frac{1}{n}) \right\|_J \|P\| \\ &\leq \xi\, e^{-\frac{\omega}{n}} \int_{\frac{1}{n}}^{\infty} e^{-\lambda s} M e^{\omega s}\, ds = \frac{M\xi}{\lambda - \omega} e^{\frac{-\lambda}{n}}. \end{aligned}$$

Since $T(t) \in J(X)$, by Lemma 3, the operator $R_n(\lambda, A) - R(\lambda, A)$,

$$(R_n(\lambda, A) - R(\lambda, A))\, x = \int_0^{\frac{1}{n}} e^{-\lambda s} T(s)x\, ds$$

is in $J(X)$ and $\|R_n(\lambda, A) - R(\lambda, A)\|_J \leq \xi \, \frac{(1-e^{-\lambda/n})}{\lambda}$ for $n \in N$, $\frac{1}{n} < \epsilon$.

Since $R_n(\lambda, A) \in J(X)$ for all n, $\lim_{n\to\infty} \frac{(1-e^{-\lambda/n})}{\lambda} = 0$ and $(J(X), \|.\|_J)$ is a Banach space, it follows that $R(\lambda, A) \in J(X)$ for $\lambda > \omega > 0$ and

$$\begin{aligned} \|R(\lambda, A)\|_J &= \left\| \lim_{n\to\infty} R_n(\lambda, A) \right\|_J \\ &= \lim_{n\to\infty} \|R_n(\lambda, A)\|_J \\ &\leq \lim_{n\to\infty} \frac{M\xi}{\lambda - \omega} e^{\frac{-\lambda}{n}} = \frac{M\xi}{\lambda - \omega}. \end{aligned}$$

The resolvent identity,

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda) R(\lambda, A) R(\mu, A),$$

implies $R(\mu, A) \in J(X)$ for all $\mu \in \rho(A)$.

Conversely $(i) \to (ii)$. Since $T(t) \in L(X)$ for all $t > 0$ and $R(\lambda, A) \in J(X)$ for all $\lambda \in \rho(A)$, it follows that $\lambda R(\lambda, A) T(t) \in J(X)$. But $\|R(\lambda, A)\| \leq \frac{M}{\lambda - \omega}$ for $\lambda > \omega$, and

$$\lim_{\lambda\to\infty} \lambda R(\lambda, A) T(t) x = T(t) x,$$

for all $x \in X$, since $T(t)$ is a c_0-semigroup,[12]. Hence

$$\|\lambda R(\lambda, A) T(t)\|_J \leq \|\lambda R(\lambda, A)\|_J \, \|T(t)\| \leq \frac{\beta\lambda}{\lambda - \omega} \, \|T(t)\| \, .$$

Consequently, there exists $\gamma > 0$ and $\varepsilon > 0$ such that

$$\|\lambda R(\lambda, A) T(t)\|_J \leq \gamma \frac{\beta\lambda}{\lambda - \omega}$$

for all $t \in (0, \varepsilon)$ and $\lambda > \omega$. The s.d. property of $J(X)$ gives $T(t) \in J(X)$ for all $t \in (0, \varepsilon)$. Lemma 2 then implies that, $T(t) \in J(X)$ for all $t > 0$. Further, since $\{\frac{\lambda}{\lambda - \omega} : \lambda > \omega \geq 0\}$ is a bounded set, it follows that for $t \in (0, \varepsilon)$, $\|T(t)\|_J \leq \gamma \sup_{\lambda} \frac{\beta\lambda}{\lambda - \omega}$. □

As a corollary we get :

THEOREM 2. *Let X be an infinite dimensional Banach space and $T(t)$ be a c_0-semigroup in $L(X)$. The Hille-Yosida inequality for the resolvent operator $R(\lambda, A)$ of the infinitesimal generator A of $T(t)$ is not true if the operator norm, $\|R(\lambda, A)\|$, of $R(\lambda, A)$ is replaced by an ideal norm $\|R(\lambda, A)\|_J$ for any Banach ideal $J(X)$, properly contained in $L(X)$, that satisfies the s.d. property.*

Proof. Since (ii) in Theorem 1 implies that $I \subseteq J(X)$, the result follows. □

In the next result, the **boundedness condition** on $\|T(t)\|$ in Theorem 1 is replaced by an **integrability condition**.

THEOREM 3. *Let $T(t)$ be a c_0-semigroup on a Banach space X with infinitesimal generator A and $J(X)$ be a Banach ideal in $L(X)$ that has the s.d. property. If $T(t) \in J(X)$ for all $t > 0$, $T \in L^1((0,t_0), J(X))$ for some $t_0 > 0$ and the integral $\int_0^{t_0} \lambda e^{-\lambda s} \|T(s)\|_J \, ds$ is bounded in λ, then $R(\lambda, A) \in J(X)$ for all $\lambda \in \rho(A)$ and $\|\lambda R(\lambda, A)\|_J$ is bounded for large λ.*

Proof. Since $T \in L^1((0,t_0), J(X))$, it follows that for $0 < t < t_0$, $T \in L^1((0,t), J(X))$ and $\lim_{t\to 0} \int_0^t \|T(s)\|_J \, ds = 0$. For $\lambda \in \rho(A)$, $\lambda \in R$, $\lambda > \omega$ and $0 < t < t_0$, define:

$$R_t(\lambda, A)x = \int_t^\infty e^{-\lambda s} T(s)x \, ds = T(t) \int_t^\infty e^{-\lambda s} T(s-t)x \, ds.$$

Since $T(t) \in J(X)$ and the operator P, $P(x) = \int_t^\infty e^{-\lambda s} T(s-t)\, x ds$, is a bounded operator in $L(X)$ for $\lambda > \omega$, the operator $R_t(\lambda, A) \in J(X)$ and

$$\|R_t(\lambda, A) - R(\lambda, A)\|_J = \left\| \int_0^t e^{-\lambda s} T(s) \, ds \right\|_J \leq \int_0^t \|T(s)\|_J \, ds,$$

noting that $\sup_{s\in(0,t)} e^{-\lambda s} \leq 1$. Consequently, $\lim_{t\to 0} \|R_t(\lambda, A) - R(\lambda, A)\|_J = 0$. Further, since $J(X)$ is a Banach space, it follows that $R(\lambda, A) \in J(X)$ for all $\lambda \in R$, $\lambda > \omega$ and

$$\begin{aligned} \|\lambda R(\lambda, A)\|_J &= \left\| \int_0^\infty \lambda e^{-\lambda s} T(s) ds \right\|_J = \left\| \sum_{n=0}^\infty \int_{nt_0}^{(n+1)t_0} \lambda e^{-\lambda s} T(s) \, ds \right\|_J \\ &= \left\| \sum_{n=0}^\infty T(nt_0) \int_{nt_0}^{(n+1)t_0} \lambda e^{-\lambda s} T(s - nt_0) \, ds \right\|_J \end{aligned}$$

$$= \left\| \sum_{n=0}^{\infty} T^n(t_0)\lambda \int_0^{t_0} e^{-n\lambda t_0} e^{-\lambda s} T(s)\, ds \right\|_J$$

$$\leq \lambda \sum_{n=0}^{\infty} \|T(t_0)\|^n e^{-n\lambda t_0} \int_0^{t_0} e^{-\lambda s} \|T(s)\|_J \, ds$$

$$= \lambda \int_0^{t_0} e^{-\lambda s} \|T(s)\|_J \, ds \sum_{n=0}^{\infty} \left(\|T(t_0)\| e^{-\lambda t_0}\right)^n$$

$$= \frac{\lambda}{1 - e^{-\lambda t_0} \|T(t_0)\|} \int_0^{t_0} e^{-\lambda s} \|T(s)\|_J \, ds,$$

noting that for large λ, $e^{-\lambda t_0} \|T(t_0)\| < 1$. But $\lambda \int_0^{t_0} e^{-\lambda s} \|T(s)\|_J \, ds$ is bounded by assumption. Thus $\|\lambda R(\lambda, A)\|_J$ is bounded for large λ.

The resolvent identity,

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda) R(\lambda, A) R(\mu, A),$$

implies that $R(\mu, A) \in J(X)$ for all $\mu \in \rho(A)$. □

4. When is a semigroup in some operator ideal?

Throughout this section, $J(X)$ is an ideal in $L(X)$ with ideal norm $\|.\|_J$ for which $(J(X), \|.\|_J)$ is a Banach space.

THEOREM 4. *Let $T(t)$ be a c_0 differentiable semigroup on a Banach space X with infinitesimal generator A. If there exists $\lambda_0 \in \rho(A)$ such that $R(\lambda_0, A) \in J(X)$, then $T(t) \in J(X)$ for all $t > 0$.*

Proof. With no loss of generality, assume $\lambda_0 = 0$. Defi ne : $B(t)x = \int_0^t T(s)x\, ds$. Then $B \in L(X)$. Using Theorem 2.4 [12] we have :

$$AB(t)x = A \int_0^t T(s)x\, ds = T(t)x - x = (T(t) - I)x$$

for all $x \in X$. Hence $-AB(t) = (0 - A)B(t) = I - T(t)$. This implies

$$B(t) = R(0, A)(I - T(t)).$$

Since $R(0, A) \in J(X)$ and $I - T(t) \in L(X)$, the operator $B(t) \in J(X)$ for all $t > 0$. But $T(t)$ is strongly continuous. So,

$$T(t)x = \frac{d}{dt}\int_0^t T(s)x\,ds = \lim_{n\to\infty} n\left(B(t+\frac{1}{n})x - B(t)x\right)$$

in X. Put

$$\begin{aligned} D_n(t)x &= n\left(B(t+\frac{1}{n})x - B(t)x\right) \\ &= n\left(R(0,A)\left(I - T(t+\frac{1}{n})\right)x - R(0,A)\,(I - T(t))\,x\right) \\ &= n\,R(0,A)\left(T(t)x - T(t+\frac{1}{n})x\right). \end{aligned}$$

Since $T(t)$ is differentiable, so $\lim_{n\to\infty} n\left(T(t) - T(t+\frac{1}{n})\right) = -T'(t)$ and

$$T(t)x = \lim_{n\to\infty} D_n(t)x = -R(0,A)T'(t).$$

Thus, $T(t) \in J(X)$. □

5. Further results

In this section we present some results on some classes of semigroups of operators on Hilbert spaces.

THEOREM 5. *Let $T(t)$ be a c_0-semigroup on a Hilbert space H. If $T(t) \in C_{p_0}(H)$ for some $p_0 > 0$, then $T(t) \in C_p(H)$ for all $p > 0$.*

Proof. If $p > p_0$, then $T(t) \in C_p(H)$ since $C_{p_0}(H) \subseteq C_p(H)$, [7].

If $p < p_0$, then choose $n \in N$ such that $\frac{p_0}{n} < p$. Since $T(\frac{t}{n}) \in C_{p_0}(H)$, then $T^n(\frac{t}{n}) \in C_{\frac{p_0}{n}}$, [4]. But

$$T(t) = T(n\frac{t}{n}) = T^n(\frac{t}{n}).$$

Hence, $T(t) \in C_{\frac{p_0}{n}}(H) \subseteq C_p(H)$. □

REMARK 1. For self adjoint operators, Theorem 5 was proved in [8].

COROLLARY 2. *Let $T(t)$ be a c_0-semigroup of compact normal operators on a Hilbert space H and (λ_n) be the eigenvalues of the infinitesimal generator of $T(t)$. The following are equivalent :*

(i) *$T(t)$ is compact and* $\sum_{n=1}^{\infty} \left|e^{\lambda_n t}\right| < \infty.$

(ii) $T(t) \in C_2(H).$

Proof. $(i) \to (ii)$. Let $T(t)$ be a compact normal operator in $L(H)$. By the spectral mapping theorem for semigroups,[1], we have $T(t) = \sum_{n=1}^{\infty} e^{\lambda_n t} e_n(t) \otimes e_n(t)$, where $(e_n(t))$ is a sequence of orthonormal vectors in H. Since $\sum_{n=1}^{\infty} \left| e^{\lambda_n t} \right| < \infty$, we have $T(t) \in C_1(H) \subseteq C_2(H)$.

Conversely $(ii) \longrightarrow (i)$. Let $T(t) \in C_2(H)$ (so $T(t)$ is compact). Theorem 5 implies $T(t) \in C_1(H)$. Consequently, $\sum_{n=1}^{\infty} \left| e^{\lambda_n t} \right| < \infty$. □

REMARK 2. For self adjoint operators, Corollary 2 is to be found in [1].

THEOREM 6. *Let $T(t)$ be a c_0-semigroup of normal operators on a Hilbert space H with infinitesimal generator A. Then for $\lambda \in \rho(A)$ and Re $\lambda > \omega$, $R(\lambda, A)$ is a normal operator.*

Proof. Since $T(t)$ is normal for all $t \geq 0$, then $T(t)T^*(t) = T^*(t)T(t)$ for all $t \geq 0$. For $s, t \geq 0$ and $s, t \in Q$ (the set of rational numbers) there exist positive integers m, n, k, and r such that $s = \frac{n}{m}$ and $t = \frac{k}{r}$. Thus

$$T(t)T^*(s) = T(\frac{k}{r})T^*(\frac{m}{n}) = \left(T(\frac{1}{rn})\right)^{kn} \left(T^*(\frac{1}{rn})\right)^{mr}.$$

Since $T(t)$ is normal for all $t \geq 0$, we have :

$$T(t)T^*(s) = \left(T^*(\frac{1}{rn})\right)^{mr} \left(T(\frac{1}{rn})\right)^{kn} = T^*(s)T(t).$$

The density of Q in R gives $T(t)T^*(s) = T^*(s)T(t)$ for all $s, t \geq 0$.

Now, for $\lambda \in \rho(A)$, Re $\lambda > \omega$ we have :

$$\begin{aligned} R(\lambda, A)(R(\lambda, A))^* &= R(\lambda, A)R(\lambda, A^*) \\ &= \int_0^{\infty} e^{-\lambda t} T(t) dt \int_0^{\infty} e^{-\lambda t} T^*(t) dt \\ &= \int_0^{\infty}\int_0^{\infty} e^{-\lambda t} e^{-\lambda s} T(t)T^*(s) dt ds \\ &= R(\lambda, A^*)R(\lambda, A) \\ &= R(\lambda, A)^* R(\lambda, A). \end{aligned}$$

Thus, $R(\lambda, A)$ is a normal operator. □

REMARK 3. $R(\lambda, A)$ need not be a unitary operator for $\lambda \in \rho(A)$ even if the c_0-semigroup, $T(t)$, is a semigroup of unitary operators. The function $T : [0, \infty) \longrightarrow L\left(L^2[0,1]\right)$ defined by : $T(t) f(s) = \Phi(t, s) f(s)$, where $\Phi(s, t) = \cos st + i \sin st$, $t > 0$ defines a c_0-semigroup of unitary operators in $L(L^2[0, 1])$. But its infinitesimal generator A, $Af(s) = isf(s)$, is not a unitary operator.

Proof. Now, for $\lambda \in \rho(A)$, and Re $\lambda > \omega$ we have :

$$\begin{aligned} R(\lambda, A) f(s) &= \int_0^\infty e^{-\lambda t} T(t) f(s) dt \\ &= \left(\frac{\lambda}{s^2 + \lambda^2} + \frac{is}{s^2 + \lambda^2}\right) f(s). \end{aligned}$$

Hence, $R(\lambda, A)$ is a multiplication operator with $\Psi(\lambda, s) = \frac{\lambda}{s^2+\lambda^2} + \frac{is}{s^2+\lambda^2}$, and

$$\|R(\lambda : A)\| = \|\Psi\|_\infty = \sup_s \frac{1}{(s^2 + \lambda^2)^{\frac{1}{2}}} = \frac{1}{|\lambda|} \neq 1.$$

So, $R(\lambda, A)$ is not a unitary operator. □

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