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ON CURVATURES OF LINEAR FRAME BUNDLES WITH NATURALLY LIFTED METRICS

Abstract. We study some simple natural metrics on the linear frame bundle LM over a Riemannian manifold (M, g) which generalize the so-called diagonal metric introduced by K. P. Mok in 1978. We derive the basic formulas for the Levi-Civita connection and the curvature tensor. Next, we limit ourselves to the base manifold (M, g) with constant sectional curvature and calculate all types of curvatures on LM in this case. Finally, we add a new result to those published in 1986 by L. A. Cordero and M. de León for the diagonal metric, but now in a more general situation.

1. Introduction

Let (M, g) be a Riemannian manifold, and TM its tangent bundle. Then the Sasaki metric g^s of TM is known as an "extremely rigid" Riemannian metric on TM. In fact, E. Musso and F. Tricerri [7] have proved that *the scalar curvature* $\mathrm{Sc}(g^s)$ *is constant if and only if* (M, g) *is flat, and hence* $\mathrm{Sc}(g^s) = 0$. Now, let

$$LM = \{(x, u) \mid x \in M,\ u \text{ is a linear frame at } x\}$$

be the linear frame bundle over M. K. P. Mok [6] has defined so-called the *diagonal lift* g^d of g to LM, which is a Riemannian metric resembles the Sasaki metric g^s of TM. It seems that phenomenon similar to geometry of (TM, g^s) happens about geometry of (LM, g^d). L. A. Cordero and M. de León [2] have shown this metric g^d is also "rigid" in some sense. Namely, they have proved

THEOREM 1. *The Riemannian manifold* (LM, g^d) *is never locally symmetric unless* (M, g) *is locally Euclidean.*

THEOREM 2. *If* (LM, g^d) *is an Einstein manifold, then* (M, g) *is flat.*

THEOREM 3. *The Riemannian manifolds* (LM, g^d) *and* (M, g) *have the same constant scalar curvature if and only if both are flat.*

It should be interesting to find "non-rigid" metrics on TM and also on LM. An example of such metrics on TM has been given by V. Oproiu [9]. He has proved that *if* (M, g), $\dim M > 2$, *is a space of negative constant sectional curvature, then* TM *equipped with any Oproiu metric is a Kaehler Einstein manifold with positive*

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constant scalar curvature. The metrics studied by V. Oproiu belong to the list of full classification of "naturally lifted" metrics given by the present authors [5].

The present authors have also given in [4] full classification of naturally lifted (possibly degenerate) pseudo-Riemannian metrics on LM which come from a second order natural transformation of a Riemannian metric g on M. We study in this paper specific Riemannian metrics from our list, which are a bit more general ones than the diagonal lift g^d. The horizontal and vertical distributions are no more orthogonal in general with respect to the metrics treated in this paper.

Usually, calculations for getting geometric objects of LM are not short. They are complicated sometimes. We shall give some simpler procedure in order to calculate them. It is hard in general to obtain explicit expressions of the inverse matrix of the lifted metric. Yet, it is not our case. So, contractions of tensors are easily calculated. We shall prove that metrics which we pick in this paper are still rigid when the base manifold is of constant curvature.

2. Preliminaries

We shall summarize briefly the basic definitions and results which will be used later.

Let M be an n-dimensional smooth manifold. Then the linear frame bundle LM over M consists of all pairs (x, u), where x is a point of M and u is a basis for the tangent space M_x of M at x. We denote by p the natural projection of LM to M defined by $p(x, u) = x$. If $(U;\ x^1, x^2, \dots, x^n)$ is a system of local coordinates in M, then a basis $u = (u_1, u_2, \dots, u_n)$ for M_x can be expressed uniquely in the form

$$u_\lambda = \sum_{i=1}^{n} u^i_\lambda \left(\frac{\partial}{\partial x^i}\right)_x \quad (\lambda = 1, 2, \dots, n), \tag{1}$$

and hence

$$(p^{-1}U;\ x^1, x^2, \dots, x^n, u^1_1, u^2_1, \dots, u^n_n)$$

is a system of local coordinates in LM.

Let ∇ be a linear connection on M. Then the tangent space $(LM)_{(x,u)}$ of LM at $(x, u) \in LM$ splits into the horizontal and vertical subspace with respect to ∇:

$$(LM)_{(x,u)} = H_{(x,u)} \oplus V_{(x,u)}.$$

If a point $(x, u) \in LM$ and a vector $X \in M_x$ are given, then there exists a unique vector $X^h \in H_{(x,u)}$ such that $p_*(X^h) = X$. We call X^h the *horizontal lift* of X to LM at (x, u). In terms of local coordinates, if $X = \sum_i \xi^i (\partial/\partial x^i)_x$, then X^h is expressed as

$$X^h = \sum_i \xi^i \left(\frac{\partial}{\partial x^i}\right)_{(x,u)} - \sum_{a,b,i,\lambda} \Gamma^i_{ab} u^a_\lambda \xi^b \left(\frac{\partial}{\partial u^i_\lambda}\right)_{(x,u)},$$

where Γ^i_{jk}'s are local components of ∇ (*cf.*, [1]).

We define naturally n different vertical lifts of $X \in M_x$. If ω is a 1-form on M, then $\iota_\mu \omega$ $(\mu = 1, 2, \dots, n)$ are functions on LM defined by $(\iota_\mu \omega)(x, u) = \omega(u_\mu)$ for all $(x, u) = (x, u_1, u_2, \dots, u_n) \in LM$. The *vertical lifts* $X^{v,\lambda}$ $(\lambda = 1, 2, \dots, n)$ of $X \in M_x$ to LM at (x, u) are the n vectors such that $X^{v,\lambda}(\iota_\mu \omega) = \omega(X)\delta^\lambda_\mu$ $(\lambda, \mu = 1, 2, \dots, n)$ hold for all 1-forms ω on M, where δ^λ_μ denotes the Kronecker's delta. The n vertical lifts are always uniquely determined, and they are linearly independent if $X \neq 0$. They are expressed in a local coordinate system as

$$X^{v,\lambda} = \sum_i \xi^i \left(\frac{\partial}{\partial u^i_\lambda}\right)_{(x,u)},$$

where $\lambda = 1, 2, \dots, n$ (*cf.*, [2]).

The *vertical lift* of a smooth function f on M is a function f^v on LM defined by $f^v = f \circ p$. We have

$$X^h(f^v) = Xf \tag{2}$$

for all vectors $X \in M_x$ and functions f on M.

Let X be a vector field on M. Then its horizontal lift X^h and n vertical lifts $X^{v,\lambda}$ are vector fields on LM which assign to each point $(x, u) \in LM$ the horizontal lift $X^h_{(x,u)}$ of X_x and n vertical lifts $X^{v,\lambda}_{(x,u)}$ of X_x, respectively. The Lie-bracket are expressed at $(x, u) \in LM$ in the form

$$\begin{aligned} [X^h,\ Y^h]_{(x,u)} &= [X, Y]_x{}^h - \sum_{\alpha=1}^{n} (R_x(X, Y)u_\alpha)^{v,\alpha}, \\ [X^h,\ Y^{v,\lambda}]_{(x,u)} &= (\nabla_X Y)_x{}^{v,\lambda}, \\ [X^{v,\lambda},\ Y^{v,\mu}]_{(x,u)} &= 0 \end{aligned} \tag{3}$$

for all vector fields X and Y on M, where $\lambda, \mu = 1, 2, \dots, n$ and R is the curvature tensor field of ∇ defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for all vector fields X, Y and Z on M (*cf.*, [2]).

We shall use later the following formulas for differentiation of the coordinate functions u^i_μ with respect to the lifts X^h and $X^{v,\lambda}$ of a vector field $X = \sum \xi^i \partial/\partial x^i \in \mathfrak{X}(M)$:

$$X^h(u^i_\mu) = -\sum_{a,b} \Gamma^i_{ab} u^a_\mu \xi^b, \tag{4}$$

$$X^{v,\lambda}(u^i_\mu) = \xi^i \delta^\lambda_\mu, \tag{5}$$

where $i, \lambda, \mu = 1, 2, \dots, n$.

3. Naturally lifted metrics of constant type

The present authors have given in [4, Theorem 3.2] the list of (possibly degenerate) pseudo-Riemannian metrics G on LM which come from a second order natural transformation of g on M. Let $\{\omega^1, \omega^2, \dots, \omega^n\}$ be the dual frame to a linear frame $\{u_1, u_2, \dots, u_n\}$ on M. The list of metrics is expressed in the following form:

$$G_{(x,u)}(X^h,\ Y^h) = \sum_{\alpha,\beta=1}^{n} \varphi_{\alpha\beta}\omega_x^\alpha(X)\omega_x^\beta(Y),$$

$$G_{(x,u)}(X^h,\ Y^{\upsilon,\mu}) = \sum_{\alpha,\beta=1}^{n} \varphi_{\alpha}{}^{\mu}_{\beta}\omega_x^\alpha(X)\omega_x^\beta(Y),$$

$$G_{(x,u)}(X^{\upsilon,\lambda},\ Y^{\upsilon,\mu}) = \sum_{\alpha,\beta=1}^{n} \varphi^{\lambda\mu}_{\alpha\beta}\omega_x^\alpha(X)\omega_x^\beta(Y)$$

for all $X, Y \in M_x$, where $\varphi_{\alpha\beta}$, $\varphi_{\alpha}{}^{\mu}_{\beta}$ and $\varphi^{\lambda\mu}_{\alpha\beta}$ $(\alpha, \beta, \lambda, \mu = 1, 2, \dots, n)$ are arbitrary smooth functions of $n(n+1)/2$ variables $w_{\rho\sigma} = g(u_\rho, u_\sigma)$ $(1 \le \rho \le \sigma \le n)$.

We take a considerably simpler case:

$$\varphi_{\alpha\beta} = w_{\alpha\beta}, \quad \varphi_{\alpha}{}^{\mu}_{\beta} = c^\mu w_{\alpha\beta} \quad \text{and} \quad \varphi^{\lambda\mu}_{\alpha\beta} = c^{\lambda\mu} w_{\alpha\beta},$$

where c^μ and $c^{\lambda\mu} = c^{\mu\lambda}$ are constants. We denote such a metric by $\bar{g}$. That is, $\bar{g}$ is given by

$$\begin{aligned} &\bar{g}_{(x,u)}(X^h,\ Y^h) = g_x(X, Y), \\ &\bar{g}_{(x,u)}(X^h,\ Y^{\upsilon,\mu}) = c^\mu g_x(X, Y), \\ &\bar{g}_{(x,u)}(X^{\upsilon,\lambda},\ Y^{\upsilon,\mu}) = c^{\lambda\mu} g_x(X, Y) \end{aligned} \tag{6}$$

for all $X, Y \in M_x$, where $\lambda, \mu = 1, 2, \dots, n$.

REMARK 1. If $c^\lambda = 0$ and $c^{\lambda\mu} = \delta^{\lambda\mu}$ in (6), where $\delta^{\lambda\mu}$ is the Kronecker's delta, then the metric $\bar{g}$ is the diagonal lift g^d of g defined by K. P. Mok.

Let $\{E_i \mid i = 1, 2, \dots, n\}$ be an orthonormal basis of a tangent space M_x at x to the base manifold (M, g). Then

$$\{(E_i{}^h, E_i{}^{\upsilon,1}, E_i{}^{\upsilon,2}, \dots, E_i{}^{\upsilon,n}) \mid i = 1, 2, \dots, , n\} \tag{7}$$

is a basis of $(LM)_{(x,u)}$ (ordered in a specific way). The components of $\bar{g}_{(x,u)}$ with respect to this basis are given by

$$\bar{g}_{ij} = \bar{g}(E_i{}^h, E_j{}^h) = \delta_{ij},$$

$$\bar{g}_{i\,j}^{\;\mu} = \bar{g}(E_i{}^h, E_j{}^{v,\mu}) = c^\mu \delta_{ij},$$

$$\bar{g}_{i\,j}^{\lambda\mu} = \bar{g}(E_i{}^{v,\lambda}, E_j{}^{v,\mu}) = c^{\lambda\mu} \delta_{ij},$$

where $i, j, \lambda, \mu = 1, 2, \ldots, n$. This implies that the matrix of $\bar{g}_{(x,u)}$ with respect to the basis (7) (which is of degree $n^2 + n$) consists of n identical blocks of the form

$$C = \begin{bmatrix} 1 & c^1 & c^2 & \ldots & c^n \\ c^1 & c^{11} & c^{12} & \ldots & c^{1n} \\ c^2 & c^{21} & c^{22} & \ldots & c^{2n} \\ & \ldots & \ldots & \ldots & \\ c^n & c^{n1} & c^{n2} & \ldots & c^{nn} \end{bmatrix}$$

along the principal diagonal. Hence $\bar{g}$ is positive definite if and only if all principal minor determinants of C are positive. In particular, the matrix $[c^{\lambda\mu}]$ is positive definite.

From now on we assume that $\bar{g}$ is positive definite. The components of the inverse matrix $\bar{g}^{-1}$ of $\bar{g}$ are

$$\bar{g}^{ij} = \frac{1}{k}\delta^{ij},$$

(8)
$$\bar{g}^{i\,j}_{\;\mu} = -\frac{1}{k}\delta^{ij} \sum_\sigma c_{\mu\sigma} c^\sigma,$$

$$\bar{g}^{i\,j}_{\lambda\mu} = \delta^{ij}\Big(c_{\lambda\mu} + \frac{1}{k}\big(\sum_\rho c_{\lambda\rho} c^\rho\big)\big(\sum_\sigma c_{\mu\sigma} c^\sigma\big)\Big),$$

where $i, j, \lambda, \mu = 1, 2, \ldots, n$,

$$k = \frac{\det C}{\det[c^{\lambda\mu}]} = 1 - \sum_{\lambda,\mu} c_{\lambda\mu} c^\lambda c^\mu$$

and $[c_{\lambda\mu}]$ is the inverse matrix of $[c^{\lambda\mu}]$.

PROPOSITION 1. *Let* $\bar{\nabla}$ *be the Levi-Civita connection of* $(LM, \bar{g})$. *If* $X, Y \in$

$\mathfrak{X}(M)$, *then we have*

$$(\bar{\nabla}_{X^h}Y^h)_{(x,u)} = (\nabla_X Y)^h_x + \frac{1}{2k}\sum_\alpha c^\alpha\big\{(R_x(u_\alpha, X)Y)^h + (R_x(u_\alpha, Y)X)^h\big\}$$
$$- \frac{1}{2k}\sum_{\alpha,\rho,\sigma} c^\alpha c_{\rho\sigma}c^\sigma\big\{(R_x(u_\alpha, X)Y)^{v,\rho} + (R_x(u_\alpha, Y)X)^{v,\rho}\big\}$$
$$- \frac{1}{2}\sum_\alpha (R_x(X, Y)u_\alpha)^{v,\alpha},$$

$$(\bar{\nabla}_{X^h}Y^{v,\mu})_{(x,u)} = \frac{1}{2k}\sum_\alpha c^{\mu\alpha}(R_x(u_\alpha, Y)X)^h$$
$$+ (\nabla_X Y)^{v,\mu}_x - \frac{1}{2k}\sum_{\alpha,\rho,\sigma} c^{\mu\alpha}c_{\rho\sigma}c^\sigma(R_x(u_\alpha, Y)X)^{v,\rho},$$

$$(\bar{\nabla}_{X^{v,\lambda}}Y^h)_{(x,u)} = \frac{1}{2k}\sum_\alpha c^{\lambda\alpha}(R_x(u_\alpha, X)Y)^h$$
$$- \frac{1}{2k}\sum_{\alpha,\rho,\sigma} c^{\lambda\alpha}c_{\rho\sigma}c^\sigma(R_x(u_\alpha, X)Y)^{v,\rho},$$

$$(\bar{\nabla}_{X^{v,\lambda}}Y^{v,\mu})_{(x,u)} = 0,$$

where $\lambda, \mu = 1, 2, \dots, n$.

Proof. We use the standard Koszul formula for the covariant differentiation applied now to the connection $\bar{\nabla}$ and to the vector fields on LM. □

Finally we get the formulas for the Riemannian curvature in full general form.

PROPOSITION 2. *Let* $\bar{R}$ *be the Riemannian curvature tensor field of* $(LM, \bar{g})$.

If $X, Y, Z \in M_x$*, then we have*

$$
\begin{aligned}
&\bar{R}_{(x,u)}(X^h, Y^h)Z^h \\
&\quad = (R_x(X,Y)Z)^h \\
&\qquad + \frac{1}{2k}\sum_{\alpha} c^{\alpha}\{((\nabla_X R)_x(u_\alpha, Y)Z)^h - ((\nabla_Y R)_x(u_\alpha, X)Z)^h \\
&\qquad\qquad + ((\nabla_X R)_x(u_\alpha, Z)Y)^h - ((\nabla_Y R)_x(u_\alpha, Z)X)^h\} \\
&\qquad + \frac{1}{4k^2}\sum_{\alpha,\beta} c^{\alpha}c^{\beta}\{(R_x(u_\beta, X)R_x(u_\alpha, Y)Z)^h - (R_x(u_\beta, Y)R_x(u_\alpha, X)Z)^h \\
&\qquad\qquad + (R_x(u_\beta, X)R_x(u_\alpha, Z)Y)^h - (R_x(u_\beta, Y)R_x(u_\alpha, Z)X)^h\} \\
&\qquad + \frac{1}{4k}\sum_{\alpha,\beta} c^{\alpha\beta}\{(R_x(u_\beta, R_x(X,Z)u_\alpha)Y)^h - (R_x(u_\beta, R_x(Y,Z)u_\alpha)X)^h \\
&\qquad\qquad + 2(R_x(u_\beta, R_x(X,Y)u_\alpha)Z)^h\} \\
&\qquad - \frac{1}{2k}\sum_{\alpha,\rho,\sigma} c^{\alpha}c_{\rho\sigma}c^{\sigma}\{((\nabla_X R)_x(u_\alpha, Y)Z)^{v,\rho} - ((\nabla_Y R)_x(u_\alpha, X)Z)^{v,\rho} \\
&\qquad\qquad + ((\nabla_X R)_x(u_\alpha, Z)Y)^{v,\rho} - ((\nabla_Y R)_x(u_\alpha, Z)X)^{v,\rho}\} \\
&\qquad - \frac{1}{4k^2}\sum_{\alpha,\beta,\rho,\sigma} c^{\alpha}c^{\beta}c_{\rho\sigma}c^{\sigma}\{(R_x(u_\beta, X)R_x(u_\alpha, Y)Z)^{v,\rho} \\
&\qquad\qquad\qquad - (R_x(u_\beta, Y)R_x(u_\alpha, X)Z)^{v,\rho} \\
&\qquad\qquad\qquad + (R_x(u_\beta, X)R_x(u_\alpha, Z)Y)^{v,\rho} \\
&\qquad\qquad\qquad - (R_x(u_\beta, Y)R_x(u_\alpha, Z)X)^{v,\rho}\} \\
&\qquad - \frac{1}{4k}\sum_{\alpha,\beta} c^{\alpha}\{(R_x(X, R_x(u_\alpha, Y)Z)u_\beta)^{v,\beta} - (R_x(Y, R_x(u_\alpha, X)Z)u_\beta)^{v,\beta} \\
&\qquad\qquad + (R_x(X, R_x(u_\alpha, Z)Y)u_\beta)^{v,\beta} - (R_x(Y, R_x(u_\alpha, Z)X)u_\beta)^{v,\beta}\} \\
&\qquad - \frac{1}{4k}\sum_{\alpha,\beta,\rho,\sigma} c^{\alpha\beta}c_{\rho\sigma}c^{\sigma}\{(R_x(u_\beta, R_x(X,Z)u_\alpha)Y)^{v,\rho} \\
&\qquad\qquad\qquad - (R_x(u_\beta, R_x(Y,Z)u_\alpha)X)^{v,\rho} \\
&\qquad\qquad\qquad + 2(R_x(u_\beta, R_x(X,Y)u_\alpha)Z)^{v,\rho}\} \\
&\qquad - \frac{1}{2}\sum_{\alpha}\{((\nabla_X R)_x(Y,Z)u_\alpha)^{v,\alpha} - ((\nabla_Y R)_x(X,Z)u_\alpha)^{v,\alpha}\},
\end{aligned}
$$

$$
\begin{aligned}
&\bar{R}_{(x,u)}(X^h, Y^h)Z^{v,\lambda} \\
&\quad = \frac{1}{2k}\sum_{\alpha} c^{\lambda\alpha}\{((\nabla_X R)_x(u_\alpha, Z)Y)^h - ((\nabla_Y R)_x(u_\alpha, Z)X)^h\} \\
&\qquad + \frac{1}{4k^2}\sum_{\alpha,\beta} c^{\lambda\alpha}c^{\beta}\{(R_x(u_\beta, X)R_x(u_\alpha, Z)Y)^h - (R_x(u_\beta, Y)R_x(u_\alpha, Z)X)^h\} \\
&\qquad + (R_x(X, Y)Z)^{v,\lambda} \\
&\qquad - \frac{1}{2k}\sum_{\alpha,\rho,\sigma} c^{\lambda\alpha}c_{\rho\sigma}c^{\sigma}\{((\nabla_X R)_x(u_\alpha, Z)Y)^{v,\rho} - ((\nabla_Y R)_x(u_\alpha, Z)X)^{v,\rho}\} \\
&\qquad - \frac{1}{4k^2}\sum_{\alpha,\beta,\rho,\sigma} c^{\lambda\alpha}c^{\beta}c_{\rho\sigma}c^{\sigma}\{(R_x(u_\beta, X)R_x(u_\alpha, Z)Y)^{v,\rho} \\
&\qquad\qquad\qquad - (R_x(u_\beta, Y)R_x(u_\alpha, Z)X)^{v,\rho}\} \\
&\qquad - \frac{1}{4k}\sum_{\alpha,\beta} c^{\lambda\alpha}\{(R_x(X, R_x(u_\alpha, Z)Y)u_\beta)^{v,\beta} - (R_x(Y, R_x(u_\alpha, Z)X)u_\beta)^{v,\beta}\},
\end{aligned}
$$

$$
\begin{aligned}
&\bar{R}_{(x,u)}(X^h, Y^{v,\lambda})Z^h \\
&\quad = \frac{1}{2k}\sum_{\alpha} c^{\lambda\alpha}((\nabla_X R)_x(u_\alpha, Y)Z)^h \\
&\qquad + \frac{1}{2k}c^{\lambda}\{(R_x(X, Y)Z)^h + (R_x(Z, Y)X)^h\} \\
&\qquad + \frac{1}{4k^2}\sum_{\alpha,\beta} c^{\lambda\alpha}c^{\beta}\{(R_x(u_\beta, X)R_x(u_\alpha, Y)Z)^h - (R_x(u_\alpha, Y)R_x(u_\beta, X)Z)^h \\
&\qquad\qquad\qquad - (R_x(u_\alpha, Y)R_x(u_\beta, Z)X)^h\} \\
&\qquad - \frac{1}{2k}\sum_{\alpha,\rho,\sigma} c^{\lambda\alpha}c_{\rho\sigma}c^{\sigma}((\nabla_X R)_x(u_\alpha, Y)Z)^{v,\rho} \\
&\qquad + \frac{1}{4k^2}\sum_{\alpha,\beta,\rho,\sigma} c^{\lambda\alpha}c^{\beta}c_{\rho\sigma}c^{\sigma}\{(R_x(u_\alpha, Y)R_x(u_\beta, X)Z)^{v,\rho} \\
&\qquad\qquad\qquad + (R_x(u_\alpha, Y)R_x(u_\beta, Z)X)^{v,\rho} \\
&\qquad\qquad\qquad - (R_x(u_\beta, X)R_x(u_\alpha, Y)Z)^{v,\rho}\} \\
&\qquad - \frac{1}{4k}\sum_{\alpha,\beta} c^{\lambda\alpha}(R_x(X, R_x(u_\alpha, Y)Z)u_\beta)^{v,\beta} \\
&\qquad - \frac{1}{2k}c^{\lambda}\sum_{\rho,\sigma} c_{\rho\sigma}c^{\sigma}\{(R_x(X, Y)Z)^{v,\rho} + (R_x(Z, Y)X)^{v,\rho}\} \\
&\qquad + \frac{1}{2}(R_x(X, Z)Y)^{v,\lambda},
\end{aligned}
$$

$$\begin{aligned}
&\bar{R}_{(x,u)}(X^h, Y^{v,\lambda})Z^{v,\mu} \\
&\quad = -\frac{1}{2k}c^{\lambda\mu}(R(Y,Z)X)^h \\
&\qquad - \frac{1}{4k^2}\sum_{\alpha,\beta} c^{\lambda\alpha}c^{\mu\beta}(R_x(u_\alpha, Y)R_x(u_\beta, Z)X)^h \\
&\qquad + \frac{1}{2k}c^{\lambda\mu}\sum_{\rho,\sigma} c_{\rho\sigma}c^{\sigma}(R(Y,Z)X)^{v,\rho} \\
&\qquad + \frac{1}{4k^2}\sum_{\alpha,\beta,\rho,\sigma} c^{\lambda\alpha}c^{\mu\beta}c_{\rho\sigma}c^{\sigma}(R_x(u_\alpha, Y)R_x(u_\beta, Z)X)^{v,\rho},
\end{aligned}$$

$$\begin{aligned}
&\bar{R}_{(x,u)}(X^{v,\lambda}, Y^{v,\mu})Z^h \\
&\quad = \frac{1}{k}c^{\lambda\mu}(R_x(X,Y)Z)^h \\
&\qquad + \frac{1}{4k^2}\sum_{\alpha,\beta} c^{\lambda\alpha}c^{\mu\beta}\{(R_x(u_\alpha, X)R_x(u_\beta, Y)Z)^h \\
&\qquad\qquad - (R_x(u_\alpha, Y)R_x(u_\beta, X)Z)^h\} \\
&\qquad - \frac{1}{k}c^{\lambda\mu}\sum_{\rho,\sigma} c_{\rho\sigma}c^{\sigma}(R_x(X,Y)Z)^{v,\rho} \\
&\qquad - \frac{1}{4k^2}\sum_{\alpha,\beta,\rho,\sigma} c^{\lambda\alpha}c^{\mu\beta}c_{\rho\sigma}c^{\sigma}\{(R_x(u_\alpha, X)R_x(u_\beta, Y)Z)^{v,\rho} \\
&\qquad\qquad - (R_x(u_\alpha, Y)R_x(u_\beta, X)Z)^{v,\rho}\},
\end{aligned}$$

$$\bar{R}_{(x,u)}(X^{v,\lambda}, Y^{v,\mu})Z^{v,\nu} = 0,$$

where $\lambda, \mu, \nu = 1, 2, \ldots, n$.

Proof. We use Proposition 1, (4), (5) and a formula

$$\begin{aligned}
&\bar{\nabla}_{\boldsymbol{X}}\big((R(u_\alpha, Y)Z)^L\big) \\
&\quad = \sum_i \boldsymbol{X}(u^i_\alpha)(R(\partial/\partial x^i, Y)Z)^L + \sum_i u^i_\alpha \bar{\nabla}_{\boldsymbol{X}}\big((R(\partial/\partial x^i, Y)Z)^L\big)
\end{aligned}$$

for all $\boldsymbol{X} \in \mathfrak{X}(LM)$ and $Y, Z \in \mathfrak{X}(M)$, where L stands for the horizontal or vertical lift. □

4. The case of base manifold with constant sectional curvature

We assume that the base manifold (M, g) is a space of constant curvature K. Then we have

$$R_x(X,Y)Z = K\{\langle Y, Z\rangle X - \langle X, Z\rangle Y\} \tag{9}$$

for all $X, Y, Z \in M_x$, where $\langle\cdot,\cdot\rangle$ stands for the scalar product g_x at $x \in M$. We shall calculate the Ricci tensor and the scalar curvature of $(LM, \bar{g})$.

First, substituting (9) into the formulas of Proposition 2, and taking scalar product, we have

PROPOSITION 3. *If $X, Y, Z \in M_x$, then we have*

$$\begin{aligned}
&\bar{g}_{(x,u)}(\bar{R}_{(x,u)}(X^h, Y^h)Z^h,\ V^h)\\
&\quad = K\{\langle Y, Z\rangle\langle X, V\rangle - \langle X, Z\rangle\langle Y, V\rangle\}\\
&\qquad + \frac{K^2}{4k}\sum_{\alpha,\beta} c^\alpha c^\beta\{5(\langle u_\alpha, X\rangle\langle Y, Z\rangle - \langle u_\alpha, Y\rangle\langle X, Z\rangle)\langle u_\beta, V\rangle\\
&\qquad\qquad - 5(\langle u_\alpha, X\rangle\langle Y, V\rangle - \langle u_\alpha, Y\rangle\langle X, V\rangle)\langle u_\beta, Z\rangle\\
&\qquad\qquad - 4\langle u_\alpha, u_\beta\rangle(\langle Y, Z\rangle\langle X, V\rangle - \langle X, Z\rangle\langle Y, V\rangle)\}\\
&\qquad + \frac{3K^2}{4}\sum_{\alpha,\beta} c^{\alpha\beta}\{(\langle u_\alpha, Y\rangle\langle X, Z\rangle - \langle u_\alpha, X\rangle\langle Y, Z\rangle)\langle u_\beta, V\rangle\\
&\qquad\qquad - (\langle u_\alpha, Y\rangle\langle X, V\rangle - \langle u_\alpha, X\rangle\langle Y, V\rangle)\langle u_\beta, Z\rangle\},
\end{aligned}$$

$$\begin{aligned}
&\bar{g}_{(x,u)}(\bar{R}_{(x,u)}(X^h, Y^h)Z^h,\ V^{v,\mu})\\
&\quad = Kc^\mu\{\langle Y, Z\rangle\langle X, V\rangle - \langle X, Z\rangle\langle Y, V\rangle\}\\
&\qquad - \frac{K^2}{4k}\sum_{\alpha,\beta} c^\alpha c^{\mu\beta}\{2(\langle u_\beta, Y\rangle\langle X, Z\rangle - \langle u_\beta, X\rangle\langle Y, Z\rangle)\langle u_\alpha, V\rangle\\
&\qquad\qquad - 2(\langle u_\beta, Y\rangle\langle X, V\rangle - \langle u_\beta, X\rangle\langle Y, V\rangle)\langle u_\alpha, Z\rangle\\
&\qquad\qquad - (\langle u_\alpha, Y\rangle\langle X, V\rangle - \langle u_\alpha, X\rangle\langle Y, V\rangle)\langle u_\beta, Z\rangle\\
&\qquad\qquad + 2\langle u_\alpha, u_\beta\rangle(\langle Y, Z\rangle\langle X, V\rangle - \langle X, Z\rangle\langle Y, V\rangle)\\
&\qquad\qquad - (\langle u_\alpha, X\rangle\langle u_\beta, Y\rangle - \langle u_\alpha, Y\rangle\langle u_\beta, X\rangle)\langle Z, V\rangle\},
\end{aligned}$$

$$\bar{g}_{(x,u)}(\bar{R}_{(x,u)}(X^h, Y^h)Z^{v,\lambda},\ V^{v,\mu})$$
$$= Kc^{\lambda\mu}\{\langle Y, Z\rangle \langle X, V\rangle - \langle X, Z\rangle \langle Y, V\rangle\}$$
$$- \frac{K^2}{4k}\sum_{\alpha,\beta} c^{\lambda\alpha}c^{\mu\beta}\{(\langle u_\beta, Y\rangle \langle X, Z\rangle - \langle u_\beta, X\rangle \langle Y, Z\rangle)\langle u_\alpha, V\rangle$$
$$+ (\langle u_\alpha, X\rangle \langle Y, V\rangle - \langle u_\alpha, Y\rangle \langle X, V\rangle)\langle u_\beta, Z\rangle$$
$$+ \langle u_\alpha, u_\beta\rangle (\langle Y, Z\rangle \langle X, V\rangle - \langle X, Z\rangle \langle Y, V\rangle)$$
$$- (\langle u_\alpha, X\rangle \langle u_\beta, Y\rangle - \langle u_\alpha, Y\rangle \langle u_\beta, X\rangle)\langle Z, V\rangle\},$$

$$\bar{g}_{(x,u)}(\bar{R}_{(x,u)}(X^h, Y^{v,\lambda})Z^{v,\mu},\ V^h)$$
$$= -\frac{K}{2}c^{\lambda\mu}\{\langle X, Z\rangle \langle Y, V\rangle - \langle X, Y\rangle \langle Z, V\rangle\}$$
$$- \frac{K^2}{4k}\sum_{\alpha,\beta} c^{\lambda\alpha}c^{\mu\beta}\{(\langle u_\beta, Y\rangle \langle X, Z\rangle - \langle u_\beta, X\rangle \langle Y, Z\rangle)\langle u_\alpha, V\rangle$$
$$- (\langle u_\alpha, u_\beta\rangle \langle X, Z\rangle - \langle u_\alpha, Z\rangle \langle u_\beta, X\rangle)\langle Y, V\rangle\},$$

$$\bar{g}_{(x,u)}(\bar{R}_{(x,u)}(X^h, Y^{v,\lambda})Z^{v,\mu},\ V^{v,\nu}) = 0,$$
$$\bar{g}_{(x,u)}(\bar{R}_{(x,u)}(X^{v,\lambda}, Y^{v,\mu})Z^{v,\nu},\ V^{v,\rho}) = 0,$$

where $\lambda, \mu, \nu, \rho = 1, 2, \dots, n$.

Let $\overline{\mathrm{Ric}}$ be the Ricci tensor of $(LM, \bar{g})$. Then, by (8), we have

$$\overline{\mathrm{Ric}}(\boldsymbol{X}, \boldsymbol{Y}) = \frac{1}{k}\sum_a \bar{g}(\bar{R}(E_a{}^h, \boldsymbol{X})\boldsymbol{Y},\ E_a{}^h)$$
$$- \frac{1}{k}\sum_{a,\mu,\sigma} c_{\mu\sigma}c^\sigma \bar{g}(\bar{R}(E_a{}^{v,\mu}, \boldsymbol{X})\boldsymbol{Y},\ E_a{}^h)$$
$$- \frac{1}{k}\sum_{a,\mu,\sigma} c_{\mu\sigma}c^\sigma \bar{g}(\bar{R}(E_a{}^h, \boldsymbol{X})\boldsymbol{Y},\ E_a{}^{v,\mu})$$
$$+ \sum_{a,\lambda,\mu} c_{\lambda\mu}\bar{g}(\bar{R}(E_a{}^{v,\lambda}, \boldsymbol{X})\boldsymbol{Y},\ E_a{}^{v,\mu})$$
$$+ \frac{1}{k}\sum_{a,\lambda,\rho,\mu,\sigma} c_{\lambda\rho}c^\rho c_{\mu\sigma}c^\sigma \bar{g}(\bar{R}(E_a{}^{v,\lambda}, \boldsymbol{X})\boldsymbol{Y},\ E_a{}^{v,\mu})$$

for all $\boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{X}(LM)$. Thus, by Proposition 3, we obtain

PROPOSITION 4. *If* $X, Y, Z \in M_x$, *then we have*

$$\begin{aligned}&\overline{\mathrm{Ric}}_{(x,u)}(X^h, Y^h)\\&= \frac{(n-1)(2k-1)K}{k}\langle X, Y\rangle\\&\quad + \frac{K^2}{2k^2}\sum_{\alpha,\beta} c^\alpha c^\beta \{\langle u_\alpha, u_\beta\rangle\langle X, Y\rangle - \langle u_\alpha, X\rangle\langle u_\beta, Y\rangle\}\\&\quad - \frac{K^2}{2k}\sum_{\alpha,\beta} c^{\alpha\beta}\{\langle u_\alpha, u_\beta\rangle\langle X, Y\rangle + (n-2)\langle u_\alpha, X\rangle\langle u_\beta, Y\rangle\},\end{aligned}$$

$$\begin{aligned}&\overline{\mathrm{Ric}}_{(x,u)}(X^h, Y^{v,\mu})\\&= -\frac{(n-1)K}{2k}c^\mu\langle X, Y\rangle\\&\quad + \frac{K^2}{2k^2}\sum_{\alpha,\beta} c^\alpha c^{\mu\beta}\{\langle u_\alpha, u_\beta\rangle\langle X, Y\rangle - \langle u_\alpha, Y\rangle\langle u_\beta, X\rangle\},\end{aligned}$$

$$\begin{aligned}&\overline{\mathrm{Ric}}_{(x,u)}(X^{v,\lambda}, Y^{v,\mu})\\&= \frac{K^2}{2k^2}\sum_{\alpha,\beta} c^{\lambda\alpha} c^{\mu\beta}\{\langle u_\alpha, u_\beta\rangle\langle X, Y\rangle - \langle u_\alpha, Y\rangle\langle u_\beta, X\rangle\},\end{aligned}$$

where $\lambda, \mu = 1, 2, \ldots, n$.

Let $\overline{\mathrm{Sc}}(\bar{g})$ be the scalar curvature of $(LM, \bar{g})$. Then, by (8), we have

$$\begin{aligned}\overline{\mathrm{Sc}}(\bar{g}) &= \frac{1}{k}\sum_a \overline{\mathrm{Ric}}(E_a{}^h,\ E_a{}^h) - \frac{2}{k}\sum_{a,\mu,\sigma} c_{\mu\sigma}c^\sigma\,\overline{\mathrm{Ric}}(E_a{}^{v,\mu},\ E_a{}^h)\\&\quad + \sum_{a,\lambda,\mu} c_{\lambda\mu}\overline{\mathrm{Ric}}(E_a{}^{v,\lambda},\ E_a{}^{v,\mu})\\&\quad + \frac{1}{k}\sum_{a,\lambda,\rho,\mu,\sigma} c_{\lambda\rho}c^\rho c_{\mu\sigma}c^\sigma\,\overline{\mathrm{Ric}}(E_a{}^{v,\lambda},\ E_a{}^{v,\mu})\end{aligned}$$

for all $\boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{X}(LM)$. Thus, by Proposition 4, we obtain

PROPOSITION 5. *The scalar curvature of the frame bundle* $(LM, \bar{g})$ *is given by*

$$\overline{\mathrm{Sc}}(\bar{g})_{(x,u)} = \frac{n(n-1)K}{k} - \frac{(n-1)K^2}{2k^2}\sum_{\alpha,\beta} c^{\alpha\beta}\langle u_\alpha, u_\beta\rangle. \tag{10}$$

As consequence of the Proposition 5, we have

THEOREM 4. *Let (M, g) be a space of constant sectional curvature. If the scalar curvature $\overline{\mathrm{Sc}}(\bar{g})$ of the frame bundle $(LM, \bar{g})$ is constant then both manifolds are flat.*

Proof. Since the matrix $[c^{\lambda\mu}]$ is positive definite, the second term of the right-hand side of (10) is constant if and only if $K = 0$. □

REMARK 2. This can be applied to the diagonal metric and hence we obtain a new result in the spirit of Theorem 1-3 from the paper by Cordero-de León.

References

[1] CORDERO L.A. AND DE LEÓN M., *Lifts of tensor fields to the frame bundle*, Rend. Circ. Mat. Palermo, **32** (1983), 236–271.

[2] CORDERO L.A. AND DE LEÓN M., *On the curvature of the induced Riemannian metric on the frame bundle of a Riemannian manifold*, J. Math. pures et appl., **65** (1986), 81–91.

[3] CORDERO L.A., DODSON C.T.J. AND DE LEÓN M., *Differential geometry of frame bundles*, Mathematics and its Applications **47**, Kluwer Academic Publishers Group, Dordrecht 1989.

[4] KOWALSKI O. AND SEKIZAWA M., *Natural transformations of Riemannian metrics on manifolds to metrics on linear frame bundles—a classification*, in: "Differential Geometry and its Applications", (Krupka D. and Švec A. Eds.) D. Reidel Publ. Comp., Dordrecht, Boston, Lancaster, Tokyo 1987, 149–178.

[5] KOWALSKI O. AND SEKIZAWA M., *Natural transformations of Riemannian metrics on manifolds to metrics on tangent bundles—A classification*, Bull. Tokyo Gakugei Univ. (4) **40** (1988), 1–29.

[6] MOK K.P., *On the differential geometry of frame bundles of Riemannian manifolds*, J. reine angew Math. **302** (1978), 16–31.

[7] MUSSO E. AND TRICERRI F. *Riemannian metrics on tangent bundles*, Ann. Mat. Pura Appl. (4) **150** (1988), 1–20.

[8] O'NEILL B., *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, London 1983.

[9] OPROIU V., *Some new geometric structures on the tangent bundle*, Math. Publ. Debrecen **55** (1999), 261–281.

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