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A REMARK ON THE AMPLE CONE OF $\overline{\mathcal{M}}_{g,n}$

Abstract. Here we address a question on the ample cone of the moduli spaces of curves with an inductive approach inspired by a paper of Arbarello and Cornalba.

1. Introduction

The moduli space $\overline{\mathcal{M}}_{g,n}$ of n-pointed stable curves of genus g is a very natural and classical object, which from Riemann's times on has been deeply investigated by several authors. Despite this fact, the birational geometry of $\overline{\mathcal{M}}_{g,n}$ still remains a mystery. Indeed, every attempt in finding a regular pattern in the geometry of such a space seems destined to fail. This long history of trials and errors begins already in 1915 with Severi: being aware of the fact that $\mathcal{M}_g$ is unirational for $g \leq 10$, he was led to conjecture that $\mathcal{M}_g$ is unirational for all genera g (see [10], end of § 2.). That this is not really the case was shown in the first eighties by Eisenbud, Harris, and Mumford, who were able to prove that $\mathcal{M}_g$ is of general type for $g \geq 24$ (see [7], [2]). Anyway, the hope for a uniform description of the birational nature of the moduli spaces $\overline{\mathcal{M}}_g$ was still alive and inspired the celebrated Slope Conjecture by Harris and Morrison (see [6]): if $a\lambda - b\delta$ is the class of an effective divisor on $\overline{\mathcal{M}}_g$, then it should be $\frac{a}{b} \geq 6 + \frac{12}{g+1}$. In particular, since the class of the canonical divisor on $\overline{\mathcal{M}}_g$ is exactly $13\lambda - 2\delta$, it would follow that $\overline{\mathcal{M}}_g$ has negative Kodaira dimension for $g \leq 22$. Unfortunately, as recently pointed out by Farkas and Popa (see [4]), the Slope Conjecture does not hold for $g = 10$: a counterexample is provided by the divisor corresponding to curves on a K3 surface. Moreover, effective divisors behave in a wild manner already in genus zero: as observed by Keel and Vermeire (see [11]), the natural guess that every effective divisor on $\overline{\mathcal{M}}_{0,n}$ is an effective linear combination of boundary classes turns out to be false for every $n \geq 6$. However, one can still hope to fix at least the geometry of ample divisors. Recall that $\overline{\mathcal{M}}_{g,n}$ has a natural stratification by topological type, the codimension k strata corresponding to curves with at least k singular points. In the paper [5] by Gibney, Keel, and Morrison, the following Conjecture is attributed to Fulton:

CONJECTURE 1. ([5] (0.2)) A divisor on $\overline{\mathcal{M}}_{g,n}$ is ample if and only if it has positive intersection with all one-dimensional strata.

The main result of [5] is that Conjecture 1 holds for all g if and only if it holds for $g = 0$. In the same paper it is also described a natural approach to the case of $\overline{\mathcal{M}}_{0,n}$: as already pointed out by Keel and McKernan in [9], Conjecture 1 would be implied by a

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positive answer to:

QUESTION 1. ([5] (0.13)) If a divisor on $\overline{\mathcal{M}}_{0,n}$ has non-negative intersection with all one-dimensional strata, does it follow that the divisor is linearly equivalent to an effective combination of boundary divisors?

Until now, the best achievement in this direction is the following:

THEOREM 1. *For $n \leq 6$, the answer to Question 1 is affirmative.*

As we shall see, it is possible to approach such a result in several different ways. First of all, as noticed in [5] (0.14), Question 1 admits a purely combinatorial reformulation, which can be checked by using a computer precisely for $n \leq 6$ (indeed, its computational complexity makes it untractable already for $n = 7$). Next, a more conceptual analysis of the case $n = 6$ has been carried out in the paper [3] by Farkas and Gibney. Essentially, their idea is to express every divisor D on $\overline{\mathcal{M}}_{0,6}$ as an explicit linear combination of boundary divisors whose coefficients are intersection numbers with one-dimensional strata.

Here instead we are going to present a new proof, which follows an inductive strategy inspired by the paper [1] by Arbarello and Cornalba. Namely, let $P := \{1, 2, \ldots, n\}$ and for every $S \subset P$ with $2 \leq |S| \leq n-2$ let $\Delta_{\{0,S\}}$ be the boundary component of $\overline{\mathcal{M}}_{0,n}$ whose general element is the union of two copies of $\mathbb{P}^1$, labelled respectively by S and $P \setminus S$, meeting at one point. We denote by δ_S the corresponding class in $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n})$ and we define inductively:

$$\begin{aligned}
\mathscr{B}_4 &:= \{\delta_{\{2,3\}}\} \\
\mathscr{B}_n &:= \mathscr{B}_{n-1} \cup \{\delta_B : B \subseteq \{1, \ldots, n\}, n \notin B \supseteq \{n-1, n-2\}\} \\
&\quad \cup \{\delta_{B^c \setminus \{n\}} : \delta_B \in \mathscr{B}_{n-1} \setminus \mathscr{B}_{n-2}\}.
\end{aligned}$$

Then we have

PROPOSITION 1. *$\mathscr{B}_n$ is a basis of* $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n})$.

Moreover, for $n \leq 5$ every divisor having non-negative intersection with all one-dimensional strata can be expressed as an effective linear combination of divisors in $\mathscr{B}_n$; for $n = 6$ this is no longer the case, but one still maintains a control on the sign of coefficients, which is strong enough to conclude the proof of Theorem 1 in a few lines. Unfortunately, as n grows up, the combinatorial complexity of the problem explodes: indeed, even the case $n = 7$ seems to be completely out of reach.

2. The tools

We are going to make essential use of the following basic facts:

LEMMA 1. *(Arbarello–Cornalba) Let* $\vartheta : \overline{\mathcal{M}}_{0,A\cup\{q\}} \longrightarrow \overline{\mathcal{M}}_{0,P}$ *be the map which associates to any* $A \cup \{q\}$*-pointed genus zero curve the* P*-pointed genus zero curve obtained by glueing to it a fixed* $A^c \cup \{r\}$*-pointed genus zero curve via identification of* q *and* r*. Then for every* $B \subset P$ *with* $B \neq A$ *and* $B \neq A^c$ *we have*

$$\vartheta^*(\delta_B) = \begin{cases} \delta_B & \text{if } B \subset A \text{ and } B \neq A \\ \delta_{B\setminus A^c \cup \{q\}} & \text{if } B \supset A^c \text{ and } B \neq A^c \\ 0 & \text{otherwise} \end{cases}$$

(see [1], Lemma 3.3).

LEMMA 2. *(Keel) If* $a, b, c, d \in \{1, 2, \dots, n\}$ *are four distinct elements, then the following relation holds in* $\overline{\mathcal{M}}_{0,n}$*:*

$$\sum_{\substack{a,b \in T \\ c,d \notin T}} \delta_T = \sum_{\substack{a,c \in T \\ b,d \notin T}} \delta_T$$

(see [8], (2) p. 550).

LEMMA 3. *(Gibney-Keel-Morrison) Let*

$$D = \sum_{|S|\geq 2} b_S \delta_S$$

be a divisor on $\overline{\mathcal{M}}_{0,n}$ *and set* $b_S := 0$ *for* $|S| = 1$*. Then* D *has non-negative intersection with all one-dimensional strata if and only if*

$$b_{I\cup J} + b_{I\cup K} + b_{I\cup L} \geq b_I + b_J + b_K + b_L$$

for every partition $I \cup J \cup K \cup L = \{1, 2, \dots, n\}$

(see [5], Theorem 2.1).

For further details, we refer the interested reader to the original papers; however, we stress that the corresponding proofs are very short and elementary.

3. The proofs

Proof of Proposition 1. From [8] it is known that $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n})$ is a free group on $2^{n-1} - \binom{n}{2} - 1$ generators. Therefore, in order to get the claim it will be sufficient to show:

(1) $$|\mathscr{B}_n| = 2^{n-1} - \binom{n}{2} - 1;$$

(2) there are no linear relations among the elements of $\mathscr{B}_n$.

We are going to check (1) by induction on n.

If $n = 4$, it is clear that $\mathscr{B}_4$ has the right cardinality.
If $n \geq 5$, by inductive assumption we have

$$|\mathscr{B}_{n-1}| = 2^{n-2} - \binom{n-1}{2} - 1$$

and

$$|\mathscr{B}_{n-1} \setminus \mathscr{B}_{n-2}| = \sum_{k=0}^{n-5} \binom{n-3}{k}.$$

It follows that

$$\begin{aligned} |\mathscr{B}_n \setminus \mathscr{B}_{n-1}| &= \sum_{k=0}^{n-4} \binom{n-3}{k} + |\mathscr{B}_{n-1} \setminus \mathscr{B}_{n-2}| = \\ &= \sum_{k=0}^{n-4} \left[\binom{n-3}{k} + \binom{n-3}{k-1} \right] = \sum_{k=0}^{n-4} \binom{n-2}{k} \end{aligned}$$

and

$$\begin{aligned} 2^{n-1} - \binom{n}{2} - 1 &= 2^{n-2} + \sum_{k=0}^{n-2} \binom{n-2}{k} - \binom{n-1}{2} - (n-1) - 1 = \\ &= 2^{n-2} - \binom{n-1}{2} - 1 + \sum_{k=0}^{n-4} \binom{n-2}{k} = |\mathscr{B}_n|. \end{aligned}$$

As for (2), let us argue by induction on n again.
If $n = 4$ there is nothing to prove.
If $n \geq 5$, let $\sum a_B \delta_B = 0$ be a linear relation in $\mathscr{B}_n$. By Lemma 1 applied to $A := P \setminus \{n, n-1\}$ we have:

$$0 = \vartheta^*(\sum a_B \delta_B) = \sum_{\delta_B \in \mathscr{B}_{n-1}} a_B \delta_B$$

hence by inductive assumption $a_B = 0$ for every $\delta_B \in \mathscr{B}_{n-1}$. Next, by Lemma 1 applied to $A := P \setminus \{n, n-2\}$ we have:

$$0 = \vartheta^*(\sum a_B \delta_B) = \sum_{\delta_{B^c \setminus \{n\}} \in \mathscr{B}_{n-1}} a_B \delta_B$$

hence by inductive assumption $a_B = 0$ for every δ_B such that $\delta_{B^c \setminus \{n\}} \in \mathscr{B}_{n-1}$. In order to conclude, we have only to show that the elements in $\mathscr{B}_n$ with $n \notin B \supseteq \{n-1, n-2\}$ are linearly independent. This fact is a direct consequence of [1], Lemma 3.9, so the proof is over.

$\square$

Proof of Theorem 1. The case $n = 4$ is obvious. Fix $n = 5$ and let D be a divisor on $\overline{\mathcal{M}}_{0,5}$ having non-negative intersection with all one-dimensional strata. Write

$$D = c_{\{2,3\}}\delta_{\{2,3\}} + c_{\{3,4\}}\delta_{\{3,4\}} + c_{\{1,5\}}\delta_{\{1,5\}} + c_{\{2,5\}}\delta_{\{2,5\}} + c_{\{1,4\}}\delta_{\{1,4\}}$$

in the basis $\mathscr{B}_5$. From Lemma 3 it follows that $c_{\{2,3\}} \geq 0$ (let $I = \{2\}$, $J = \{3\}$, $K = \{1\}$, $L = \{4, 5\}$); $c_{\{3,4\}} \geq 0$ (let $I = \{3\}$, $J = \{4\}$, $K = \{5\}$, $L = \{1, 2\}$); $c_{\{1,5\}} \geq 0$ (let $I = \{1\}$, $J = \{5\}$, $K = \{3\}$, $L = \{2, 4\}$); $c_{\{2,5\}} \geq 0$ (let $I = \{2\}$, $J = \{5\}$, $K = \{4\}$, $L = \{1, 3\}$); $c_{\{1,4\}} \geq 0$ (let $I = \{1\}$, $J = \{4\}$, $K = \{2\}$, $L = \{3, 5\}$). Hence the case $n = 5$ is over. Fix now $n = 6$, let D be a divisor on $\overline{\mathcal{M}}_{0,6}$ having non-negative intersection with all one-dimensional strata and express $D = \sum c_B\delta_B$ in the basis $\mathscr{B}_6$. ¿From Lemma 1 applied to $A := P \setminus \{6, 5\}$ and to $A := P \setminus \{6, 4\}$ as in the proof of Proposition 1 and from Lemma 3 it follows that all coefficients are non-negative, with the unique possible exception of $\delta_{\{1,4,5\}}$. However, if $c_{\{1,4,5\}} < 0$, then by applying the relation:

$$\sum_{\substack{4,5 \in B \\ 2,3 \notin B}} \delta_B = \sum_{\substack{2,5 \in B \\ 3,4 \notin B}} \delta_B$$

(see Lemma 2), we can replace $\delta_{\{1,4,5\}}$ with $\delta_{\{1,2,3\}}$. If we express

$$D = \sum_{B \neq \{1,4,5\}} c'_B\delta_B + c'_{\{1,2,3\}}\delta_{\{1,2,3\}}$$

we have $c'_B \neq c_B$ only for $B = \{3, 4\}$, $B = \{1, 3, 4\}$, $B = \{2, 5\}$, and $B = \{1, 2, 5\}$; in all these cases, Lemma 1 shows that $c'_B \geq 0$. Since $c_B \geq 0$ for every $B \neq \{1, 2, 3\}$ and either $c_{\{1,4,5\}}$ or $c'_{\{1,2,3\}}$ is non-negative, it follows that Question 1 has a positive answer also for $n = 6$.

□

REMARK 1. In the case $n = 6$, one may wonder whether the sign of $c_{\{1,4,5\}}$ is actually ambiguous or not. Indeed, it is possible to construct explicit examples with $c_{\{1,4,5\}} < 0$ (for instance, take $c_{\{1,4,5\}} = -1$, $c_{\{2,3,4\}} = c_{\{1,2,5\}} = c_{\{3,4\}} = c_{\{2,3,5\}} = c_{\{1,3,4\}} = c_{\{2,5\}} = c_{\{2,4,5\}} = c_{\{3,4,5\}} = c_{\{2,3,4,5\}} = 1$, $c_{\{1,5\}} = c_{\{2,3\}} = c_{\{1,2,4,5\}} = c_{\{1,3,4,5\}} = c_{\{1,4\}} = c_{\{4,5\}} = 0$ and use Lemma 3) and with $c_{\{1,4,5\}} > 0$ (for instance, take $c_{\{1,4,5\}} = c_{\{4,5\}} = c_{\{2,3\}} = c_{\{2,3,4,5\}} = c_{\{2,3,4\}} = c_{\{2,3,5\}} = c_{\{2,4,5\}} = c_{\{3,4,5\}} = 1$, $c_{\{1,2,5\}} = c_{\{3,4\}} = c_{\{1,3,4\}} = c_{\{2,5\}} = c_{\{1,5\}} = c_{\{1,2,4,5\}} = c_{\{1,3,4,5\}} = c_{\{1,4\}} = 0$ and use Lemma 3). We also point out that the number of indeterminate signs grows up with n (for instance, in the case $n = 7$, none of the coefficients $c_{\{1,4,5\}}$, $c_{\{2,3,6\}}$, $c_{\{1,2,5,6\}}$, $c_{\{2,5,6\}}$, $c_{\{2,3,5,6\}}$, and $c_{\{1,4,5,6\}}$ is forced to be non-negative by Lemma 3); moreover, for $n \geq 7$ there seems to be no uniform way to apply a relation from Lemma 2 in order to remove a negative sign without introducing any other one.

References

[1] ARBARELLO E. AND CORNALBA M., *Calculating cohomology groups of moduli spaces of curves via algebraic geometry*, Inst. Hautes Études Sci. Publ. Math. **88** (1998), 97–127.

[2] Eisenbud D. and Harris J., *The Kodaira dimension of the moduli space of curves of genus* ≥ 23, Invent. Math. **90** (1987), 359–387.

[3] Farkas G. and Gibney A., *The Mori cones of moduli spaces of pointed curves of small genus*, Trans. Amer. Math. Soc. **355** (2003), 1183–1199.

[4] Farkas G. and Popa M., *Effective divisors on* $\bar{M}_g$*, curves on* $K3$ *surfaces and the slope conjecture*, J. Alg. Geom. (to appear).

[5] Gibney A., Keel S. and Morrison I., *Towards the ample cone of* $\overline{M}_{g,n}$, J. Amer. Math. Soc. **15** (2002), 273–294.

[6] Harris J. and Morrison I., *Slopes of effective divisors on the moduli space of stable curves*, Invent. Math. **99** (1990), 321–355.

[7] Harris J. and Mumford D., *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** (1982), 23–88.

[8] Keel S., *Intersection theory of moduli space of stable n-pointed curves of genus zero*, Trans. Amer. Math. Soc. **330** (1992), 545–574.

[9] Keel S. and McKernan J., *Contractible extremal rays on* $\overline{\mathcal{M}}_{0,n}$, preprint alg-geom/9607009 (1996).

[10] Severi F., *Sulla classificazione delle curve algebriche e sul teorema di esistenza di Riemann*, Rend. R. Acc. Naz. Lincei **24** (5) (1915), 877–888.

[11] Vermeire P., *A Counterexample to Fulton's Conjecture on* $\overline{\mathcal{M}}_{0,n}$, J. Algebra **248** (2002), 780–784.

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