Rend. Sem. Mat. Univ. Pol. Torino - Vol. 63, 1 (2005)
Polynom. Interp. Proceed.

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QUERIES ON FAMILIES OF FAT POINTS

Abstract. Here we raise some questions on the postulation of non-generic unions of fat points in $\mathbf{P}^n$, e.g. for a fixed integer $z > 0, t > 0$ the dimension of all such Z's with $h^i(\mathbf{P}^n, \mathcal{I}_Z(t)) \geq z$, $i = 0, 1$.

Let X be an integral n-dimensional projective variety. Fix positive integers k and m_i, $1 \leq i \leq k$, and set $z = z(n; m_1, \ldots, m_k) := \sum_{i=1}^{k} \binom{n + m_i - 1}{n}$. Let $A(X; m_1, \ldots, m_k)$ be the set of all zero-dimensional subschemes of X of the form $\cup_{i=1}^{k} m_i P_i$, where $P_1, \ldots, P_k$ are k distinct points of X_{reg}. Let $B(X; m_1, \ldots, m_k)$ be the closure of $A(X; m_1, \ldots, m_k)$ in the Hilbert scheme $\mathrm{Hilb}^z(X)$ of all length z zero-dimensional subschemes of X. Since $\mathrm{Hilb}^z(X)$ is a projective scheme, the variety $B(X; m_1, \ldots, m_k)$ is complete.

QUESTION 1. Fix $L \in \mathrm{Pic}(X)$ and set $\alpha_i := h^i(X, \mathcal{I}_Z \otimes L)$, $i = 0, 1$, where Z is the general element of $A(X; m_1, \ldots, m_k)$.

(a) Find upper and lower bounds for the dimension of integral subvarieties T of $A(X; m_1, \ldots, m_k)$ such that $h^i(X, \mathcal{I}_W \otimes L) > \alpha_i$ for every $W \in T$; more generally, fix an integer $a > 0$ and find bounds for $\dim(T)$ such that $h^i(X, \mathcal{I}_W \otimes L) \geq \alpha_i + a$ for every $W \in T$. More generally, do the same simultaneously for finitely many line bundles L.

(b) Find closed subvarieties $\bar{T}$ (if possible with large dimension) of $B(X; m_1, \ldots, m_k)$ which intersect $A(X; m_1, \ldots, m_k)$ and such that $h^i(X, \mathcal{I}_A \otimes L) = \alpha_i$ for every $A \in \bar{T}$ (or such that $h^i(X, \mathcal{I}_A \otimes L) \leq \alpha_i + a$ for every $A \in \bar{T}$).

(c) As in part (a) or (b) do the same taking a vector bundle E instead of L.

Part (b) means to find families of finite unions of fat points such that all their limits have good postulation or such that we may control the postulation of all their limits. For part (c) it is essential to consider only "important " or "nice " examples, e.g. sufficiently general stable vector bundles. When $X = \mathbf{P}^n$ part (c) for the bundles $\Omega^i_{\mathbf{P}^n}(t)$ is important for the minimal free resolution of finite unions of fat points in $\mathbf{P}^n$ and of their limits.

We are working on these questions, but our preliminary results do not kill the topic. For the case $X = \mathbf{P}^n$, $m_1 = \cdots = m_k = 1$ and $L = \mathcal{O}_{\mathbf{P}^2}(t)$, see [1].

*The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

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AMS Subject Classification: 14N05.

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