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LARGE DEVIATIONS ESTIMATES FOR SOME LEVEL CROSSING PROBABILITIES IN BAYESIAN SETTING

Abstract. The probabilities of rare events have interest in some network and risk problems and, in this paper, we concentrate our attention on some level crossing probabilities for Brownian Motion and Poisson Process. We are interested in large deviations estimates and we present the Bayesian and frequentist approaches which lead to different conclusions; more precisely, as argued in other works in the literature on this topic, the Bayesian approach is more conservative. In the Bayesian framework we always deal with conjugate families of prior distributions.

1. Introduction

The study of level crossing probabilities has interest in several fields (mathematical risk theory connected with insurance problems, queueing theory and others). In this paper we consider the probabilities that some real valued stochastic processes (starting at zero) cross a positive level; moreover we are interested in large deviations estimates.

Stochastic processes can be used to model some phenomena. We are able to choose the stochastic process (in a suitable class) which better models the phenomena if we know all its relevant characteristics. On the other hand, without this knowledge, we have some unknown parameter to estimate.

The methods proposed so far tipically separate the point estimation of the parameter and the estimation of the level crossing probability. Anyway this approach can give a very misleading inference; for instance see the part of Introduction in [6] on the gambler's ruin.

On the other hand one can integrate the point estimation of the parameter and the estimation of level crossing probability using the Bayesian approach as in [6]. More precisely, instead of considering point estimation, our aim is to produce a full description of the uncertainty about the parameter, providing more information and coherence for the level crossing probabilities estimation. The main tool in the Bayesian approach is the so called *predictive level crossing probability* which is

$$\int_{\Theta} p(Q,\theta)\pi(d\theta|X_1,\ldots,X_n)$$

where:

- $p(Q,\theta)$ is the probability that the process crosses the positive level Q when the

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parameter is $\theta \in \Theta$ (Θ is the set of all possible values for θ);
• $\pi(\cdot|X_1, \ldots, X_n)$ is the posterior distribution of the parameter θ given an i.i.d. n-sample $(X_1, \ldots, X_n)$ drawn from the stochastic process.

The stochastic processes considered in this paper are Brownian Motion (with unknown drift or with unknown variance) and Poisson Process (with unknown intensity and with known negative drift). We present two kinds of large deviations estimates for level crossing probabilities: the first kind concerns the case $Q = qn$ for some fixed $q > 0$ and $n \to \infty$; the second kind is inspired by the so called *slow Markov walk limit* in large deviations theory (see e.g. [2], Example 1 and Theorem 2). We point out that (1) and (3) provide the frequentist estimates concerning slow Markov walk limit.

Let us conclude with the outline of the paper. Section 2 is devoted to recall some preliminaries; in particular some level crossing probabilities estimates are recalled, which can be considered as the frequentist estimates. In section 3 we present a Bayesian version of the estimates presented in section 2; we are not aware of any work on this topic in which Bayesian estimates concerning the slow Markov walk limit appear. Some concluding remarks are presented in section 4 where we point out the analogies with some other works cited in the bibliography.

2. Preliminaries

2.1. Preliminaries on large deviations

The main topic in this paper is large deviations, so let us recall some basic definitions (see e.g. [4], Chapter 1). Let Ω be a Hausdorff topological space with Borel σ-algebra $\mathcal{B}_\Omega$ and let us call rate function a lower semicontinuous function $I : \Omega \to [0, \infty]$; then a family of probability measures $(\nu_n)_{n\geq 1}$ on $(\Omega, \mathcal{B}_\Omega)$ satisfies the large deviations principle (LDP) with rate function I if

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu_n(F) \leq - \inf_{\omega\in F} I(\omega) \ (\forall F \text{ closed})$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu_n(G) \geq - \inf_{\omega\in G} I(\omega) \ (\forall G \text{ open}).$$

We also say that a family of Ω-valued random variables (Y_n) satisfies the LDP if the inequalities above hold with $\nu_n \equiv P(Y_n \in \cdot)$. We point out that in each proposition of section 3 we have a family of random probability measures (namely the posterior distributions) which satisfies the LDP almost surely.

2.2. Preliminaries on Brownian Motion

Let $(B_t)_{t\geq 0}$ be a standard Brownian Motion and let $(Z_t^{(\mu,\sigma^2)})_{t\geq 0}$ be defined as follows:

$$Z_t^{(\mu,\sigma^2)} \equiv B_{\sigma^2 t} + \mu t;$$

thus $(Z_t^{(\mu,\sigma^2)})_{t\geq 0}$ is a Brownian Motion with drift $\mu \in \mathbb{R}$ and variance parameter $\sigma^2 > 0$.

In order to introduce the level crossing probabilities it is useful to consider the random variables $(T_Q(\mu,\sigma^2) : Q > 0)$ defined by

$$T_Q(\mu,\sigma^2) \equiv \inf\{t \geq 0 : Z_t^{(\mu,\sigma^2)} > Q\};$$

then $(Z_t^{(\mu,\sigma^2)})_{t\geq 0}$ crosses any positive level Q with probability $P(T_Q(\mu,\sigma^2) < \infty)$ and

$$P(T_Q(\mu,\sigma^2) < \infty) \equiv \begin{cases} 1 & \mu \geq 0 \\ e^{2Q\frac{\mu}{\sigma^2}} & \mu < 0 \end{cases} \equiv e^{-Qw_{\mu,\sigma^2}},$$

where $w_{\mu,\sigma^2} \equiv \max\{0, -2\frac{\mu}{\sigma^2}\}$.

In conclusion, for all $q > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log P(T_{qn}(\mu,\sigma^2) < \infty) \equiv \lim_{n\to\infty} \frac{1}{n} \log e^{-qnw_{\mu,\sigma^2}} \equiv -qw_{\mu,\sigma^2};$$

$$\text{(1)} \qquad \lim_{n\to\infty} \frac{1}{n} \log P(T_q(\mu,\frac{\sigma^2}{n}) < \infty) \equiv \lim_{n\to\infty} \frac{1}{n} \log e^{-qw_{\mu,\sigma^2/n}} \equiv -qw_{\mu,\sigma^2}.$$

2.3. Preliminaries on Poisson Process

Let $(N_t)_{t\geq 0}$ be a standard homogeneous Poisson Process (namely with intensity equal to 1) and let $(Z_t^{(\lambda)}(\varepsilon))_{t\geq 0}$ be defined as follows:

$$Z_t^{(\lambda)}(\varepsilon) \equiv \varepsilon N_{\frac{\lambda}{\varepsilon}t} - ct;$$

thus $(Z_t^{(\lambda)}(\varepsilon))_{t\geq 0}$ is a suitable compound Poisson Process with drift. More precisely we have a compound Poisson Process which can have upwards jumps equal to ε only and the intensity for these jumps is $\frac{\lambda}{\varepsilon}$; moreover, for all $\lambda > 0$, $(Z_t^{(\lambda)}(\varepsilon))_{t\geq 0}$ crosses any positive level with probability 1 when $c \leq 0$ and so that we choose $c > 0$.

In order to introduce the level crossing probabilities it is useful to consider the random variables $(T_Q^{(\varepsilon)}(\lambda) : Q > 0)$ defined by

$$T_Q^{(\varepsilon)}(\lambda) \equiv \inf\{t \geq 0 : Z_t^{(\lambda)}(\varepsilon) > Q\};$$

then $(Z_t^{(\lambda)}(\varepsilon))_{t\geq 0}$ crosses any positive level Q with probability $P(T_Q^{(\varepsilon)}(\lambda) < \infty)$.

Furthermore let $(F_\varepsilon : \varepsilon > 0)$ be the function defined by

$$F_\varepsilon(\gamma) \equiv \frac{\lambda}{\varepsilon}(e^{\varepsilon\gamma} - 1) - c\gamma$$

and let $w_\lambda^{(\varepsilon)}$ be defined by

$$w_\lambda^{(\varepsilon)} \equiv \sup\{\gamma \geq 0 : F_\varepsilon(\gamma) \leq 0\};$$

then, since we have

$$\sup\{\gamma \geq 0 : F_\varepsilon(\gamma) \leq 0\} \equiv \sup\{\gamma \geq 0 : F_1(\varepsilon\gamma) \leq 0\},$$

we get the identity $\varepsilon w_\lambda^{(\varepsilon)} \equiv w_\lambda^{(1)}$.

By adapting some standard topic (see e.g. the proof of Theorem 5.1 in [1], Chapter XII, Section 5), we have

$$P(T_Q^{(\varepsilon)}(\lambda) < \infty) \equiv e^{-w_\lambda^{(\varepsilon)} Q} \mathbb{E}_{P_{w_\lambda^{(\varepsilon)}}} \left[e^{-w_\lambda^{(\varepsilon)} [N_{T_Q^{(\varepsilon)}(\lambda)} - cT_Q^{(\varepsilon)}(\lambda) - Q]} \right]$$

where $P_{w_\lambda^{(\varepsilon)}}$ is a suitable probability measure such that $P_{w_\lambda^{(\varepsilon)}}(T_Q^{(\varepsilon)}(\lambda) < \infty) = 1$ for all $Q > 0$; then, since we have

$$P_{w_\lambda^{(\varepsilon)}}(0 < N_{T_Q^{(\varepsilon)}(\lambda)} - cT_Q^{(\varepsilon)}(\lambda) - Q < \varepsilon) = 1,$$

we get the following refinement of Lundberg inequality:

$$e^{-w_\lambda^{(\varepsilon)}(Q+\varepsilon)} \leq P(T_Q^{(\varepsilon)}(\lambda) < \infty) \leq e^{-w_\lambda^{(\varepsilon)} Q}. \tag{2}$$

In conclusion (2) provides the following limits for all $q > 0$ (the first one is trivial while for the second one we have to employ $\varepsilon w_\lambda^{(\varepsilon)} \equiv w_\lambda^{(1)}$ with $\varepsilon = \frac{1}{n}$):

$$\lim_{n\to\infty} \frac{1}{n} \log P(T_{qn}^{(1)}(\lambda) < \infty) \equiv -q w_\lambda^{(1)};$$

$$\lim_{n\to\infty} \frac{1}{n} \log P(T_q^{(\frac{1}{n})}(\lambda) < \infty) \equiv -q w_\lambda^{(1)}. \tag{3}$$

3. Bayesian estimates

In this section we present a Bayesian version of the estimates presented above. More precisely we observe the processes $(Z_t^{(\mu,\sigma^2)})$ and $(Z_t^{(\lambda)}(1) + ct)$ when t is an integer and one of the parameters is unknown. Thus we consider some suitable statistical models and, for each one, we consider the conjugate family of prior distributions presented in [3] (Chapter 9); furthermore we prove an almost sure LDP for the posterior distributions applying Gärtner Ellis Theorem (see e.g. [4], Chapter 2, Section 3); finally we derive the estimates for level crossing probabilities using Varadhan's Lemma (see e.g. [4], Chapter 4, Section 3). We point out that posterior distributions can be expressed in a closed form since we deal with conjugate families; this fact leads to an easy application of Gärtner Ellis Theorem for posterior distributions.

In view of what follows, we introduce some useful notation.

- Let $H(\nu_1|\nu_2)$ be the relative entropy of ν_1 with respect to ν_2, namely

$$H(\nu_1|\nu_2) = \begin{cases} \int_\Omega \log(\frac{d\nu_1}{d\nu_2}(\omega))\nu_1(d\omega) & \nu_1 \| \nu_2 \\ \infty & \text{otherwise} \end{cases};$$

it is known that $H(\nu_1|\nu_2)$ is nonnegative and it is equal to zero if and only if $\nu_1 = \nu_2$ (see e.g. [9], Chapter 2, Section 3, Theorem 3.1).
• Let N_{μ,σ^2} be the Normal distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$.
• Let P_λ be the Poisson distribution with parameter $\lambda > 0$.
• Let $G_{\alpha,\beta}$ be the Gamma distribution with parameters $\alpha, \beta > 0$.

3.1. Brownian Motion with unknown drift

In this subsection we consider $(Z_t^{(\mu,\sigma^2)})$ where μ is unknown and $\sigma^2 = \frac{1}{r}$ is known. In such a case it is useful to consider the function $-\overline{w}_{\hat{\mu},\frac{1}{r}}$ defined by

$$-\overline{w}_{\hat{\mu},\frac{1}{r}}(q) \equiv \sup_{\mu\in\mathbb{R}}[-qw_{\mu,\frac{1}{r}} - H(N_{\hat{\mu},\frac{1}{r}}|N_{\mu,\frac{1}{r}})].$$

PROPOSITION 1. *Let (X_n) be the sequence defined by*

$$X_n \equiv Z_n^{(\mu,\frac{1}{r})} - Z_{n-1}^{(\mu,\frac{1}{r})}$$

and let $(\overline{X}_n)$ be the sequence of sample means, i.e. $\overline{X}_n \equiv \frac{1}{n}\sum_{k=1}^n X_k \equiv \frac{Z_n^{(\mu,\frac{1}{r})}}{n}$. Furthermore let $N_{\mu_0,\frac{1}{r_0}}$ be any prior Normal distribution for the parameter μ and let $N_{\mu_0,\frac{1}{r_0}}(\cdot|X_1,\dots,X_n)$ be the corresponding posterior distribution.
Then $\overline{X}_n$ converges to $\hat{\mu}$ almost surely (where $\hat{\mu}$ is the true value of the parameter) and we have:

$$\lim_{n\to\infty}\frac{1}{n}\log\int_{\mathbb{R}} P(T_{qn}(\mu,\frac{1}{r}) < \infty)N_{\mu_0,\frac{1}{r_0}}(d\mu|X_1,\dots,X_n) \equiv -\overline{w}_{\hat{\mu},\frac{1}{r}}(q);$$
$$\lim_{n\to\infty}\frac{1}{n}\log\int_{\mathbb{R}} P(T_q(\mu,\frac{1}{rn}) < \infty)N_{\mu_0,\frac{1}{r_0}}(d\mu|X_1,\dots,X_n) \equiv -\overline{w}_{\hat{\mu},\frac{1}{r}}(q).$$

Proof. It is known (see e.g. [3], 1970, Chapter 9, Section 5, Theorem 1) that

$$N_{\mu_0,\frac{1}{r_0}}(\cdot|X_1,\dots,X_n) = N_{\frac{r_0\mu_0+nr\overline{X}_n}{r_0+nr},\frac{1}{r_0+nr}}.$$

Moreover, since $\overline{X}_n$ converges to $\hat{\mu}$, in general we have

$$\Lambda_{\hat{\mu}}(\gamma) \equiv \lim_{n\to\infty}\frac{1}{n}\log\int_{\mathbb{R}} e^{n\gamma\mu}N_{\mu_0,\frac{1}{r_0}}(d\mu|X_1,\dots,X_n) \equiv$$
$$\equiv \lim_{n\to\infty}\frac{1}{n}(n\gamma\frac{r_0\mu_0+nr\overline{X}_n}{r_0+nr} + \frac{(n\gamma)^2}{2(r_0+nr)}) \equiv \gamma\hat{\mu} + \frac{\gamma^2}{2r};$$

thus, by Gärtner Ellis Theorem, $N_{\mu_0,\frac{1}{r_0}}(\cdot|X_1,\dots,X_n)$ satisfies the LDP with rate function

$$\Lambda_{\hat{\mu}}^*(\mu) \equiv \sup_{\gamma\in\mathbb{R}}[\gamma\mu - \Lambda_{\hat{\mu}}(\gamma)] \equiv H(N_{\hat{\mu},\frac{1}{r}}|N_{\mu,\frac{1}{r}})$$

(the latter equality can be checked). In conclusion we obtain the desidered limits by Varadhan's Lemma since the function $\mu \mapsto -w_{\mu,\frac{1}{r}}$ is continuous. □

3.2. Brownian Motion with unknown variance

In this subsection we consider $(Z_t^{(\mu,\sigma^2)})$ where μ is known (and the inequality $\mu < 0$ avoids trivial cases) and $\sigma^2 = \frac{1}{r}$ is unknown. In such a case it is useful to consider the function $-\overline{w}_{\mu,\frac{1}{\hat{r}}}$ defined by

$$-\overline{w}_{\mu,\frac{1}{\hat{r}}}(q) \equiv \sup_{r>0}[-qw_{\mu,\frac{1}{r}} - H(G_{\frac{1}{2},\frac{\hat{r}}{2}}|G_{\frac{1}{2},\frac{r}{2}})].$$

PROPOSITION 2. *Let (X_n) be the sequence defined by*

$$X_n \equiv (Z_n^{(\mu,\frac{1}{r})} - Z_{n-1}^{(\mu,\frac{1}{r})} - \mu)^2$$

and let $(\overline{X}_n)$ be the sequence of sample means, i.e. $\overline{X}_n \equiv \frac{1}{n}\sum_{k=1}^n X_k$. Furthermore let $G_{\alpha,\beta}$ be any prior Gamma distribution for the parameter r and let $G_{\alpha,\beta}(\cdot|X_1,\dots,X_n)$ be the corresponding posterior distribution.
Then $\overline{X}_n$ converges to $\frac{1}{\hat{r}}$ almost surely (where $\hat{r}$ is the true value of the parameter) and we have:

$$\lim_{n\to\infty}\frac{1}{n}\log\int_0^\infty P(T_{qn}(\mu,\frac{1}{r}) < \infty)G_{\alpha,\beta}(dr|X_1,\dots,X_n) \equiv -\overline{w}_{\mu,\frac{1}{\hat{r}}}(q);$$

$$\lim_{n\to\infty}\frac{1}{n}\log\int_0^\infty P(T_q(\mu,\frac{1}{rn}) < \infty)G_{\alpha,\beta}(dr|X_1,\dots,X_n) \equiv -\overline{w}_{\mu,\frac{1}{\hat{r}}}(q).$$

Proof. It is known (see e.g. [3], 1970, Chapter 9, Section 5, Theorem 2) that

$$G_{\alpha,\beta}(\cdot|X_1,\dots,X_n) = G_{\alpha+\frac{n}{2},\beta+\frac{n}{2}\overline{X}_n}.$$

Moreover, since $\overline{X}_n$ converges to $\frac{1}{\hat{r}}$ almost surely, in general we have

$$\Lambda_{\hat{r}}(\gamma) \equiv \lim_{n\to\infty}\frac{1}{n}\log\int_0^\infty e^{n\gamma r}G_{\alpha,\beta}(dr|X_1,\dots,X_n) \equiv$$

$$\equiv \lim_{n\to\infty}\begin{cases}\frac{\alpha+\frac{n}{2}}{n}\log(\frac{\beta+\frac{n}{2}\overline{X}_n}{\beta+\frac{n}{2}\overline{X}_n-n\gamma}) & n\gamma < \beta+\frac{n}{2}\overline{X}_n\\ \infty & n\gamma \geq \beta+\frac{n}{2}\overline{X}_n\end{cases} \equiv \begin{cases}\frac{1}{2}\log(\frac{\frac{1}{2\hat{r}}}{\frac{1}{2\hat{r}}-\gamma}) & \gamma < \frac{1}{2\hat{r}}\\ \infty & \gamma \geq \frac{1}{2\hat{r}}\end{cases};$$

thus, by Gärtner Ellis Theorem, $G_{\alpha,\beta}(\cdot|X_1,\dots,X_n)$ satisfies the LDP with rate function

$$\Lambda_{\hat{r}}^*(r) \equiv \sup_{\gamma\in\mathbb{R}}[\gamma r - \Lambda_{\hat{r}}(\gamma)] \equiv \begin{cases} H(G_{\frac{1}{2},\frac{\hat{r}}{2}}|G_{\frac{1}{2},\frac{r}{2}}) & r > 0\\ \infty & r \leq 0\end{cases}$$

(the latter equality can be checked). In conclusion we obtain the desidered limits by Varadhan's Lemma since the function $r \mapsto -w_{\mu,\frac{1}{r}}$ is continuous. □

3.3. Poisson Process

In this subsection we consider $(Z_t^{(\lambda)}(1))$ where λ is unknown. In such a case it is useful to consider the function $-\overline{w}_{\hat{\lambda}}$ defined by

$$-\overline{w}_{\hat{\lambda}}(q) \equiv \sup_{\lambda>0}[-qw_\lambda^{(1)} - H(P_{\hat{\lambda}}|P_\lambda)].$$

PROPOSITION 3. *Let (X_n) be the sequence defined by*

$$X_n \equiv Z_n^{(\lambda)}(1) - Z_{n-1}^{(\lambda)}(1) + c$$

and let $(\overline{X}_n)$ be the sequence of sample means, i.e. $\overline{X}_n \equiv \frac{1}{n}\sum_{k=1}^n X_k \equiv \frac{Z_n^{(\lambda)}(1)+cn}{n}$. Furthermore let $G_{\alpha,\beta}$ be any prior Gamma distribution for the parameter λ and let $G_{\alpha,\beta}(\cdot|X_1,\ldots,X_n)$ be the corresponding posterior distribution.
Then $\overline{X}_n$ converges to $\hat{\lambda}$ almost surely (where $\hat{\lambda}$ is the true value of the parameter) and we have:

$$\lim_{n\to\infty}\frac{1}{n}\log\int_0^\infty P(T_{qn}^{(1)}(\lambda)<\infty)G_{\alpha,\beta}(d\lambda|X_1,\ldots,X_n) \equiv -\overline{w}_{\hat{\lambda}}(q);$$

$$\lim_{n\to\infty}\frac{1}{n}\log\int_0^\infty P(T_q^{(\frac{1}{n})}(\lambda)<\infty)G_{\alpha,\beta}(d\lambda|X_1,\ldots,X_n) \equiv -\overline{w}_{\hat{\lambda}}(q).$$

Proof. It is known (see e.g. [3], 1970, Chapter 9, Section 4, Theorem 1) that

$$G_{\alpha,\beta}(\cdot|X_1,\ldots,X_n) = G_{\alpha+n\overline{X}_n,\beta+n}.$$

Moreover, since $\overline{X}_n$ converges to $\hat{\lambda}$ almost surely, in general we have

$$\Lambda_{\hat{\lambda}}(\gamma) \equiv \lim_{n\to\infty}\frac{1}{n}\log\int_0^\infty e^{n\gamma\lambda}G_{\alpha,\beta}(d\lambda|X_1,\ldots,X_n) \equiv$$

$$\equiv \lim_{n\to\infty}\begin{cases}\frac{\alpha+n\overline{X}_n}{n}\log(\frac{\beta+n}{\beta+n-n\gamma}) & n\gamma<\beta+n\\ \infty & n\gamma\geq\beta+n\end{cases} \equiv \begin{cases}\hat{\lambda}\log(\frac{1}{1-\gamma}) & \gamma<1\\ \infty & \gamma\geq 1\end{cases};$$

thus, by Gärtner Ellis Theorem, $G_{\alpha,\beta}(\cdot|X_1,\ldots,X_n)$ satisfies the LDP with rate function

$$\Lambda_{\hat{\lambda}}^*(\lambda) \equiv \sup_{\gamma\in\mathbb{R}}[\gamma\lambda - \Lambda_{\hat{\lambda}}(\gamma)] \equiv \begin{cases}H(P_{\hat{\lambda}}|P_\lambda) & \lambda>0\\ \infty & \lambda\leq 0\end{cases}$$

(the latter equality can be checked).
In view of what follows let us recall the following inequalities which follow from (2) (for $P(T_{qn}^{(1)}(\lambda)<\infty)$ is trivial while for $P(T_q^{(\frac{1}{n})}(\lambda)<\infty)$ we employ the identity $\varepsilon w_\lambda^{(\varepsilon)} \equiv w_\lambda^{(1)}$ with $\varepsilon=\frac{1}{n}$):

$$e^{-w_\lambda^{(1)}(qn+1)} \leq P(T_{qn}^{(1)}(\lambda)<\infty),\ P(T_q^{(\frac{1}{n})}(\lambda)<\infty) \leq e^{-w_\lambda^{(1)}qn}.$$

Moreover for all $\varepsilon > 0$ there exists $n_\varepsilon \geq 1$ such that the next inequalities hold for all $n \geq n_\varepsilon$:

$$(4) \qquad e^{-(q+\varepsilon)w_\lambda^{(1)}n} \leq P(T_{qn}^{(1)}(\lambda) < \infty), P(T_q^{(\frac{1}{n})}(\lambda) < \infty) \leq e^{-qw_\lambda^{(1)}n}.$$

Now let us point out what follows: the function

$$\gamma \mapsto \lambda_\gamma = \begin{cases} \frac{c\gamma}{e^\gamma - 1} & \gamma \neq 0 \\ c & \gamma = 0 \end{cases}$$

is continuous so that its inverse $\lambda \mapsto \gamma_\lambda$ is also continuous and then we can say that $\lambda \mapsto w_\lambda^{(1)}$ is continuous since the identity $w_\lambda^{(1)} \equiv \max\{\gamma_\lambda, 0\}$ holds.

The continuity of $\lambda \mapsto w_\lambda^{(1)}$ allows to apply Varadhan's Lemma and, by taking into account (4), we have

$$-\overline{w}_{\hat{\lambda}}(q+\varepsilon) \leq \liminf_{n\to\infty} \frac{1}{n} \log \int_0^\infty P(T_{qn}^{(1)}(\lambda) < \infty) G_{\alpha,\beta}(d\lambda|X_1, \ldots, X_n) \leq$$

$$\leq \limsup_{n\to\infty} \frac{1}{n} \log \int_0^\infty P(T_{qn}^{(1)}(\lambda) < \infty) G_{\alpha,\beta}(d\lambda|X_1, \ldots, X_n) \leq -\overline{w}_{\hat{\lambda}}(q)$$

and

$$-\overline{w}_{\hat{\lambda}}(q+\varepsilon) \leq \liminf_{n\to\infty} \frac{1}{n} \log \int_0^\infty P(T_q^{(\frac{1}{n})}(\lambda) < \infty) G_{\alpha,\beta}(d\lambda|X_1, \ldots, X_n) \leq$$

$$\leq \limsup_{n\to\infty} \frac{1}{n} \log \int_0^\infty P(T_q^{(\frac{1}{n})}(\lambda) < \infty) G_{\alpha,\beta}(d\lambda|X_1, \ldots, X_n) \leq -\overline{w}_{\hat{\lambda}}(q).$$

Thus we conclude the proof showing that $-\overline{w}_{\hat{\lambda}}$ is a continuous function. In order to do that one can check that $-\overline{w}_{\hat{\lambda}}$ is a convex function which assumes finite values since we have

$$0 \geq -\overline{w}_{\hat{\lambda}}(q) \geq -qw_{\hat{\lambda}}^{(1)} \quad (\forall q > 0);$$

then, since $\lim_{q\to 0^+} -\overline{w}_{\hat{\lambda}}(q) = -\overline{w}_{\hat{\lambda}}(0) = 0$, the continuity of $-\overline{w}_{\hat{\lambda}}$ follows from a well known result for convex functions (see e.g. [11], Chapter 3, Theorem 3.2). □

4. Concluding remarks

Let us consider the sequences (X_n) in each proposition in section 3 and assume that all the parameters are known. Then we can say what follows:

if μ in Proposition 1 is known, $(\overline{X}_n)$ satisfies the LDP with rate function $\hat{\mu} \mapsto H(N_{\hat{\mu},\frac{1}{r}}|N_{\mu,\frac{1}{r}})$;

if r in Proposition 2 is known, $(\overline{X}_n)$ satisfies the LDP with rate function $\hat{r} \mapsto H(G_{\frac{1}{2},\frac{\hat{r}}{2}}|G_{\frac{1}{2},\frac{r}{2}})$;

if λ in Proposition 3 is known, $(\overline{X}_n)$ satisfies the LDP with rate function $\hat{\lambda} \mapsto H(P_{\hat{\lambda}}|P_\lambda)$.

Then, if we compare the rate functions above with the rate functions for posterior distributions presented in each proposition of section 3, the roles played by the two arguments of $H(\cdot|\cdot)$ are interchanged.

This fact has some analogies with other works in the literature: Theorem 1 in [7] deals with random variables taking values in a finite set, Theorem 1 in [8] deals with random variables taking values in a compact metric space and a Dirichlet process as prior distribution, Theorem IV.3 in [10] and Theorem 1 in [5] deal with discrete time Markov chains.

In order to explain the interchange of arguments in the relative entropy, let us use the symbol θ for the parameters μ, r and λ which appear in each one of the propositions of section 3. Then: in the LDP of sample means $(\overline{X}_n)$ we ask how likely it is that $\overline{X}_n$ is close to some $\hat{\theta}$ given that the true value of the parameter is θ; in the LDP of posterior distributions we ask how likely it is that true value of the parameter is close to θ given that we observe $\overline{X}_n$ close to some $\hat{\theta}$.

The author thinks that the results in this paper could be extended to a wider class of Lévy processes by referring to appropriate exponential statistical models.

Furthermore let us point out that

$$(5)\qquad -\overline{w}_{\hat{\mu},\frac{1}{r}}(q) - (-qw_{\hat{\mu},\frac{1}{r}});\ -\overline{w}_{\mu,\frac{1}{\hat{r}}}(q) - (-qw_{\mu,\frac{1}{\hat{r}}});\ -\overline{w}_{\hat{\lambda}}(q) - (-qw^{(1)}_{\hat{\lambda}})$$

are nonnegative and nondecreasing functions (of q). The nonnegativeness is trivial; moreover the functions $-\overline{w}_{\hat{\mu},\frac{1}{r}}, -\overline{w}_{\mu,\frac{1}{\hat{r}}}$ and $-\overline{w}_{\hat{\lambda}}$ are nonpositive and convex functions (this can be checked) so that the functions (5) are nondecreasing as a consequence of the next proposition.

PROPOSITION 4. *Let $f : [0,\infty[\to \mathbb{R}$ be a convex function such that*

$$0 \geq f(q) \geq -qm \ \ (\forall q \in [0,\infty[)$$

for some $m > 0$. Then the function g defined by $g(q) \equiv f(q) - (-qm)$ is nondecreasing.

Proof. Let us consider $0 < x < y$ and set $t = \frac{x}{y}$; thus we have

$$f(x) = f(ty + (1-t)0) \leq tf(y) + (1-t)f(0) = tf(y)$$

since $x = ty + (1-t)0$, $t \in]0,1[$ and f is convex. Moreover we have

$$(1-t)f(x) = f(x) - tf(x) \leq tf(y) - tf(x) = t(f(y) - f(x))$$

whence we obtain

$$f(y) - f(x) \geq \underbrace{\frac{1-t}{t}}_{>0} f(x) \geq \frac{1-t}{t}(-xm) = -\frac{x}{t}m + xm = -ym + xm$$

which is equivalent to $g(y) \geq g(x)$. □

In conclusion the Bayesian estimates of all the level crossing probabilities in this paper are asymptotically guaranteed to be more conservative to a degree which becomes more pronunced as q increases. This fact agrees with the discussion presented in [6] on some general network and risk management problems in a Bayesian context.

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