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**A. Sandovici**

# A SMOOTH GAUGE MODEL ON TANGENT BUNDLE

**Abstract.** The tangent manifold $TM$ of a smooth, paracompact manifold $M$, fibered over $M$ by the natural projection $\pi$, carries an integrable distribution $\ker\pi_*$, called vertical distribution. If one takes a supplementary distribution of it, called horizontal, an almost complex structure $F_S$ appears. One endowes the vertical distribution with a Riemmanian metric $\gamma$. Then $\gamma$ can be prolonged to a Riemannian metric $G_S$ on $TM$ such that the pair $(F_S, G_S)$ becomes an almost Hermitian structure.

In this paper some deformations of $F_S$ and $G_S$ are proposed and new almost Hermitian structures are determined. With respect to some gauge objects, some properties of these structures are pointed out. For a smooth gauge geometrical model determined by one of these almost Hermitian structures, the general form of the corresponding Einstein-Yang Mills equations is obtained.

## 1. Introduction

In the last century, a lot of geometrical models for gravitational and electromagnetic theories have been proposed. Especially, there are some Riemannian, Finslerian or, more general, Lagrangian geometrical models which live on the tangent manifold $TM$ of a smooth, finite-dimensional and paracompact manifold $M$ [1, 4, 10, 13, 14, 22]. The differential geometry of the Lagrange spaces is now considerable developed and used in various fields to study the natural processes where the dependence of position, velocity or momentum are involved. The geometry of Lagrange spaces gives few models for both the gravitational and electromagnetic fields in a very natural blending of the geometrical structure of the space with the characteristic properties of the physical fields [15, 18, 19, 22, 23].

In this paper a new gauge geometrical model on the total space of the tangent bundle is proposed. This model appears in a natural way from the study of some almost Hermitian structures on $TM$.

The paper is organized as follows. In section 2 some natural geometrical objects, as the vertical distribution, almost tangent structure, nonlinear connection, Lagrange metric, d-connection and almost Hermitian structure are presented. The next section deals with some gauge objects on $TM$. The set of all gauge d-connections is caracterized in Theorem 2. In Lagrangian approach, the EYM equations are some tensorial equations which are obtained from a variational problem associated to a complete Lagrangian. A first form of these equations is obtained in Section 4. In the next two sections a class of locally conformal Kähler structures on tangent manifold is pointed out. This class is found in a study of a deformed Sasaki metric. Also, the problem of d-connections compatible with this class of almost Hermitian structures is solved. Moreover, from these investigations two remarcable geometrical models on $TM$ appear. One of them, called the homegeneous model has been intensively studied in [20]. The other is used in the last section in order to obtain a general form of EYM equations.

## 2. Preliminaries

As the Lagrange geometry is a part of the geometry of the tangent bundle of a manifold $M$, some natural geometrical objects, as the vertical distribution and the almost tangent structure, that live in $TM$ will be presented first. Also, an important tool in the geometry of the tangent bundle is the nonlinear connection.

### 2.1. Nonlinear connection on tangent manifold

Let $M$ be a real $n-$dimensional manifold of $C^{\infty}-$class. Denote by $(TM, \pi, M)$ the tangent bundle of the base manifold $M$ and by $(T_0M, \pi, M)$, the tangent bundle with the null cross-section removed. For every point $p \in M$, there exist local charts $\left(U, \varphi = \left(x^i\right)\right)$ on $p \in M$ and $\left(\pi^{-1}(U), \phi = \left(x^i, y^i\right)\right)$ on $u \in \pi^{-1}(p) \subset TM$ such that with respect to these, the canonical submersion $\pi$ has the equation $\pi : \left(x^i, y^i\right) \in \pi^{-1}(U) \to \left(x^i\right) \in U$. The local charts on $TM$ of the form $\left(\pi^{-1}(U), \phi = \left(x^i, y^i\right)\right)$ are called induced local charts, $\left(y^i\right)$ are coordinates of vectors $y^i \frac{\partial}{\partial x^i}|_p$ from $T_pM$, and $\frac{\partial}{\partial x^i}|_p$ is the natural basis of $T_pM$.

Denote by $\pi_*$ the linear map induced by the canonical submersion $\pi : TM \to M$. As for every $u \in TM$, $\pi_{*,u} : T_uTM \to T_{\pi(u)}M$ is an epimorphism, then the kernel determines a $n-$dimensional distribution $V : u \in TM \to V_uTM = ker\pi_{*,u} \subset T_uTM$. This distribution will be called the vertical distribution of the tangent bundle. In fact this is the tangent space to the natural foliation induced by the submersion $\pi$. Consequently, the vertical distribution is integrable. The natural basis of $T_uTM$ induced by a local chart $\left(\pi^{-1}(U), \phi = \left(x^i, y^i\right)\right)$ at $u$ is denoted by $\left\{\frac{\partial}{\partial x^i}|_u, \frac{\partial}{\partial y^i}|_u\right\}$. It follows that $\left\{\frac{\partial}{\partial y^i}|_u\right\}$ is a basis of $V_uTM$.

For every $u \in TM$ there exists the linear map $J_u : T_uTM \to T_uTM$, $J_u = \frac{\partial}{\partial y^i}|_u \otimes dx^i|_u$. It is called the almost tangent structure of the tangent bundle or the vertical endomorphism and it has two important properties $J_u^2 = 0$ and $\mathrm{Ker}J_u = \mathrm{Im}J_u = V_uTM$. Let $\mathcal{F}(TM)$ and $\mathcal{X}(TM)$ be the ring of $C^{\infty}-$functions over $TM$ and the $\mathcal{F}(TM)-$module of vector field over $TM$, respectively. With respect to the usual Poisson brackets, $\mathcal{X}(TM)$ is a real Lie algebra. Having this aspect in mind, the almost tangent structure $J$ may be taught as an $\mathcal{F}(TM)-$ linear map $J : \mathcal{X}(TM) \to \mathcal{X}(TM)$ with the local expresion $J = \frac{\partial}{\partial y^i} \otimes dx^i$.

A nonlinear connection on $TM$ is a $n-$dimensional distribution $HTM : u \in TM \to H_uTM \subset T_uTM$ that is supplementary to the vertical distribution, which means that the direct sum

$$T_uTM = H_uTM \oplus V_uTM, \quad \forall u \in TM, \tag{1}$$

holds. As $\pi_{*,u} : T_uTM \to T_{\pi(u)}TM$ is an epimorphism for every $u \in TM$, then the restriction of it to $H_uTM$ determines an isomorphism between $H_uTM$ and $T_{\pi(u)}M$. The inverse map of this isomorphism is denoted by $l_{h,u} : T_{\pi(u)}M \to H_uTM$ and it is called the horizontal lift induced by the given nonlinear connection $HTM$. If an induced local chart $\left(\pi^{-1}(U), \phi = \left(x^i, y^i\right)\right)$ at $u \in TM$ is fixed, then because of $\pi_{*,u} \circ l_{h,u} = \mathrm{Id}_{H_u TM}$ it follows that

$$l_{h,u}\left(\frac{\partial}{\partial x^i}|_{\pi(u)}\right) = \frac{\partial}{\partial x^i}|_u - N_i^j(u)\frac{\partial}{\partial y^j}|_u =: \frac{\delta}{\delta x^i}|_u.$$

The functions $N_j^i$ are defined over $\pi^{-1}(U)$ and are called the local coefficients of the nonlinear connection $HTM$.

For every $u \in TM$ and a local chart $\left(\pi^{-1}(U), \phi = \left(x^i, y^i\right)\right)$ at $u$ there exists now a basis $\left\{\frac{\delta}{\delta x^i}|_u, \frac{\partial}{\partial y^i}|_u\right\}$ of $T_u TM$ adapted to the decomposition (1). This basis will be called the adapted basis of the given nonlinear connection. If a change of induced local charts from

$$\left(\pi^{-1}(U), \phi = \left(x^i, y^i\right)\right) \quad \text{to} \quad \left(\pi^{-1}(V), \psi = \left(x'^i, y'^i\right)\right)$$

is performed

$$\text{(2)} \qquad \begin{cases} x'^i = x'^i(x^1, ..., x^n), & \mathrm{rank}(\frac{\partial x'^i}{\partial x^k}) = n, \\ y'^i = \frac{\partial x'^i}{\partial x^j} \cdot y^j, & \end{cases}$$

then the corresponding adapted basis and the local coefficients of the nonlinear connection are related as follows

$$\frac{\delta}{\delta x^i} = \frac{\partial x'^j}{\partial x^i}\frac{\delta}{\delta x'^j},$$

$$\frac{\partial}{\partial y^i} = \frac{\partial x'^j}{\partial x^i}\frac{\partial}{\partial y'^j}, \quad \mathrm{rank}\left(\frac{\partial x'^j}{\partial x^i}\right) = n,$$

$$\text{(3)} \qquad N_i^p \frac{\partial x'^j}{\partial x^p} = \frac{\partial x'^p}{\partial x^i} N'^j_p + \frac{\partial y'^j}{\partial x^i}.$$

At every point $u \in TM$, $T_u^* TM$ stands for the cotangent space at $u$ to $TM$, that is the dual space of $T_u TM$ over the field of real numbers $\mathbb{R}$. Then

$$\left\{dx^i|_u, \delta y^i|_u = dy^i|_u + N_j^i(u) dx^j|_u\right\},$$

is a basis of $T_u^* TM$, that is called the adapted cobasis of the nonlinear connection (in fact it is the dual of the adapted basis).

An important geometrical object for a nonlinear connection $HTM$ is the map $\theta : \mathcal{X}(TM) \to \mathcal{X}(TM)$, locally given by

$$\theta = \frac{\delta}{\delta x^i} \otimes \delta y^i.$$

The linear map $\theta$ is globally defined and it has the properties $\theta^2 = 0$, $\mathrm{Ker}\theta = \mathrm{Im}\theta = HTM$. The maps $h_u = \theta_u \circ J_u$ and $v_u = J_u \circ \theta_u$ are the horizontal and the vertical projectors that correspond to the Whitney sum (1).

### 2.2. Generalized Lagrange metrics

A generalized Lagrange metric (or a GL-metric for short) is a metric $g$ on the vertical subbundle $VTM$ of the tangent space $TM$ This means that for every $u \in TM$, $g_u : V_uTM \times V_uTM \to R$ is bilinear, symmetric, of rank $n$ and of constant signature. A pair $GL^n = (M, g)$, with a GL-metric is called a generalized Lagrange space, or a GL-space for short.

If $\left(\pi^{-1}(U), \phi = \left(x^i, y^i\right)\right)$ is an induced local chart at $u = (x, y) \in TM$, then the notation $g_{ij}(u) = g_u\left(\frac{\partial}{\partial y^i}|_u, \frac{\partial}{\partial y^j}|_u\right)$ will be used from now on. Therefore a GL-metric may be given by a set of functions $g_{ij}(x, y)$ such that the following three properties are verified

1. rank $\left(g_{ij}\right) = n$, $g_{ij}(x, y) = g_{ji}(x, y)$;
2. The quadratic form $g_{ij}(x, y)\xi^i\xi^j$ has constant signature on $TM$;
3. If another local chart $\left(\pi^{-1}(V), \psi = \left(x'^i, y'^i\right)\right)$ at $u \in TM$ is given and $g'_{kh}(x', y') = g_u\left(\frac{\partial}{\partial y'^k}|_u, \frac{\partial}{\partial y'^h}|_u\right)$ then $g_{ij}$ and $g'_{kh}$ are related by

$$g_{ij} = \frac{\partial x'^k}{\partial x^i}\frac{\partial x'^h}{\partial x^j}g'_{kh}. \tag{4}$$

A tensor field of $(r, s)$-type on $TM$ whose components transform under a change of local coordinates on $TM$ like the components of a tensor field of the same type on the base manifold $M$ is called a distinguished tensor field, or for short a d-tensor field. For instance, from (4) it follows that a GL-metric is a d-tensor field of (0, 2)-type.

Two important particular cases of GL-metrics will be mentioned bellow, namely Finsler and Lagrange metrics. A Finsler metric on $TM$ is a function $F : TM \to \mathbb{R}$ with the folowing properties

1. $F$ is a positive function of $C^\infty$-class on $T_0M$ and only continuous on the null cross-section of the tangent bundle;
2. $F$ is positively homogeneous of degree one on $T_0M$ with respect to the velocity $y^i$;
3. The matrix with the entries

$$g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}, \tag{5}$$

   has rank $n$ on $T_0M$ and the quadratic form $g_{ij}(x, y)\xi^i\xi^j$ has constant signature on $T_0M$.

A Finsler space is a pair $F^n = (M, F)$ with $F$ a Finsler metric. The tensor field with the components given by (5) is called the metric tensor of the Finsler space $F^n$. Denote by $g^{jk}$ the local components of the inverse matrix of $g_{ij}$ (that means $g_{ij}g^{jk} = \delta_i^k$).

If the homogeneity condition 2° is not asked, then $L := F^2$ is called a Lagrange metric and the pair $(M, L)$ is called a Lagrange space. The geometry of these spaces was intensively studied by R. Miron and M. Anastasiei in [17].

### 2.3. Almost Hermitian structures on $TM$

Let $I$ be the identity tensor field of (1, 1)-type on $TM$. An almost complex structure is a tensor field $F$ of (1, 1)-type on $TM$ such that $F^2 = -I$. In the geometry of the tangent bundle there exists a special almost complex structure, which in the adapted basis $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$, is given by

$$F_S\left(\frac{\delta}{\delta x^i}\right) = -\frac{\partial}{\partial y^i}, \quad F_S\left(\frac{\partial}{\partial y^i}\right) = \frac{\delta}{\delta x^i}. \tag{6}$$

Let $G$ be a metrical structure and $F$ be an almost complex structure on $TM$, respectively. A pair $(G, F)$ on $TM$ is said to be an almost Hermitian structure if the 2-form $\Omega(X, Y) := G(FX, Y)$ is closed and the following identity holds

$$G(FX, FY) = G(X, Y), \quad \forall X, Y \in \chi(TM). \tag{7}$$

For any almost complex structure $F$ its Nijenhuis tensor field is given by

$$N_F(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] - [X, Y], \quad X, Y \in \chi(TM).$$

An almost Hermitian structure is called locally conformal almost Kähler structure if there exists a 1-form $\theta$ such that $d\Omega = \Omega \wedge \theta$ and it is called almost Kähler structure if $d\Omega = 0$. A locally conformal almost Kähler structure $F$ is called locally conformal Kähler structure if its Nijenhuis tensor field $N_F$ is zero.

### 2.4. d-connections

A distinguished linear connection on $TM$ (a d-connection for short) is a linear connection $D$ on $TM$ which preserves by parallelism the horizontal distribution $H$ and the vertical distribution $V$, respectively. Working in the adapted basis $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$, the local coefficients $D\Gamma = (\overset{(H)}{F}{}^i_{jk}, \overset{(V)}{F}{}^i_{jk}, \overset{(H)}{C}{}^i_{jk}, \overset{(V)}{C}{}^i_{jk})$ of a d-connection on $TM$ are as follows

$$\begin{cases} D_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta x^j} = \overset{(H)}{F}{}^i_{jk} \frac{\delta}{\delta x^i}, & D_{\frac{\partial}{\partial y^k}} \frac{\delta}{\delta x^j} = \overset{(H)}{C}{}^i_{jk} \frac{\delta}{\delta x^i}, \\ D_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial y^j} = \overset{(V)}{F}{}^i_{jk} \frac{\partial}{\partial y^i}, & D_{\frac{\partial}{\partial y^k}} \frac{\partial}{\partial y^j} = \overset{(V)}{C}{}^i_{jk} \frac{\partial}{\partial y^i}. \end{cases}$$

Every d-connection on $TM$ is caracterized by its local coefficients $D\Gamma = (\overset{(H)}{F}{}^i_{jk}, \overset{(V)}{F}{}^i_{jk}, \overset{(H)}{C}{}^i_{jk}, \overset{(V)}{C}{}^i_{jk})$. With respect to (2), these local coefficients are some real functions on $TM$ such that the following laws of transformations

$$\overset{(H)}{F}{}^{\prime i}_{jk} = \frac{\partial x^{\prime i}}{\partial x^p}\frac{\partial x^q}{\partial x^{\prime j}}\frac{\partial x^r}{\partial x^{\prime k}}\overset{(H)}{F}{}^p_{qr} - \frac{\partial^2 x^{\prime i}}{\partial x^q \partial x^r}\frac{\partial x^q}{\partial x^{\prime j}}\frac{\partial x^r}{\partial x^{\prime k}},$$

$$\overset{(V)}{F}{}^{\prime i}_{jk} = \frac{\partial x^{\prime i}}{\partial x^p}\frac{\partial x^q}{\partial x^{\prime j}}\frac{\partial x^r}{\partial x^{\prime k}}\overset{(V)}{F}{}^p_{qr} - \frac{\partial^2 x^{\prime i}}{\partial x^q \partial x^r}\frac{\partial x^q}{\partial x^{\prime j}}\frac{\partial x^r}{\partial x^{\prime k}},$$

$$\overset{(H)}{C}{}^{\prime i}_{jk} = \frac{\partial x^{\prime i}}{\partial x^p}\frac{\partial x^q}{\partial x^{\prime j}}\frac{\partial x^r}{\partial x^{\prime k}}\overset{(H)}{C}{}^p_{qr},$$

$$\overset{(V)}{C}{}^{\prime i}_{jk} = \frac{\partial x^{\prime i}}{\partial x^p}\frac{\partial x^q}{\partial x^{\prime j}}\frac{\partial x^r}{\partial x^{\prime k}}\overset{(V)}{C}{}^p_{qr},$$

hold. Every vector field $X \in \mathcal{X}(TM)$ can be decomposed as $X = hX + \upsilon X$ such that, in local adapted coordinates $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$, the identities $hX = X^i \frac{\delta}{\delta x^i}$, $\upsilon X = \dot{X}^i \frac{\partial}{\partial y^i}$, hold. Every d-connection $D$ on $TM$ determines an algorithm of h-covariant derivative and an algorithm of v-covariant derivative in the algebra of d-tensor fields on $M$. Conversely, having two algorithms of h- and v- covariant derivatives, there exists a unique d-connection on $TM$ which generates exactly these derivative algorithms. The theory of such covariant derivatives has been developed in [17]. For instance, if $K^{ij}_{pq}(x, y)$ is a d-tensor field for which the first indices are of horizontality and the last indices are of verticality, then its h-covariant derivative is given by

$$K^{ij}_{pq|k} = \frac{\delta K^{ij}_{pq}}{\delta x^k} + \overset{(H)}{F}{}^i_{hk}K^{hj}_{pq} + \overset{(V)}{F}{}^j_{hk}K^{ih}_{pq} - \overset{(H)}{F}{}^h_{pk}K^{ij}_{hq} - \overset{(V)}{F}{}^h_{qk}K^{ij}_{ph},$$

and its v-covariant derivative is as folows

$$K^{ij}_{pq}\mid_k = \frac{\partial K^{ij}_{pq}}{\partial y^k} + \overset{(H)}{C}{}^i_{hk}K^{hj}_{pq} + \overset{(V)}{C}{}^j_{hk}K^{ih}_{pq} - \overset{(H)}{C}{}^h_{pk}K^{ij}_{hq} - \overset{(V)}{C}{}^h_{qk}K^{ij}_{ph}.$$

As usual, the torsion of a d-connection $D$ on $TM$ is determined by the formula

$$T(X, Y) = D_X Y - D_Y X - [X, Y], \quad \forall\, X, Y \in \mathcal{X}(TM).$$

Moreover, in the adapted basis $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$, the torsion tensor field $T$ can be expressed as

$$T\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^j}\right) = T^i_{jk}\frac{\delta}{\delta x^i} + R^i_{jk}\frac{\partial}{\partial y^i},$$

$$T\left(\frac{\partial}{\partial y^k}, \frac{\delta}{\delta x^j}\right) = \overset{(H)}{P}{}^i_{jk}\frac{\delta}{\delta x^i} + \overset{(V)}{P}{}^i_{jk}\frac{\partial}{\partial y^i},$$

$$T\left(\frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^j}\right) = S^i_{jk}\frac{\partial}{\partial y^i},$$

where

$$T^i_{jk} = \overset{(H)^i}{F}_{jk} - \overset{(H)^i}{F}_{kj}, \quad R^i_{jk} = \tfrac{\delta N^i_j}{\delta x^k} - \tfrac{\delta N^i_k}{\delta x^j},$$

$$\overset{(H)^i}{P}_{jk} = \overset{(H)^i}{C}_{jk} \quad \overset{(V)^i}{P}_{jk} = \tfrac{\partial N^i_j}{\partial y^k} - \overset{(V)^i}{F}_{kj},$$

$$S^i_{jk} = \overset{(V)^i}{C}_{jk} - \overset{(V)^i}{C}_{kj}.$$

Furthermore, the curvature of a d-connection $D$ on $TM$ is determined by the usual formula

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z, \quad \forall X, Y, Z \in \chi(TM).$$

In the adapted basis $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ the curvature $R$ has 16 local components but only 6 are essential, namely

$$R\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^h} = \overset{(H)^i}{R}_{hjk}\frac{\delta}{\delta x^i}, \quad R\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^h} = \overset{(V)^i}{R}_{hjk}\frac{\partial}{\partial y^i},$$

$$R\left(\frac{\partial}{\partial y^k}, \frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^h} = \overset{(H)^i}{P}_{hjk}\frac{\delta}{\delta x^i}, \quad R\left(\frac{\partial}{\partial y^k}, \frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^h} = \overset{(V)^i}{P}_{hjk}\frac{\partial}{\partial y^i},$$

$$R\left(\frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^j}\right)\frac{\delta}{\delta x^h} = \overset{(H)^i}{S}_{hjk}\frac{\delta}{\delta x^i}, \quad R\left(\frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^j}\right)\frac{\partial}{\partial y^h} = \overset{(V)^i}{S}_{hjk}\frac{\partial}{\partial y^i}.$$

The essential local components of the curvature tensor field of a d-connection $D$ on $TM$ are given by

$$\overset{(H)^i}{R}_{hjk} = \frac{\delta \overset{(H)^i}{F}_{hj}}{\delta x^k} - \frac{\delta \overset{(H)^i}{F}_{hk}}{\delta x^j} + \overset{(H)^m}{F}_{hj}\overset{(H)^i}{F}_{mk} - \overset{(H)^m}{F}_{hk}\overset{(H)^i}{F}_{mj} + R^m_{jk}\overset{(H)^i}{C}_{hm},$$

$$\overset{(V)^i}{R}_{hjk} = \frac{\delta \overset{(V)^i}{F}_{hj}}{\delta x^k} - \frac{\delta \overset{(V)^i}{F}_{hk}}{\delta x^j} + \overset{(V)^m}{F}_{hj}\overset{(V)^i}{F}_{mk} - \overset{(V)^m}{F}_{hk}\overset{(V)^i}{F}_{mj} + R^m_{jk}\overset{(V)^i}{C}_{hm},$$

$$\begin{aligned}\overset{(H)^i}{P}_{hjk} \;&=\; \frac{\partial \overset{(H)^i}{F}_{hj}}{\partial y^k} - \frac{\delta \overset{(H)^i}{C}_{hk}}{\delta x^j} + \overset{(H)^m}{F}_{hj}\overset{(H)^i}{C}_{mk} - \overset{(H)^m}{C}_{hk}\overset{(H)^i}{F}_{mj} \\ &\quad + \left(\overset{(V)^m}{P}_{jk} + \overset{(V)^m}{F}_{kj}\right)\overset{(H)^i}{C}_{hm},\end{aligned}$$

$$\begin{aligned}\overset{(V)^i}{P}_{hjk} \;&=\; \frac{\partial \overset{(V)^i}{F}_{hj}}{\partial y^k} - \frac{\delta \overset{(V)^i}{C}_{hk}}{\delta x^j} + \overset{(V)^m}{F}_{hj}\overset{(V)^i}{C}_{mk} - \overset{(V)^m}{C}_{hk}\overset{(V)^i}{F}_{mj} \\ &\quad + \left(\overset{(V)^m}{P}_{jk} + \overset{(V)^m}{F}_{kj}\right)\overset{(V)^i}{C}_{hm},\end{aligned}$$

$$\overset{(H)}{S}{}^{i}_{hjk} = \frac{\partial \overset{(H)}{C}{}^{i}_{hj}}{\partial y^k} - \frac{\partial \overset{(H)}{C}{}^{i}_{hk}}{\partial y^j} + \overset{(H)}{C}{}^{m}_{hj}\overset{(H)}{C}{}^{i}_{mk} - \overset{(H)}{C}{}^{m}_{hk}\overset{(H)}{C}{}^{i}_{mj},$$

$$\overset{(V)}{S}{}^{i}_{hjk} = \frac{\partial \overset{(V)}{C}{}^{i}_{hj}}{\partial y^k} - \frac{\partial \overset{(V)}{C}{}^{i}_{hk}}{\partial y^j} + \overset{(V)}{C}{}^{m}_{hj}\overset{(V)}{C}{}^{i}_{mk} - \overset{(V)}{C}{}^{m}_{hk}\overset{(V)}{C}{}^{i}_{mj}.$$

### 2.5. Metrical structure on $TM$

Let $N$ be a fixed nonlinear connection on $TM$. A h-metrical structure on $TM$ is a d-tensor field $hG = g_{ij}(x,y)dx^i \otimes dx^j$ with $g_{ij}(x,y) = g_{ji}(x,y)$, det $\big(g_{ij}(x,y)\big) \neq 0$ and such that the quadratic form $g_{ij}X^iX^j$ has constant signature. Furthermore, a v-metrical structure on $TM$ is a d-tensor field $\upsilon G = h_{ij}(x,y)\delta y^i \otimes \delta y^i$ with $h_{ij}(x,y) = h_{ji}(x,y)$, det $\big(h_{ij}(x,y)\big) \neq 0$ and such that the quadratic form $h_{ij}X^iX^j$ has constant signature. A (h,v)-metrical structure on $TM$ is a tensor field $G = hG + \upsilon G$ such that $hG$ is a h-metrical structure and $\upsilon G$ is a v-metrical structure on $TM$, respectively. Thus $G$ can be written as

$$G = g_{ij}(x,y)dx^i \otimes dx^j + h_{ij}(x,y)\delta y^i \otimes \delta y^j.$$

A d-connection $D$ on $TM$ is calleed compatible with the metrical structure $G$ if $D_X G = 0, \forall X \in \mathfrak{X}(TM)$. The next result can be found in [17]. It is mentioned here because it will be used in the next section.

THEOREM 1. *The d-connection $D$ on $TM$ having the local coefficients $D\Gamma(N) = (\overset{(H)}{F}{}^{i}_{jk}, \overset{(V)}{F}{}^{i}_{jk}, \overset{(H)}{C}{}^{i}_{jk}, \overset{(V)}{C}{}^{i}_{jk})$ as follows*

$$\overset{(H,c)}{F}{}^{i}_{jk} = \frac{1}{2}g^{ip}\left(\frac{\delta g_{pj}}{\delta x^k} + \frac{\delta g_{pk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^p}\right),$$

$$\overset{(V,c)}{F}{}^{i}_{jk} = \frac{\partial N^i_k}{\partial y^j} + \frac{1}{2}h^{ip}\left(\frac{\delta h_{jp}}{\delta x^k} - \frac{\partial N^q_k}{\partial y^j}h_{qp} - \frac{\partial N^q_k}{\partial y^p}h_{jq}\right),$$

$$\overset{(H,c)}{C}{}^{i}_{jk} = \frac{1}{2}g^{ip}\frac{\partial g_{jp}}{\partial y^k},$$

$$\overset{(V,c)}{C}{}^{i}_{jk} = \frac{1}{2}h^{ip}\left(\frac{\partial h_{pj}}{\partial y^k} + \frac{\partial h_{pk}}{\partial y^j} - \frac{\partial h_{jk}}{\partial y^p}\right),$$

*is compatible with the metrical structure $G$ and depends only on the nonlinear connection $N$ and on the metrical structure $G$.*

The d-connection which has been determined in Theorem 1 will be called the canonical connection of the metrical structure $G$ and will be denoted by $C\Gamma(N)$. The structures $hG$ and $vG$ generate two operators (called the Obata operators) which have the following local coefficients

$$\Omega^{ih}_{rj} = \frac{1}{2}\left(\delta^i_r\delta^h_j - g_{rj}g^{ih}\right), \quad \Theta^{ih}_{rj} = \frac{1}{2}\left(\delta^i_r\delta^h_j - h_{rj}h^{ih}\right).$$

COROLLARY 1. *The set of all d-connections compatible with metrical structure G is given as follows*

$$\overset{(H)}{F}{}^i_{jk} = \overset{(H,c)}{F}{}^i_{jk} + \Omega^{ih}_{rj}X^r_{hk}, \quad \overset{(V)}{F}{}^i_{jk} = \overset{(V,c)}{F}{}^i_{jk} + \Theta^{ih}_{rj}Y^r_{hk},$$

$$\overset{(H)}{C}{}^i_{jk} = \overset{(H,c)}{C}{}^i_{jk} + \Omega^{ih}_{rj}U^r_{hk}, \quad \overset{(V)}{C}{}^i_{jk} = \overset{(V,c)}{C}{}^i_{jk} + \Theta^{ih}_{rj}V^r_{hk},$$

*where* $X^r_{hk}$, $Y^r_{hk}$, $U^r_{hk}$ *and* $V^r_{hk}$ *are arbitrary d-tensor fields.*

## 3. Gauge objects on TM

The basic ideas for a geometrical point of view concerning the gauge theories on $TM$ can be found in the papers of Asanov [3, 4]. After that many authors used this approach for the developement of some physical theories having $TM$ as geometrical model (see for instance [1, 12, 13, 19]). This section is devoted to a Lagrangian gauge theory on the total space of the tangent bundle.

### 3.1. Gauge transformations

Let $\xi = (TM, \pi, M)$ be the tangent bundle of a finite-dimensional (dim$M = n$), paracompact, smooth real manifold $M$. A gauge transformation on $\xi$ is a pair of diffeomorphisms $(f_1, f_2)$, $f_1 : TM \to TM$, $f_2 : M \to M$ such that the following diagram

$$\begin{array}{ccccc} & & f_1 & & \\ & TM & ----\to & TM & \\ \pi & \downarrow & & \downarrow & \pi \\ & M & ----\to & M & \\ & & f_2 & & \end{array}$$

is commutative. Using the local coordinates, a gauge transformation can be given as

$$\text{(8)} \qquad \begin{cases} \widetilde{x}^i = X^i(x^1, ..., x^n), & rank(\frac{\partial X^i}{\partial x^k}) = n, \\ \widetilde{y}^i = Y^i(x^1, ..., x^n, y^1, ..., y^n), & rank(\frac{\partial Y^i}{\partial y^k}) = n, \end{cases}$$

Since $f_1$ and $f_2$ are globaly defined, the next compatibility condition follows from (2) and (8)

$$\widetilde{x}'^i\left(X^k(x^1, ..., x^n)\right) = X'^i(x'^k\left(x^1, ..., x^n\right)),$$

$$\frac{\partial x'^i}{\partial x^j}\left(X^k(x^1, ..., x^n)\right)Y^j(x, y) = Y'^i(x'^k(x), \frac{\partial x'^k}{\partial x^j}\cdot y^j),$$

where $X^i$, $Y^i$ are the local coordinates of $(f_1, f_2)$ in a local chart

$$\left(\pi^{-1}(U), \left(x^i, y^i\right)\right),$$

and $X'^i$, $Y'^i$ are the local coordinates of the same gauge transformation but with respect to the local chart

$$\left(\pi^{-1}(U'), \left(x'^i, y'^i\right)\right),$$

such that $U \cap U' \neq \phi$. Let $X^i_j$ be $\frac{\partial X^i}{\partial x^j}$ and $Y^i_j$ be $\frac{\partial Y^i}{\partial y^j}$. The notations $\overset{-}{X}{}^j_k$, $\overset{-}{Y}{}^j_k$ stand for the entries of the inverse of the matrices having the entries $X^i_j$ and $Y^i_j$, respectively. Let $(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i})$ be the natural basis of $T_u TM$, $u \in TM$ with respect to the local coordinates $\left(x^i, y^i\right)$. Concerning (2), the local components of the natural basis are tranformed as

$$\text{(9)} \qquad \begin{cases} \frac{\partial}{\partial x^i} = \frac{\partial x'^k}{\partial x^i} \cdot \frac{\partial}{\partial x'^k} + \frac{\partial^2 x'^k}{\partial x^i \partial x^j} y^j \frac{\partial}{\partial y'^k}, \\ \frac{\partial}{\partial y^i} = \frac{\partial y'^j}{\partial y^i} \cdot \frac{\partial}{\partial y'^j}. \end{cases}$$

Clearly, (8) and (2) lead to the following law of transformation

$$\text{(10)} \qquad \begin{cases} \frac{\partial}{\partial x^i} = X^k_i \cdot \frac{\partial}{\partial \widetilde{x}^k} + \frac{\partial Y^k}{\partial x^i} \cdot \frac{\partial}{\partial \widetilde{y}^k}, \\ \frac{\partial}{\partial y^i} = Y^j_i (x, y) \cdot \frac{\partial}{\partial \widetilde{y}^j}, \end{cases}$$

which shows the link between the natural bases $\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right)$ and $\left(\frac{\partial}{\partial \widetilde{x}^i}, \frac{\partial}{\partial \widetilde{y}^i}\right)$.

A d-tensor field on $TM$ is completely determined by a set of real functions $W^{\dots i \dots a \dots}_{\dots j \dots b \dots}$ defined on $\pi^{-1}(U)$ which with respect to (2) have the following law of transformation

$$W'^{\dots h \dots c \dots}_{\dots k \dots d \dots}\left(x', y'\right) = \dots \frac{\partial x'^h}{\partial x^i} \cdot \dots \cdot \frac{\partial x'^c}{\partial x^a} \cdot \dots \cdot \frac{\partial x^j}{\partial x'^k} \cdot \dots \cdot \frac{\partial x^b}{\partial x'^d} \cdot \dots \cdot W^{\dots i \dots a \dots}_{\dots j \dots b \dots}(x, y).$$

Moreover, a d-tensor field given by its local components $\left(W^{\dots i \dots a \dots}_{\dots j \dots b \dots}\right)$ is called a gauge d-tensor field if with respect to (8), the folowing law of transfomation

$$\widetilde{W}^{\dots h \dots c \dots}_{\dots k \dots d \dots}(\widetilde{x}, \widetilde{y}) = \dots \cdot X^h_i \cdot \dots \cdot Y^c_a \cdot \dots \cdot \overset{-}{X}{}^j_k \cdot \dots \cdot \overset{-}{Y}{}^b_d \cdot \dots \cdot W^{\dots i \dots a \dots}_{\dots j \dots b \dots}(x, y),$$

holds. It follows from (9) and (10) that $\left(\frac{\partial}{\partial y^i}\right)$ are gauge d-vector fields. A scalar function $L : TM \to \mathbb{R}$ is called a gauge scalar if the following identities $L'(x', y') = L(x, y)$, $\widetilde{L}(\widetilde{x}, \widetilde{y}) = L(x, y)$ hold true. It is important to remark that

1. $\frac{\partial L}{\partial y^i}$ is a gauge d-tensor field of (0,1)-type;

2. In general $\frac{\partial^2 L}{\partial y^i \partial y^j}$ is a d-tensor field but is not a gauge d-tensor field;

3. In general $\frac{\delta}{\delta x^i}$ is a d-vector field but is not a gauge d-vector field.

The above remarks lead to the necessity of some gauge covariant derivative operators. Before they will be introduced, some important gauge transformations - which are used in physics - are presented below.

$$\begin{cases} \overset{\sim}{x}^i = x^i, \\ \overset{\sim}{y}^i = Y^i(x,y), \quad \operatorname{rank}(\frac{\partial Y^i}{\partial y^j}) = n. \end{cases} \tag{11}$$

$$\begin{cases} \overset{\sim}{x}^i = X^i(x), \\ \overset{\sim}{y}^i = Y^i_j(x) \cdot y^j. \end{cases} \tag{12}$$

$$\begin{cases} \overset{\sim}{x}^i = x^i, \\ \overset{\sim}{y}^i = Y^i_j(x) \cdot y^j. \end{cases} \tag{13}$$

$$\begin{cases} \overset{\sim}{x}^i = X^i(x), \\ \overset{\sim}{y}^i = \varphi(x) \cdot \delta^i_j \cdot y^j, \end{cases} \tag{14}$$

where $\varphi$ is a real function on $M$. Also, the homotheties of $\xi$ are gauge transformations

$$\begin{cases} \overset{\sim}{x}^i = x^i, \\ \overset{\sim}{y}^i = \lambda \cdot y^j \;\;, \;\; \lambda \in (0,\infty). \end{cases} \tag{15}$$

REMARK 1. The set of all gauge transformations on $TM$ together with the usual law of composition form a group structure.

### 3.2. Gauge covariant derivatives

As have seen before, if the tangent bundle is endowed with a nonlinear connection, then two operators can be defined, namely h- and v- covariant derivatives, which transform every tensor field of (p,q)-type in a tensor field of (p,q+1)-type. More precisely, the algebra of d-tensor fields is closed with respect to these two operators. The aim of this subsection is to obtain some conditions concerning these operators such that their actions preserve the gauge character of d-tensor fields.

Let $N$ be a nonlinear connection in $\xi$ which is given by its local coefficients $\left(N^i_j(x,y)\right)$. With respect to (2) these coefficients verify (3). Let $L$ be a gauge scalar.

It is easy to see that $\frac{\delta L}{\delta x^i}$ este un d-covector field which have gauge character if and only if

$$\frac{\delta}{\delta x^i} = X_i^k \cdot \frac{\delta}{\delta \widetilde{x}^k}, \tag{16}$$

where

$$\frac{\delta}{\delta \widetilde{x}^i} = \frac{\partial}{\partial \widetilde{x}^i} - \widetilde{N}_i^j(\widetilde{x}, \widetilde{y}) \cdot \frac{\partial}{\partial \widetilde{y}^j}.$$

Here $\widetilde{N}_i^j$ is just $N_i^j$ calculated in the point $(\widetilde{x}, \widetilde{y})$. Moreover, it follows that the next identity

$$\widetilde{N}_j^i(\widetilde{x}, \widetilde{y}) \cdot X_k^j(x) = Y_j^i(x, y) \cdot N_k^j(x, y) - \frac{\partial Y^i(x, y)}{\partial x^k}, \tag{17}$$

must be satisfied. A nonlinear connection generated by a set of functions $\left(N_i^j(x, y)\right)$ which satisfy (3) and (17) is called a gauge nonlinear connection on $TM$.

Let $D\Gamma = (\overset{(H)}{F}{}^i_{jk}, \overset{(V)}{F}{}^i_{jk}, \overset{(H)}{C}{}^i_{jk}, \overset{(V)}{C}{}^i_{jk})$ be a d- connection on $TM$. Denoting $\overset{(H)}{\widetilde{F}}{}^i_{jk} = \overset{(H)}{F}{}^i_{jk}(\widetilde{x}, \widetilde{y})$ and so on for $\overset{(V)}{\widetilde{F}}{}^i_{jk}$, $\overset{(H)}{\widetilde{C}}{}^i_{jk}$, $\overset{(V)}{\widetilde{C}}{}^i_{jk}$, respectively, the following laws of transformations will be imposed with respect to (8)

$$\overset{(H)}{\widetilde{F}}{}^i_{jk} = X_h^i \cdot \overline{X}{}_j^l \cdot \overline{X}{}_k^m \cdot \overset{(H)}{F}{}^h_{lm} - \overline{X}{}_j^m \cdot \overline{X}{}_k^l \cdot \frac{\partial X_l^i}{\partial x^m}, \tag{18}$$

$$\overset{(V)}{\widetilde{F}}{}^i_{jk} = Y_h^i \cdot \overline{Y}{}_j^l \cdot \overline{X}{}_k^m \cdot \overset{(V)}{F}{}^h_{lm} - \overline{Y}{}_j^p \cdot \overline{X}{}_k^q \cdot \frac{\delta Y_p^i}{\delta x^q}, \tag{19}$$

$$\overset{(H)}{\widetilde{C}}{}^i_{jk} = X_r^i \cdot \overline{X}{}_j^p \cdot \overline{Y}{}_k^q \cdot \overset{(H)}{C}{}^r_{pq}, \tag{20}$$

$$\overset{(V)}{\widetilde{C}}{}^i_{jk} = Y_r^i \cdot \overline{Y}{}_j^p \cdot \overline{Y}{}_k^q \cdot \overset{(V)}{C}{}^r_{pq} - \overline{Y}{}_j^p \cdot \overline{Y}{}_k^q \cdot \frac{\partial Y_p^i}{\partial y^q}. \tag{21}$$

A d-connection $D\Gamma = (\overset{(H)}{F}{}^i_{jk}, \overset{(V)}{F}{}^i_{jk}, \overset{(H)}{C}{}^i_{jk}, \overset{(V)}{C}{}^i_{jk})$ for which the laws (18)-(21) hold is called a gauge d-connection on $TM$.

REMARK 2. 1. Using a direct computation, it is easy to see that the torsion and curvature tensor fields associated to a gauge d-connection have gauge character, too;

2. With respect to the gauge transformation (12), the Liouville vector field $\Gamma = y^i \cdot \frac{\partial}{\partial y^i}$ is a gauge d-vector field. Therefore, the corresponding deflection tensor fields have a gauge character, too;

3. The notations $d_k W^{\dots i \dots a \dots}_{\dots j \dots b \dots}$ and $D_k W^{\dots i \dots a \dots}_{\dots j \dots b \dots}$ stand for h- and v- covariant derivatives of a gauge d-tensor field $W^{\dots i \dots a \dots}_{\dots j \dots b \dots}$, respectively.

### 3.3. Gauge d-metrical connections

Assume that the nonlinear connection $N = (N^i_j(x, y))$ lives on $TM$. All gauge objects in this subsection are with respect to the gauge transformation (12). A gauge d-tensor field $g_{ij}(x, y)$ symmetric, nondegenerate ($det g_{ij}(x, y) \neq 0$) and such that the quadratic form $g_{ij}(x, y)\xi^i \xi^j$ has constant signature is called a gauge h-metrical structure on $TM$. A gauge d-tensor field $h_{ij}(x, y)$ having similar properties with those of $g_{ij}(x, y)$ is called a gauge v-metrical structure on $TM$. A pair $(g_{ij}(x, y), h_{ij}(x, y))$, where $g_{ij}(x, y)$ is a gauge h-metrical structure and $h_{ij}(x, y)$ is a gauge v-metrical structure is called a gauge (h,v)-metrical structure on $TM$.

A gauge d-connection $D\Gamma = (\overset{(H)}{F}{}^i_{jk}, \overset{(V)}{F}{}^i_{jk}, \overset{(H)}{C}{}^i_{jk}, \overset{(V)}{C}{}^i_{jk})$ is called (h,v)-metrical, or simply metric if the following relations

$$d_k g_{ij} = 0\,, \quad D_k g_{ij} = 0\,, \quad d_k h_{ij} = 0\,, \quad D_k h_{ij} = 0\,,$$

hold true. Moreover, there exist some remarcable gauge d-connections on $TM$. Indeed, the d-connection $\overset{(B)}{D}\Gamma = (\overset{(H,c)}{F}{}^i_{jk}, \frac{\partial N^i_j}{\partial y^k}, 0, \overset{(V,c)}{C}{}^i_{jk})$, where $\overset{(H,c)}{F}{}^i_{jk}$ and $\overset{(V,c)}{C}{}^i_{jk}$ are given by Theorem 1 is a gauge one. This gauge d-conection has some "regular" properties

$$\overset{(B)}{d}_k g_{ij} = 0\,, \quad \overset{(B)}{D}_k h_{ij} = 0\,, \quad \overset{(H,c)}{F}{}^i_{jk} = \overset{(H,c)}{F}{}^i_{kj}\,, \quad \overset{(V,c)}{C}{}^i_{jk} = \overset{(V,c)}{C}{}^i_{kj}.$$

The d-connection which appear in Theorem 1 is a gauge d-connection and depends only on the nonlinear connection $N$ and on the metrical structures $g_{ij}(x, y)$, $h_{ij}(x, y)$. Its h(hh)– and v(vv)– torsions are zero. Concerning the general form of the gauge d-connections, a result will be proved below

THEOREM 2. *The set of all gauge d-connections* $D\Gamma = (\overset{(H)}{F}{}^i_{jk}, \overset{(V)}{F}{}^i_{jk}, \overset{(H)}{C}{}^i_{jk}, \overset{(V)}{C}{}^i_{jk})$ *on* $TM$ *are those from Corollary 1 such that the gauge d-tensor fields* $(X^i_{jk}, Y^i_{jk}, U^i_{jk}, V^i_{jk})$ *verify the following identities*

$$\tilde{\Omega}^{ip}_{rj} \cdot \tilde{X}^r_{pk} = X^i_h \cdot \bar{X}^l_j \cdot \bar{X}^m_k \cdot \Omega^{hp}_{rl} \cdot X^r_{pm}, \tag{22}$$

$$\tilde{\Omega}^{ip}_{rj} \cdot \tilde{U}^r_{pk} = X^i_h \cdot \bar{X}^l_j \cdot \bar{Y}^m_k \cdot \Omega^{hp}_{rl} \cdot U^r_{pm}, \tag{23}$$

$$\overset{\sim}{\Theta}{}^{ip}_{rj} \cdot \overset{\sim}{Y}{}^{r}_{pk} = Y^i_h \cdot \overset{-}{Y}{}^l_j \cdot \overset{-}{X}{}^m_k \cdot \Theta^{hp}_{rl} \cdot Y^r_{pm}, \tag{24}$$

$$\overset{\sim}{\Theta}{}^{ip}_{rj} \cdot \overset{\sim}{V}{}^{r}_{pk} = Y^i_h \cdot \overset{-}{Y}{}^l_j \cdot \overset{-}{Y}{}^m_k \cdot \Theta^{hp}_{rl} \cdot V^r_{pm}. \tag{25}$$

*Proof.* It follows from Theorem 1 that $(\overset{(H)}{F}{}^i_{jk}, \overset{(V)}{F}{}^i_{jk}, \overset{(H)}{C}{}^i_{jk}, \overset{(V)}{C}{}^i_{jk})$ are the local coefficients of a metrical d-connection. It is remains to prove the gauge character of these coefficients. Applying (18) for $\overset{\sim}{\overset{(H,c)}{F}}{}^i_{jk}$ it follows that

$$\overset{\sim}{\overset{(H,c)}{F}}{}^i_{jk} = X^i_h \cdot \overset{-}{X}{}^l_j \cdot \overset{-}{X}{}^m_k \cdot \overset{(H,c)}{F}{}^h_{lm} - \overset{-}{X}{}^m_j \cdot \overset{-}{X}{}^l_k \cdot \frac{\partial X^i_l}{\partial x^m}. \tag{26}$$

The identity (18) follows by adding (22) and (26). In the same manner the proof of gauge character for $\overset{(V)}{F}{}^i_{jk}, \overset{(H)}{C}{}^i_{jk}, \overset{(V)}{C}{}^i_{jk}$ can be done. This final remark completes the proof. □

## 4. Einstein - Yang Mills equations

In Lagrangian approach, the Einstein - Yang Mills equations (brifly EYM equations) are some tensorial equations which are obtained from a variational problem associated to a complete Lagrangian determined by torsion and curvature tensors fields of a gauge d-connection $D$ on $TM$.

As usual, the tangent bundle is endowed with a gauge nonlinear connection $N$. Furthermore, assume that on $TM$ live a gauge (h,v)-metric structure and a gauge d-connection. The next partial Lagrangians are defined by

$$L_1 = T_{ijk} \cdot T^{ijk}, \quad L_2 = R_{ijk} \cdot R^{ijk}, \quad L_3 = \overset{(H)}{P}_{ijk} \cdot \overset{(H)}{P}{}^{ijk},$$
$$L_4 = \overset{(V)}{P}_{ijk} \cdot \overset{(V)}{P}{}^{ijk}, \quad L_5 = S_{ijk} \cdot S^{ijk},$$

$$L_{11} = R^{ij}_{ij}, \quad L_{12} = \overset{(V)}{R}_{ijkl} \cdot \overset{(V)}{R}{}^{ijkl}, \quad L_{13} = \overset{(H)}{P}_{ijkl} \cdot \overset{(H)}{P}{}^{ijkl},$$
$$L_{14} = \overset{(V)}{P}_{ijkl} \cdot \overset{(V)}{P}{}^{ijkl}, \quad L_{15} = \overset{(H)}{S}_{ijkl} \cdot \overset{(H)}{S}{}^{ijkl}, \quad L_{16} = S^{ij}_{ij},$$
$$L_{21} = \overset{(H)}{R}_{ijkl} \cdot \overset{(H)}{R}{}^{ijkl}, \quad L_{22} = \overset{(V)}{S}_{ijkl} \cdot \overset{(V)}{S}{}^{ijkl}.$$

By definition, a complete Lagrangian is given as

$$L = \sum_{k \in I} n_k \cdot L_k, \quad n_k \in R, \quad k \in \{1, 2, 3, 4, 5, 11, 12, 13, 14, 15, 16, 21, 22\}.$$

Since this Lagrangian $L$ is a gauge scalar, a Lagrangian density $\mathcal{L} = L \cdot G$ ($G = \left[\det\left(g_{ij}(x,y)\right)\right]^{\frac{1}{2}} \cdot \left[\det\left(h_{ij}(x,y)\right)\right]^{\frac{1}{2}}$) can be associated with it. This $\mathcal{L}$ depends on a set of gauge fields

$$\Phi \in \left\{ g_{ij},\ h_{ij},\ N^i_j,\ \overset{(H)}{F}{}^i_{jk},\ \overset{(V)}{F}{}^i_{jk},\ \overset{(H)}{C}{}^i_{jk},\ \overset{(V)}{C}{}^i_{jk} \right\}. \tag{27}$$

>From now on the gauge transformation (12) is implied. With respect to the gauge fields (27), the solutions of the variational problem $\delta \int \mathcal{L} d^n x d^n y = 0$ are as follows

$$\frac{\delta\mathcal{L}}{\delta\Phi} \equiv \frac{\partial}{\partial x^j}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\Phi}{\partial x^j}\right)}\right) + \frac{\partial}{\partial y^j}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\Phi}{\partial y^j}\right)}\right) - \frac{\partial\mathcal{L}}{\partial\Phi} = 0. \tag{28}$$

Moreover, the Lagrangian derivatives $\frac{\delta\mathcal{L}}{\delta\Phi}$ have gauge character.

REMARK 3.

1. For an arbitrary gauge d-connection, the form of EYM equations has been obtained by Balan in [5];
2. The particular case $\overset{(H)}{C}{}^i_{jk} = 0$, $n_{21} = n_{22} = n_3 = n_{15} = 0$ has been studied in [8];
3. In [4] Asanov has solved EYM equations for the Finslerian gauge approach of the gravitational field having spherical symmetries;
4. The case of the generalized Lagrange metric $g_{ij}(x,y) = e^{2\sigma(x,y)} \cdot \gamma_{ij}(x)$ has been studied in [6] from a gauge point of view.

## 5. Some almost Hermitian structures on $TM$

A set of locally conformal Kähler structures on tangent manifold $TM$ of a smooth, finite-dimensional manifold $M$ is pointed out. This is found in a study of Sasaki metric whose the terms are special deformations of a Riemannian metric on $M$. Introducing an adequate almost complex structure, a large class of locally conformal almost Kähler structures on $TM$ is obtained. When $M$ is a space form, a subset of it determines a locally conformal Kähler structure. Some classes of d-connections compatible with these structures are obtained.

### 5.1. The geometrical framework

Let $(M, \gamma)$ be a finite-dimensional Riemannian manifold and let $\nabla$ be its Levi - Civita connection. In a local chart $(U, (x^i))$ on $M$, $\gamma_{ij}$ stands for $\gamma(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ and $\gamma^i_{jk}(x)$ stands for the Christoffel symbols of the Levi - Civita connection. On $TM$ the functions

$N^i_j(x,y) := \gamma^{\ i}_{jk}(x)y^k$ are the local coefficients of a nonlinear connection. This one will be used throughout this section. The Sasaki metric on $TM$ is as follows

$$G_S = \gamma_{ij}(x)dx^i \otimes dx^j + \gamma_{ij}(x)\delta y^i \otimes \delta y^j. \tag{29}$$

Let $L^2$ be $\gamma_{ij}(x)y^iy^j$, let $y_i$ be $\gamma_{ij}(x)y^j$ and let $a,b,c,d : Im(L^2) \subseteq \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be some real, differentiable functions such that $a > 0, b \geq 0, c > 0, d \geq 0$. If $\gamma_{ij}(x)$ is now replaced by the local components $g_{ij}(x,y)$ and $h_{ij}(x,y)$ of some generalized Lagrange metrics, respectively, a new Sasaki type metric is obtained

$$G(x,y) = g_{ij}(x,y)dx^i \otimes dx^j + h_{ij}(x,y)\delta y^i \otimes \delta y^j. \tag{30}$$

When

$$g_{ij}(x,y) = a(L^2)\gamma_{ij}(x) + b(L^2)y_iy_j, \tag{31}$$

and

$$h_{ij}(x,y) = c(L^2)\gamma_{ij}(x) + d(L^2)y_iy_j, \tag{32}$$

the metrical structure (30) will be studied in this section.

REMARK 4. It is easy to see that the reciprocal entries of $g_{ij}(x,y)$ and $h_{ij}(x,y)$ are given by

$$g^{jk}(x,y) = \frac{1}{a}\left(\gamma^{\,jk} - \frac{b}{a+bL^2}y^jy^k\right),$$

and

$$h^{jk}(x,y) = \frac{1}{c}\left(\gamma^{\,jk} - \frac{d}{c+dL^2}y^jy^k\right).$$

For $b = d = 0$ , $a = 1$ , $c = \frac{\alpha^2}{L^2}$ with $\alpha$ nonzero real number, the metrical structure given by (30), (31) and(32) has been introduced by Miron in [16] as "the homogeneous lift of a Riemannian metric". For $a = 1$, $b = 0$, the above defined metrical structure has been studied by Anastasiei in [2].

### 5.2. Locally conformal almost Kähler structures on $TM$

The pair $(G,\ F_S)$, where the metrical structure $G$ is defined by (30), (31) and (32) is not an almost Hermitian structure on $TM$. The first aim of this section is to find a new almost complex structure $F$ which paired with $G$ to provide an almost Hermitian structure. The clasical almost structure $F_S$ is modified to a linear map $F$ given in the adapted basis $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$ as follows

$$F(\frac{\delta}{\delta x^i}) = (\alpha\delta^k_i + \beta y_iy^k)\frac{\partial}{\partial y^k}, \quad F(\frac{\partial}{\partial y^i}) = (\gamma\,\delta^k_i + \delta y_iy^k)\frac{\delta}{\delta x^k}, \tag{33}$$

where $\alpha, \beta, \gamma, \delta$ are differentiable functions on $TM$ to be determined. The condition for $F$ to be an almost complex structure, that is $F^2 = -I$, is given by

$$\alpha\gamma = -1, \quad \alpha\delta + \beta\gamma + \beta\delta L^2 = 0. \tag{34}$$

The compatitibility condition (7) can be written as

$$\begin{array}{ll} \alpha^2 c = a\,, & (2\alpha\beta + \beta^2 L^2)(c + dL^2) + \alpha^2 d = b, \\ \gamma^2 a = c\,, & (2\gamma\delta + \delta^2 L^2)(a + bL^2) + b\gamma^2 = d. \end{array} \tag{35}$$

The solution of the system (34) and (35) is given by

$$\begin{array}{ll} \alpha = -\sqrt{\frac{a}{c}}\,, & \beta = \frac{\sqrt{a(c+dL^2)} + \sqrt{c\left(a+bL^2\right)}}{L^2 \cdot \sqrt{c(c+dL^2)}}, \\ \gamma = \sqrt{\frac{c}{a}}\,, & \delta = -\frac{\sqrt{a(c+dF^2)} + \sqrt{c\left(a+bL^2\right)}}{L^2 \cdot \sqrt{a\left(a+bL^2\right)}}. \end{array} \tag{36}$$

If $b = d = 0$, the solution given in (36) can be written as

$$\begin{array}{ll} \alpha = -\sqrt{\frac{a}{c}}\,, & \beta = \frac{2}{L^2} \cdot \sqrt{\frac{a}{c}}, \\ \gamma = \sqrt{\frac{c}{a}}\,, & \delta = -\frac{2}{L^2} \cdot \sqrt{\frac{c}{a}}. \end{array} \tag{37}$$

When $b = d = 0$, the system (34), (35) admits one more solution different from (37). This new solution is given by

$$\alpha = -\sqrt{\frac{a}{c}}, \quad \gamma = \sqrt{\frac{c}{a}}, \quad \beta = 0, \quad \delta = 0. \tag{38}$$

Making the substitutions $a \longrightarrow \frac{a^2}{L^2}$, $b \longrightarrow \frac{b^2 - a^2}{L^4}$, $c \longrightarrow \frac{c^2}{L^2}$, $d \longrightarrow \frac{d^2 - c^2}{L^4}$ then it is easy to see that (36) and (37) are unified to

$$\begin{array}{ll} \alpha = -\frac{a}{c}\,, & \beta = \frac{ad+bc}{dcL^2}, \\ \gamma = \frac{c}{a}\,, & \delta = -\frac{ad+bc}{abL^2}. \end{array} \tag{39}$$

The solution (38) takes the form

$$\alpha = -\frac{a}{c}, \quad \beta = 0, \quad \gamma = \frac{c}{a}, \quad \delta = 0. \tag{40}$$

In this new setting, the metrical structure $G$ can be written as

$$\begin{aligned} G_{a,b,c,d}(x, y) &= \left(\frac{a^2}{L^2}\gamma_{ij}(x) + \frac{b^2 - a^2}{L^4} y_i y_j\right) dx^i \otimes dx^j \\ &+ \left(\frac{c^2}{L^2}\gamma_{ij}(x) + \frac{d^2 - c^2}{L^4} y_i y_j\right) \delta y^i \otimes \delta y^j, \end{aligned}$$

such that $b \geq a > 0$, $d \geq c > 0$.

Let $F_{a,b,c,d}$ be the almost complex structure defined by (33) and (39), and let $F_{a,c}$ be given by (33) and (40), respectively. Then the pairs $(G_{a,b,c,d},\ F_{a,b,c,d})$ and $(G_{a,a,c,c},\ F_{a,c})$ are almost Hermitian structures on the tangent bundle, respectively. In fact, the new almost complex structures are as follows

$$\begin{aligned} F_{a,b,c,d} &= \left(-\frac{a}{c}\delta_i^k + \frac{ad+bc}{dcL^2}y_i y^k\right)\frac{\partial}{\partial y^k}\otimes dx^i \\ &\quad + \left(\frac{c}{a}\delta_i^k - \frac{ad+bc}{abL^2}y_i y^k\right)\frac{\delta}{\delta x^k}\otimes \delta y^i, \end{aligned} \tag{41}$$

and

$$F_{a,c} = -\frac{a}{c}\frac{\partial}{\partial y^k}\otimes dx^k + \frac{c}{a}\frac{\delta}{\delta x^k}\otimes \delta y^k. \tag{42}$$

REMARK 5. Some particular cases are given below. All of these have important applications in some physical theories.

1. For $a = L,\ b = L,\ c^2 = \frac{L^2}{1+L^2},\ d^2 = L^2$, the Cheeger - Gromoll metrical is obtained
$$G_{CG}(x,y) = \gamma_{ij}(x)dx^i \otimes dx^j + \frac{1}{1+L^2}(\gamma_{ij}(x) + y_i y_j)\delta y^i \otimes \delta y^j.$$
2. If $a = L,\ b = L,\ c^2 = \varphi' L^2,\ d^2 = L^2(\varphi' + 2\varphi'' L^2)$ , where $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a real differentiable function with $\varphi'(t) \neq 0, t \in \operatorname{Im}(L^2)$, one obtains the Antonelli - Hrimiuc metrical structure:
$$G_{AH}(x,y) = \gamma_{ij}(x)dx^i \otimes dx^j + (\varphi'\gamma_{ij}(x) + 2\varphi'' y_i y_j)\delta y^i \otimes \delta y^j.$$
3. For $a = L, b = L$ one obtains the Anastasiei metrical structure
$$G_A(x,y) = \gamma_{ij}(x)dx^i \otimes dx^j + (\frac{c^2}{L^2}\gamma_{ij}(x) + \frac{d^2-c^2}{L^4}y_i y_j)\delta y^i \otimes \delta y^j.$$
4. If $a = L, b = L$ , $c = d = k, k \in R$ one obtains the Miron metrical structure
$$\overset{(\circ)}{G}(x,y) = \gamma_{ij}(x)dx^i \otimes dx^j + \frac{k^2}{L^2}\gamma_{ij}(x)\delta y^i \otimes \delta y^j.$$

THEOREM 3. *The almost Hermitian structures* $(G_{a,b,c,d}, F_{a,b,c,d})$ *are locally conformal almost Kahlerian structures.*

*Proof.* The relation (33) can be rewritten as

$$F\left(\frac{\delta}{\delta x^i}\right) = A_i^k \frac{\partial}{\partial y^k}, \quad F\left(\frac{\partial}{\partial y^i}\right) = B_i^k \frac{\delta}{\delta x^k}, \tag{43}$$

where

$$A_i^k = -\frac{a}{c}\delta_i^k + \frac{ad+bc}{dcL^2} y_i y^k, \quad B_i^k = \frac{c}{a}\delta_i^k - \frac{ad+bc}{abL^2} y_i y^k, \tag{44}$$

By a direct calculation one gets

$$\begin{aligned} \Omega\left(\tfrac{\delta}{\delta x^i}, \tfrac{\delta}{\delta x^j}\right) &= 0\,, \qquad \Omega\left(\tfrac{\delta}{\delta x^i}, \tfrac{\partial}{\partial y^j}\right) = -\Omega_{ij}, \\ \Omega\left(\tfrac{\partial}{\partial y^i}, \tfrac{\delta}{\delta x^j}\right) &= \Omega_{ij}\,, \quad \Omega\left(\tfrac{\partial}{\partial y^i}, \tfrac{\partial}{\partial y^j}\right) = 0, \end{aligned} \tag{45}$$

where

$$\Omega_{ij} = \frac{\gamma a^2}{L^2}\gamma_{ij} + \left(\gamma\,\frac{b^2-a^2}{L^4} + \delta\frac{b^2}{L^2}\right) y_i y_j = \Omega_{ji}. \tag{46}$$

The essential components of $d\Omega$ can be written as

$$d\Omega(\tfrac{\delta}{\delta x^i}, \tfrac{\delta}{\delta x^j}, \tfrac{\partial}{\partial y^k}) = \Omega_{ik|j} - \Omega_{jk|i}, \tag{47}$$

and

$$d\Omega(\tfrac{\delta}{\delta x^i}, \tfrac{\partial}{\partial y^j}, \tfrac{\partial}{\partial y^k}) = \tfrac{\partial \Omega_{ik}}{\partial y^j} - \tfrac{\partial \Omega_{ij}}{\partial y^k}, \tag{48}$$

where by "|" the h- covariant derivative with respect to Berwald connection $B\Gamma(N) = (N^i_j = \gamma^i_{jk}(x)y^k, \gamma^i_{jk}(x), 0)$ is denoted. The following formulas can be verified by a direct computation

$$\gamma_{ki|j} = 0, \quad y^j_{|k} = 0, \quad y_{j|k} = 0, \quad \frac{\delta\Theta(L^2)}{\delta x^k} = 0, \tag{49}$$

and

$$\frac{\partial y_i}{\partial y^j} = \gamma_{ij}, \quad \frac{\partial L^2}{\partial y^j} = 2y_j, \quad \frac{\partial \Theta(L^2)}{\partial y^j} = 2y_j \cdot \Theta'(L^2), \tag{50}$$

for any real differentiable function $\Theta : \mathrm{Im}(L^2) \subseteq \mathbb{R}_+ \to \mathbb{R}_+$. Using (49) and (50) it immediately results $\Omega_{ki|j} = 0$, so that $d\Omega(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}) = 0$. Consequently, $d\Omega$ is completely determined by

$$d\Omega(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}) = \left[2\cdot\left(\frac{\gamma a^2}{L^2}\right)' - \left(\gamma\,\frac{b^2-a^2}{L^4} + \delta\frac{b^2}{L^2}\right)\right]\cdot\left(\gamma_{ik}y_j - \gamma_{ij}y_k\right). \tag{51}$$

Defining 1-form $\theta_0 = dL^2 = 2y_i\delta y^i$ and evaluand $\Omega \wedge \theta_0$ on the basis $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$ one finds the essential component

$$(\Omega \wedge \theta_0)\,(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}) = 2\frac{\gamma a^2}{L^2}(\gamma_{ik}y_j - \gamma_{ij}y_k). \tag{52}$$

Comparing (51) with (52) one obtains

$$d\Omega = \Omega \wedge \theta,$$

where

$$\theta = \frac{2(ac)'L^2 - (ac - bd)}{acL} dL.$$

The last two identities complete the proof. □

The next result follows from the proof of Theorem 3.

COROLLARY 2. *The almost Hermitian structures* $(G_{a,b,c,d}, F_{a,b,c,d})$ *are almost Kähler structures if and only if the identity*

$$2(ac)'L^2 = ac - bd, \tag{53}$$

*holds.*

REMARK 6. 1. The almost Hermitian structure $(G_{a,a,a,a}, F_{a,a,a,a})$ is an almost Kähler structure if and only if $a(t) = k$ $(k = const)$, where $t = L^2$.

2. The relation (53) can be written in the following form

$$c(t) = \frac{\sqrt{t}}{a(t)} \left( a_0 - \int \frac{d(t)b(t)}{2t\sqrt{t}} dt \right), \tag{54}$$

with the following restrictions

$$a_0 > 0, \quad \int \frac{d(t)b(t)}{2t\sqrt{t}} dt < a_0, \quad d \geqslant c > 0, \tag{55}$$

and with the same meaning for $t$, that is $t = L^2$.

Using similar arguments as in the proof of Theorem 3, the next two results can be proved.

THEOREM 4. *The almost Hermitian structures* $(G_{a,a,c,c}, F_{a,c})$ *are locally conformal almost Kählerian structures. More precisely, the following identity*

$$d\Omega = \Omega \wedge \theta,$$

*holds with*

$$\theta = 2 \cdot \frac{(ac)'L^2 - ac}{acL} dL.$$

COROLLARY 3. *The almost Hermitian structure* $(G_{a,a,c,c}, F_{a,c})$ *is an almost Kähler structure, if and only if the next identity*

$$c(L^2) = \frac{c_0 \cdot L^2}{a(L^2)}, c_0 > 0$$

*holds.*

### 5.3. Locally conformal Kähler structures on TM

Let $r^i_{pjk}(x)$ be the local components of the curvature tensor field of the Riemannian space $(M, \gamma)$. The Lie brackets corresponding to the adapted basis $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$ are as follows

$$\left[\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right] = R^i_{jk} \cdot \frac{\partial}{\partial y^i}, \quad \left[\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}\right] = B^i_{jk} \cdot \frac{\partial}{\partial y^i}, \quad \left[\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\right] = 0,$$

where,

$$R^i_{jk} = r^i_{pjk} \cdot y^p, \quad B^i_{jk} = \gamma^i_{jk}.$$

Evaluating $N_F$ on the adapted basis the next result is obtained.

LEMMA 1. *The almost Hermitian structures* $(G_{a,b,c,d},\ F_{a,b,c,d})$ *are locally conformal Kähler structures if and only if*

$$R^i_{jk} = f \cdot (y_j \cdot \delta^i_k - y_k \cdot \delta^i_j), \tag{56}$$

*where*

$$f = 2\alpha\alpha' - \alpha\beta + 2\alpha'\beta L^2.$$

Obviously, the relation (56) can be rewritten as

$$r^i_{sjk} \cdot y^s = f \cdot (y_j \cdot \delta^i_k - y_k \cdot \delta^i_j). \tag{57}$$

The equation (57) looks like the condition that $(M, \gamma)$ is of constant curvature. Now it is easy to see that

1. In the case of the almost Hermitian structures $(G_{a,a,a,a}, F_{a,a,a,a})$ the condition $f = k$ $(k = const.)$ gives $L = \sqrt{\frac{2}{k}}$, that means the associated Finsler function of the Riemannian space $(M, \gamma)$ is a constant;

2. For the almost Hermitian structures $(G_{a,a,a,a}, F_{a,a})$, the condition $f = k$ $(k = const.)$ is verified if and only if the Riemannian space $(M, \gamma)$ is an Euclidean space;

3. In the case of the almost Hermitian structures $(G_{a,a,c,c}, F_{a,c})$, the folowing result holds.

THEOREM 5. *If the Riemannian manifild* $(M, \gamma)$ *is of constant curvature* $k \in R$, *then for* $c(t) = \frac{1}{a(t)\sqrt{x_0 - \int \frac{k}{a^4(t)}dt}}$ *with* $x_0$ *a real constant such that* $\int \frac{k}{a^4(t)}dt < x_0$, *the almost Hermitian structures* $(G_{a,a,c,c},\ F_{a,c})$ *are locally conformal Kähler structures on* $TM$.

## 6. d-connections compatible with some almost Hermitian structures

Every linear connection on $TM$ can be expressed by means of some local coefficients as follows

$$D_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta x^j} = \overset{(H)}{F}{}^i_{jk} \frac{\delta}{\delta x^i} + \overset{(1)}{F}{}^i_{jk} \frac{\partial}{\partial y^i}, \quad D_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial y^j} = \overset{(2)}{F}{}^i_{jk} \frac{\delta}{\delta x^i} + \overset{(V)}{F}{}^i_{jk} \frac{\partial}{\partial y^i},$$

$$D_{\frac{\partial}{\partial y^k}} \frac{\delta}{\delta x^j} = \overset{(H)}{C}{}^i_{jk} \frac{\delta}{\delta x^i} + \overset{(1)}{C}{}^i_{jk} \frac{\partial}{\partial y^i}, \quad D_{\frac{\partial}{\partial y^k}} \frac{\partial}{\partial y^j} = \overset{(2)}{C}{}^i_{jk} \frac{\delta}{\delta x^i} + \overset{(V)}{C}{}^i_{jk} \frac{\partial}{\partial y^i}.$$

The next formulas follow from the theory of linear connections on tangent manifold

$$D_{\frac{\delta}{\delta x^k}} dx^i = -\overset{(H)}{F}{}^i_{jk} dx^j - \overset{(2)}{F}{}^i_{jk} \delta y^j, \quad D_{\frac{\delta}{\delta x^k}} \delta y^i = -\overset{(1)}{F}{}^i_{jk} dx^j - \overset{(V)}{F}{}^i_{jk} \delta y^j,$$

$$D_{\frac{\partial}{\partial y^k}} dx^i = -\overset{(H)}{C}{}^i_{jk} dx^j - \overset{(2)}{C}{}^i_{jk} \delta y^j, \quad D_{\frac{\partial}{\partial y^k}} \delta y^i = -\overset{(1)}{C}{}^i_{jk} dx^j - \overset{(V)}{C}{}^i_{jk} \delta y^j.$$

The Levi - Civita connection is an important tool in every physical theory on $TM$. In fact, the Levi - Civita connection of $G_{a,b,c,d}$ is a linear connection $D$ on $TM$ such that the next two identities

$$XG_{a,b,c,d}(Y, Z) - G_{a,b,c,d}(D_X Y, Z) - G_{a,b,c,d}(Y, D_X Y) = 0, \tag{58}$$

$$D_X Y - D_Y X - [X, Y] = 0 \tag{59}$$

hold true in $\chi(TM)$.

THEOREM 6. *The local coefficients of the Levi -Civita connection of the metrical structure $G_{a,b,c,d}$ are as follows*

$$\overset{(H)}{F}{}^i_{jk} = \frac{1}{2} g^{im} \cdot \left( \frac{\delta g_{jm}}{\delta x^k} + \frac{\delta g_{mk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^m} \right),$$

$$\overset{(1)}{F}{}^i_{jk} = \frac{1}{2} \cdot \left( R^i_{kj} - h^{ip} \cdot \frac{\partial g_{kp}}{\partial y^j} \right),$$

$$\overset{(2)}{F}{}^i_{jk} = \overset{(H)}{C}{}^i_{kj},$$

$$\overset{(V)}{F}{}^i_{jk} = \frac{1}{2} \cdot \left( \gamma^i_{jk} + h^{is} \cdot \frac{\delta h_{sj}}{\delta x^k} - h_{jm} \cdot \gamma^m_{sk} \cdot h^{si} \right),$$

$$\overset{(H)}{C}{}^i_{jk} = -\frac{1}{2} \cdot \left( h_{mk} \cdot g^{qi} \cdot R^m_{jq} - g^{qi} \cdot \frac{\partial g_{jk}}{\partial y^q} \right),$$

$$\overset{(1)}{C}{}^{i}_{jk} = \frac{1}{2} \cdot \left( h^{is} \cdot \frac{\delta h_{sk}}{\delta x^j} - \gamma^{\ i}_{jk} - h_{km} \cdot \gamma^{\ m}_{sj} \cdot h^{si} \right),$$

$$\overset{(2)}{C}{}^{i}_{jk} = \frac{1}{2} \cdot \left( g^{iq} \cdot \gamma^{\ m}_{kh} \cdot h_{mj} + g^{iq} \cdot \gamma^{\ m}_{jq} \cdot h_{km} - g^{iq} \cdot \frac{\delta h_{jk}}{\delta x^q} \right),$$

$$\overset{(V)}{C}{}^{i}_{jk} = \frac{1}{2} \cdot h^{im} \cdot \left( \frac{\partial h_{jm}}{\partial y^k} + \frac{\partial h_{mk}}{\partial y^j} - \frac{\partial h_{jk}}{\partial y^m} \right).$$

*Proof.* The above formulas follow by expressing in local coordinates the relations (58) and (59) and solving the tensorial system which appears. □

In what follows all d-connections compatible with the metrical structure $G_{a,b,c,d}$ are determined. More precisely, using some clasical methods of Lagrange geometry the following three results hold.

THEOREM 7. *Let $a$, $b$, $c$ and $d$ arbitrarily chosen. Then there exist d-connections compatible with the metrical structure $G_{a,b,c,d}$. One of them has the folowing coefficients*

$$\overset{(H,c)}{F}{}^{i}_{jk} = \overset{(V,c)}{F}{}^{i}_{jk} = \gamma^{\ i}_{jk},$$

$$\overset{(H,c)}{C}{}^{i}_{jk} = A \cdot \delta^i_j \cdot y_k + A_1 \cdot \delta^i_k \cdot y_j + A_2 \cdot y^i \cdot \gamma_{jk} + A_3 \cdot y^i \cdot y_j \cdot y_k,$$

$$\overset{(V,c)}{C}{}^{i}_{jk} = B \cdot \Lambda^i_{jk} + B_1 \cdot y^i \cdot \gamma_{jk} + B_2 \cdot y^i \cdot y_j \cdot y_k,$$

*where*

$$\Lambda^i_{jk} = \delta^i_j \cdot y_k + \delta^i_k \cdot y_j - y^i \cdot \gamma_{jk}, \quad A = \frac{2a'L^2 - a}{aL^2},$$

$$A_1 = \frac{b^2 - a^2}{2a^2L^2}, \quad A_2 = \frac{b^2 - a^2}{b^2L^2}, \quad A_3 = \frac{a^4 - b^4 + 4ab(ab' - ba')L^2}{2a^2b^2L^4},$$

$$B = \frac{2c'L^2 - c}{cL^2}, \quad B_1 = \frac{2c'\left(d^2 - c^2\right)}{cd^2}, \quad B_2 = \frac{2\left(c^2c' + cdd' - 2d^2c'\right)}{cd^2}.$$

COROLLARY 4. *The set of all d-connections compatible with the metrical structure $G_{a,b,c,d}$ is given by the following coefficients*

$$\overset{(H)}{F}{}^{i}_{jk} = \overset{(H,c)}{F}{}^{i}_{jk} + \Omega^{ei}_{jm} \cdot X^m_{ek}, \quad \overset{(V)}{F}{}^{i}_{jk} = \overset{(V,c)}{F}{}^{i}_{jk} + \Theta^{ei}_{jm} \cdot Y^m_{ek}, \tag{60}$$

$$\overset{(H)}{C}{}^{i}_{jk} = \overset{(H,c)}{C}{}^{i}_{jk} + \Omega^{ei}_{jm} \cdot U^m_{ek},, \quad \overset{(V)}{C}{}^{i}_{jk} = \overset{(V,c)}{C}{}^{i}_{jk} + \Theta^{ei}_{jm} \cdot V^m_{ek}, \tag{61}$$

*where $X^m_{ek}$, $Y^m_{ek}$, $U^m_{ek}$, $V^m_{ek}$ are arbitrary d-tensor fields and $\Omega^{ei}_{jm}$, $\Theta^{ei}_{jm}$ are the Obata operators corresponding to the GL-metrics $g_{ij}(x,y)$ and $h_{ij}(x,y)$, respectively.*

COROLLARY 5. *The set of all d-connections compatible with the metrical structure $G_{a,a,c,c}$ is given by the following coefficients*

$$\overset{(H)}{F}{}^{i}_{jk} = \gamma^{\ i}_{jk} + \Omega^{ei}_{jm} \cdot X^{m}_{ek}, \quad \overset{(V)}{F}{}^{i}_{jk} = \gamma^{\ i}_{jk} + \Omega^{ei}_{jm} \cdot Y^{m}_{ek}, \tag{62}$$

$$\overset{(H)}{C}{}^{i}_{jk} = A \cdot \delta^{i}_{j} \cdot y_k + \Omega^{ei}_{jm} \cdot U^{m}_{ek}, \quad \overset{(V)}{C}{}^{i}_{jk} = B \cdot \Lambda^{\ i}_{jk} + \Omega^{ei}_{jm} \cdot V^{m}_{ek}, \tag{63}$$

*where $X^{m}_{ek}$ , $Y^{m}_{ek}$ , $U^{m}_{ek}$ , $V^{m}_{ek}$ are arbitrary d-tensor field and $\Omega^{ei}_{jm}$ is the Obata operator of the Riemannian structure $\gamma$ .*

The aim of the next investigations is to find some conditions such that the almost Hermitian structure $(G_{a,b,c,d}, F_{a,b,c,d})$ admits compatible d-connections. The first step consists in the study of the linear connections compatible with the almost complex structure $F_{a,b,c,d}$. Let $D$ be an arbitrary linear connection on $TM$. In the adapted basis, the compatibility condition $DF_{a,b,c,d} = 0$ can be written as

$$A^{p}_{j} \cdot \overset{(V)}{F}{}^{i}_{pk} - A^{i}_{q} \cdot \overset{(H)}{F}{}^{q}_{jk} = 0, \quad \frac{\partial A^{i}_{j}}{\partial y^{k}} + A^{p}_{j} \cdot \overset{(V)}{C}{}^{i}_{pk} - A^{i}_{q} \cdot \overset{(H)}{C}{}^{q}_{jk} = 0, \tag{64}$$

$$A^{p}_{j} \cdot \overset{(2)}{F}{}^{i}_{pk} - B^{i}_{q} \cdot \overset{(1)}{F}{}^{q}_{jk} = 0, \quad A^{p}_{j} \cdot \overset{(2)}{C}{}^{i}_{pk} - B^{i}_{q} \cdot \overset{(1)}{C}{}^{q}_{jk} = 0 \tag{65}$$

Investigating now d-connections compatible with $F_{a,b,c,d}$ then only (64) must be studied. Since the final aim is to obtain d-connections compatible with almost Hermitian structure $(G_{a,b,c,d}, F_{a,b,c,d})$, the local coefficients $(\overset{(H)}{F}{}^{i}_{jk}, \overset{(V)}{F}{}^{i}_{jk}, \overset{(H)}{C}{}^{i}_{jk}, \overset{(V)}{C}{}^{i}_{jk})$ from Corollary 4 are used. Therefore, the first identity of (64) can be written as

$$A^{i}_{q} \cdot \Omega^{eq}_{jm} \cdot X^{m}_{ek} - A^{p}_{j} \cdot \Theta^{ei}_{pm} \cdot Y^{m}_{ek} = \beta \cdot y^{p} \cdot y_{q} \cdot E^{qi}_{jpk}, \tag{66}$$

where

$$E^{qi}_{jpk} = \delta^{q}_{j} \cdot \gamma^{\ i}_{pk} - \delta^{i}_{p} \cdot \gamma^{\ q}_{jk}, \ \beta = \frac{ad + bc}{dcL^{2}}. \tag{67}$$

The lefthand side of the relation (66) has a d-tensorial character. Since $E^{qi}_{jpk}$ is not a d-tensor field it follows that the relation (66) is not true. Therefore do not exist d-connections compatible with the almost Hermitian structure $(G_{a,b,c,d}, F_{a,b,c,d})$. Fortunately, for the almost Hermitian structure $(G_{a,a,c,c}, F_{a,c})$ there exist compatible d-connections. Repeating the above considerations and after some computations in local coordinates, one obtains the following result

THEOREM 8. *The set of all d-connections compatible with the almost Hermitian structure $(G_{a,a,c,c}, F_{a,c})$ is given by the following local coefficients*

$$\overset{(H)}{F}{}^{i}_{jk} = \gamma^{\ i}_{jk} + \Omega^{ei}_{jm} \cdot X^{m}_{ek}, \quad \overset{(V)}{F}{}^{i}_{jk} = \gamma^{\ i}_{jk} + \Omega^{ei}_{jm} \cdot X^{m}_{ek},$$

$$\overset{(H)}{C}{}^{i}_{jk} = A \cdot \delta^i_j \cdot y_k + \Omega^{ei}_{jm} \cdot U^m_{ek}, \quad \overset{(V)}{C}{}^{i}_{jk} = B \cdot \delta^i_j \cdot y_k + \Omega^{ei}_{jm} \cdot U^m_{ek},$$

*where $X^m_{ek}$, $U^m_{ek}$ are arbitrary d-tensor fields and $\Omega^{ei}_{jm}$ are the local components of the Obata operator of the Riemannian structure $\gamma$.*

In the case $X^m_{ek} = U^m_{ek} = 0$ one obtains a d-conection compatible with the almost Hermitian structure $(G_{a,a,c,c}, F_{a,c})$, which depends only on the Riemannian structure $\gamma$ and on the functions $a$ and $c$. The local coefficients of this d-connection are as follows

$$\overset{(H)}{F}{}^{i}_{jk} = \overset{(V)}{F}{}^{i}_{jk} = \gamma^i_{jk}, \quad \overset{(H)}{C}{}^{i}_{jk} = A \cdot \delta^i_j \cdot y_k, \quad \overset{(V)}{C}{}^{i}_{jk} = B \cdot \delta^i_j \cdot y_k.$$

This one will be called the canonical d-connection of the space $(T_0M, G_{a,a,c,c}, F_{a,c})$.

If $a = L$, $c = k$, $k \in \mathbb{R}\backslash\{0\}$, one obtains the so called homogeneous almost Hermitian structure $(\overset{(0)}{G}, \overset{(0)}{F})$. Here, the metrical structure $\overset{(0)}{G}$ is the Miron metrical structure and the almost complex structure $\overset{(0)}{F}$ is defined by

$$\overset{(0)}{F} = -\frac{L}{k} \cdot \frac{\partial}{\partial y^i} \otimes dx^i + \frac{k}{L} \cdot \frac{\delta}{\delta x^i} \otimes \delta y^i.$$

Furthermore, the canonical d-conection of the space $(T_0M, \overset{(0)}{G}, \overset{(0)}{F})$ is given by

$$\overset{(H)}{F}{}^{i}_{jk} = \overset{(V)}{F}{}^{i}_{jk} = \gamma^i_{jk}, \quad \overset{(H)}{C}{}^{i}_{jk} = 0, \quad \overset{(V)}{C}{}^{i}_{jk} = -\frac{1}{L^2} \cdot \delta^i_j \cdot y_k.$$

The space$(T_0M, \overset{(0)}{G}, \overset{(0)}{F})$ has been intensively studied in [20]. Its generalization, namely the space $\overset{(ac)}{M}{}^{2n} = (T_0M, G_{a,a,c,c}, \Phi_{a,c})$ will be the subject of the next section.

## 7. EYM equations on $\overset{(ac)}{M}{}^{2n}$

>From now on the metrical structure of the model is given by

$$G_{A,B}(x, y) = A \cdot \gamma_{ij}(x) dx^i \otimes dx^j + B \cdot \gamma_{ij}(x) \delta y^i \otimes \delta y^j, \tag{68}$$

where $A, B : \mathbb{R}_+ \to \mathbb{R}_+$ are real differentiable functions in the variable $t = L^2$. With this notations assumed, the canonical connection of the model is expressed by

$$\begin{cases} \overset{(H)}{F}{}^{i}_{jk} = \overset{(V)}{F}{}^{i}_{jk} = \gamma^i_{jk}(x), \\ \overset{(H)}{C}{}^{i}_{jk} = A' \cdot \delta^i_j \cdot y_k, \ \overset{(V)}{C}{}^{i}_{jk} = B' \cdot \delta^i_j \cdot y_k. \end{cases} \tag{69}$$

such that $A'(t) \neq 0$, $B'(t) \neq 0$, $\forall t \in \mathbb{R}_+$. Some calculations in local coordinates lead to the next result.

LEMMA 2. *The local components of the torsion of the canonical connection of the space* $\overset{(ac)^{2n}}{M}$ *are given by*

$$T^i_{jk} = 0, \quad R^i_{jk} = r^i_{0jk},$$

$$\overset{(H)^i}{P}{}_{jk} = A^{'} \cdot \delta^i_j \cdot y_k, \quad \overset{(V)^i}{P}{}_{jk} = 0, \quad S^i_{jk} = B^{'} \cdot \left(\delta^i_j \cdot y_k - \delta^i_k \cdot y_j\right).$$

*The local components of the curvature of the canonical connection of the space* $\overset{(ac)^{2n}}{M}$ *are given by*

$$\overset{(H)^i}{R}{}_{hjk} = r^i_{hjk} + r^m_{0jk} \cdot \gamma^i_{hm}, \quad \overset{(V)^i}{R}{}_{hjk} = r^i_{hjk} + B^{'} \cdot r^m_{0jk} \cdot \delta^i_h \cdot y_m$$

$$\overset{(H)^i}{P}{}_{hjk} = 0, \quad \overset{(V)^i}{P}{}_{hjk} = 0, \quad \overset{(H)^i}{S}{}_{hjk} = 0, \quad \overset{(V)^i}{S}{}_{hjk} = 0.$$

Having a complete imagine of the geometrical objects of this space it is now possible to determine the EYM equations of the model. These equations will be considered with respect to $\Phi_1 = y_a$ and $\Phi_2 = \gamma^a_{bc}$ as generalized gauge fields. Supposing that all the geometrical objects have gauge character it is easy to see that the EYM equations for the complete Lagrangian $L$ can be written in the following form

$$\frac{\partial L}{\partial \Phi} - \frac{\delta}{\delta x^i}\left(\frac{\partial L}{\partial\left(\frac{\delta\Phi}{\delta x^i}\right)}\right) + \frac{\partial N^i_j}{\partial y^i} \cdot \frac{\partial L}{\partial\left(\frac{\delta\Phi}{\delta x^j}\right)} - \frac{\partial}{\partial y^i}\left(\frac{\partial L}{\partial\left(\frac{\partial\Phi}{\partial y^i}\right) \mid \frac{\delta\Phi}{\delta x^i} = const}\right)$$

$$-\frac{1}{G} \cdot \frac{\delta G}{\delta x^i} \cdot \frac{\partial L}{\partial\left(\frac{\delta\Phi}{\delta x^i}\right)} - \frac{1}{G} \cdot \frac{\partial G}{\partial y^i} \cdot \frac{\partial L}{\partial\left(\frac{\partial\Phi}{\partial y^i}\right) \mid \frac{\delta\Phi}{\delta x^i} = const} = 0.$$

Let $\Gamma$ be $\det(\gamma_{ij})$. The identities

$$G = \sqrt{A^n \cdot B^n \cdot \Gamma^2} = (AB)^{n/2} \cdot \Gamma,$$

$$\frac{1}{G} \cdot \frac{\delta G}{\delta x^i} = \frac{1}{\Gamma} \cdot \frac{\partial \Gamma}{\partial x^i},$$

$$\begin{aligned} \frac{1}{G} \cdot \frac{\partial G}{\partial y^i} &= 2 \cdot y_i \cdot \frac{1}{(AB)^{n/2} \cdot \Gamma} \cdot \Gamma \cdot \left[(AB)^{n/2}\right]' \\ &= \frac{\frac{n}{2} \cdot (AB)^{\frac{n}{2}-1} \cdot 2 \cdot y_i \cdot (AB)'}{(AB)^{n/2}} \\ &= \frac{n \cdot y_i \cdot (AB)'}{AB}, \end{aligned}$$

show that for the model which is now studied, the EYM equations have now an equivalent form

$$\frac{\partial L}{\partial \Phi} - \frac{\delta}{\delta x^i}\left(\frac{\partial L}{\partial\left(\frac{\delta \Phi}{\delta x^i}\right)}\right) + \gamma^{\,m}_{jm} \cdot \frac{\partial L}{\partial\left(\frac{\delta \Phi}{\delta x^j}\right)} - \frac{\partial}{\partial y^i}\left(\frac{\partial L}{\partial\left(\frac{\partial \Phi}{\partial y^i}\right) \mid \frac{\delta \Phi}{\delta x^i} = const}\right)$$

$$-\frac{1}{\Gamma} \cdot \frac{\partial \Gamma}{\partial x^i} \cdot \frac{\partial L}{\partial\left(\frac{\delta \Phi}{\delta x^i}\right)} - \frac{n \cdot y_i \cdot (AB)'}{AB} \cdot \frac{\partial L}{\partial\left(\frac{\partial \Phi}{\partial y^i}\right) \mid \frac{\delta \Phi}{\delta x^i} = const} = 0. \tag{70}$$

The geometrical expressions of EYM equations can be now obtained from the general form (70).

## 8. Conclusions

In this paper a new smooth gauge geometrical model is proposed. This one is defined at the end of Section 6 and it is studied in the last section of the paper. Its construction is based on a new class of Hermitian structure on the tangent bundle of a smooth, finite-dimensional differentiable manifold. This Hermitian structure can be viewed as a "deformation" of the clasical Hermitian structure of the tangent bundle of a Riemannian manifold. Moreover, this model is a generalization of the so called "homogeneous geometrical model of a Riemannian space", which was studied in [20]. The equation of fields, namely (70), is written in the adapted basis. This fact is very useful in applications, cf. [17, 14, 12, 19, 13]. Mention that the study in detail of the field equation (70) will be done elsewhere.

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Adrian SANDOVICI, Department of Mathematics and Computing Science, University of Groningen, P.O. Box 800, 9700 AV Groningen, THE NETHERLANDS
e-mail: `adrian@math.rug.nl`

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