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Control Theory and Stabil., I

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A REDUCTION METHOD IN OPTIMAL CONTROL FOR THE MAYER PROBLEM

Abstract. The Mayer Problem is treated looking for extremals in $W^{1,2}$ and using controls in L^2. We face the original problem finding critical points of the Action Functional related to the the pre-Hamiltonian $h = h(x, p, u)$. In this approach we show how it is possible to apply the Amann-Conley-Zehnder reduction involving not only the velocity of the curves, but also the controls: this permits us to study the solutions in terms of truncated Fourier series.

1. Introduction

We start from a classical model which is given by a control system

$$\dot{x} = f(x(t), u(t)),$$

where

$$f : \begin{array}{ccc} \mathbb{R}^n \times \mathbb{R}^m & \longrightarrow & \mathbb{R}^n \\ (x, u) & \longmapsto & f(x, u) \end{array}$$

is a $\mathcal{C}^2$ function in all variables. Let us consider the following set of *admissible controls*

$$\mathcal{U} = \{u : [0, T] \longrightarrow \mathbb{R}^m \text{ s.t. } u(\cdot) \text{ bounded and measurable}\}.$$

Given an initial condition $x_0 \in \mathbb{R}^n$ and a control $u(\cdot) \in \mathcal{U}$, we denote by $t \mapsto x(t, u(\cdot))$ the unique Carathéodory solution of the Cauchy problem

$$(1) \qquad \begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ x(0) = x_0. \end{cases}$$

For the existence and uniqueness of the Carathéodory solution of (1) we refer the reader, for example, to [10], [14] or [15].

Once we choose a function $\Psi \in \mathcal{C}^3(\mathbb{R}^n, \mathbb{R})$, we define the *cost functional* in the following way

$$\forall u(\cdot) \in \mathcal{U} \qquad J(u(\cdot)) := \Psi(x(T, u(\cdot))).$$

To solve the Mayer Problem with free terminal point (and with final time T) means that one has to find an optimal control $u^*(\cdot)$ which maximizes the functional $J(\cdot)$ among all $u(\cdot) \in \mathcal{U}$. The corresponding solution of the Cauchy problem (1), $x^*(t) = x(t, u^*(\cdot))$, is called optimal trajectory.

Let us consider the Hamiltonian function

$$\mathcal{H}(x, p, u) = \langle p,\ f(x, u)\rangle,$$

where $\langle p, f(x,u)\rangle := p \cdot f(x,u)$ is the usual scalar product in $\mathbb{R}^n$. The function $\mathcal{H} = \mathcal{H}(x, p, u)$ is also called pre-Hamiltonian in order to distinguish it from the maximized Hamiltonian given by $H(x, p) = \sup_{\omega\in\mathbb{R}^m} \mathcal{H}(x, p, \omega)$. But, here, we deal only with $\mathcal{H} = \mathcal{H}(x, p, u)$, that we will call simply Hamiltonian.

By using the Pontryagin Maximum Principle (see for instance [14] or more recent books [2] and [12]), we can associate a Hamiltonian system with boundary conditions mixed in time to the function $\mathcal{H} = \mathcal{H}(x, p, u)$ (cf. Section 2). The extremals satisfy this Hamiltonian system. Our purpose is to find the solutions of such system by studying the critical points of the Action Functional related to the Hamiltonian (see Section 3)

$$\int_0^T [p\dot{x} - \mathcal{H}(x, p, u)]\, dt.$$

In particular, these solutions satisfy the condition $\frac{\partial \mathcal{H}}{\partial u}(x, p, u) = 0$, which is weaker than the so-called maximality condition (see Section 2).

Our main idea is to follow techniques which are very common in symplectic geometry and mainly due to C. Viterbo (see [17], [18], [19] and cf. also [1]). The same techniques are applied in optimal control problems by the author and by F. Cardin in [4]. But, in [4] the necessary conditions given by the Pontryagin Maximum Principle are used in order to obtain the Hamiltonian system connected with the maximized Hamiltonian function; moreover, the Action Functional related to the maximized Hamiltonian is the main ingredient for the construction of the generating function of the initial Lagrangian submanifold $\Lambda = \{(x(0), p(0))\} \subset T^*\mathbb{R}^n$, which collects all the initial data. This procedure moves away from the controls. Nevertheless, in our work we also want to get some information on the controls, which we explicitly handle in finding critical points of the Action Functional. Finally, in our approach we deal with the Hamiltonian function h directly, which generally has a good regularity property; instead, the maximized Hamiltonian H is usually far from being regular.

In Section 4, we apply the so-called Amann-Conley-Zehnder reduction in order to obtain a reduced problem: roughly speaking, we look for stationary points of the Action Functional in the finite dimensional space of truncated Fourier series. Not always is it possible to simplify the problem in such a way; in fact, we can do it only when we get a condition on the u-component of a fixed point map, which plays a crucial role in the reduction.

In the last Section, we discuss how we apply this approach in the class of linear quadratic (L-Q) problems in order to understand some properties about the fixed point map and the reduced Action Functional. It is interesting to notice that some L-Q problems, singular as well (i.e., such that $\frac{\partial^2 \mathcal{H}}{\partial u^2}$ fails to be strictly positive), can be reduced considering only trigonometrical polynomials. This result might be useful in applications when, in particular, the state equations of an optimal control problem are given in terms of the truncated Fourier series (see e.g. [9]).

2. Preliminaries and justifications

It is well known that the necessary conditions for optimality are classically given by the Pontryagin Maximum Principle, we write it here in a simple formulation (see books [14], [2] or [12]).

Pontryagin Maximum Principle. Let $u^*(\cdot)$ be an admissible control whose corresponding trajectory $x^*(t) = x(t, u^*(\cdot))$ is optimal. Then, there exists a vector-function $p(\cdot)$, $p(t) \in \mathbb{R}^n$, which is the solution of the adjoint linear system

$$\dot{p}(t) = -\frac{\partial \mathcal{H}}{\partial x}(x^*(t), p(t), u^*(t)), \qquad p(T) = \nabla\Psi(x^*(T)), \tag{2}$$

and, moreover, the *maximality condition*

$$\mathcal{H}(x^*(t), p(t), u^*(t)) = \sup_{\omega \in \mathbb{R}^m} \mathcal{H}(x^*(t), p(t), \omega) \tag{3}$$

holds true for almost every $t \in [0, T]$.

The triple $(x(\cdot), p(\cdot), u(\cdot))$ is said to satisfy PMP or *extremal* whenever $u(\cdot)$ is an admissible control, $x(\cdot)$ is the corresponding solution of system (1) and $p(\cdot)$ is such that (2)-(3) are satisfied.

We are going to investigate the solutions of the (controlled) Hamiltonian system associated to $\mathcal{H}$

$$\begin{cases} \dot{x} = \dfrac{\partial \mathcal{H}}{\partial p}(x, p, u) \\ \dot{p} = -\dfrac{\partial \mathcal{H}}{\partial x}(x, p, u) \\ 0 = \dfrac{\partial \mathcal{H}}{\partial u}(x, p, u) \ , \end{cases} \tag{4}$$

with a boundary condition which is mixed in time, that is

$$\begin{cases} x(0) = x_0 \\ p(T) = \nabla\Psi(x(T)) \ . \end{cases}$$

Notice that the solutions of system (4) satisfy the condition $\frac{\partial \mathcal{H}}{\partial u}(x, p, u) = 0$, which is weaker than the maximality one (3). Therefore, we are looking for triples of functions $(x(\cdot), p(\cdot), u(\cdot))$ in a set bigger than extremals set.

Let us denote by $\mathbb{J}$ the $(2n + m) \times (2n + m)$-matrix

$$\mathbb{J} := \begin{pmatrix} \mathbb{O} & \mathbb{I}_{n\times n} & \mathbb{O} \\ -\mathbb{I}_{n\times n} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{I}_{m\times m} \end{pmatrix}.$$

Thus, the Hamiltonian system (4) can be briefly written as follows

$$\begin{pmatrix} \dot{x} \\ \dot{p} \\ 0 \end{pmatrix} = \mathbb{J}\nabla\mathcal{H}(x, p, u),$$

where

$$\nabla\mathcal{H}(x,p,u)=\begin{pmatrix}\frac{\partial\mathcal{H}}{\partial x}(x,p,u)\\ \frac{\partial\mathcal{H}}{\partial p}(x,p,u)\\ \frac{\partial\mathcal{H}}{\partial u}(x,p,u)\end{pmatrix}.$$

In order to solve our problem, we first consider a canonical transformation $(x, p) \mapsto (\tilde{x}, \tilde{p})$ in $T^*\mathbb{R}^n$ given by

$$\begin{cases} p = \tilde{p} + \nabla\Psi(x) \\ x = \tilde{x}\ , \end{cases}$$

which produces the following transformed Hamiltonian:

$$\tilde{\mathcal{H}}(\tilde{x}, \tilde{p}, u) = \mathcal{H}(\tilde{x}, \tilde{p} + \nabla\Psi(\tilde{x}), u).$$

REMARK 1. For any fixed control $u(\cdot)$, the characteristics of the vector field associated to the Hamiltonian $\tilde{\mathcal{H}}$ coincide with the characteristics of the vector field associated to $\mathcal{H}$ up to the above-mentioned canonical transformation. It allows us to study the characteristic curves $(\tilde{x}(\cdot), \tilde{p}(\cdot))$ which end at the zero-section of $T^*\mathbb{R}^n$ for $t = T$, instead of curves $(x(\cdot), p(\cdot))$, which end at Graph$(\nabla\Psi)$ at time $t = T$ (cf. also [4]).

The new boundary conditions in the $(\tilde{x}, \tilde{p})$-coordinates become

$$\begin{cases} \tilde{p}(T) = 0 \\ \tilde{x}(0) = x_0, \end{cases}$$

while the transformed Hamiltonian system is similar:

$$\begin{pmatrix}\dot{\tilde{x}}\\ \dot{\tilde{p}}\\ 0\end{pmatrix} = \mathbb{J}\nabla\tilde{\mathcal{H}}(\tilde{x}, \tilde{p}, u).$$

Notations. We drop the “tilde” from the transformed quantities in order to simplify the notations, writing (x, p) instead of $(\tilde{x}, \tilde{p})$ again.

Hence, our purpose is to find the solutions of system

$$\begin{pmatrix}\dot{x}\\ \dot{p}\\ 0\end{pmatrix} = \mathbb{J}\nabla\mathcal{H}(x, p, u), \qquad \begin{cases} p(T) = 0 \\ x(0) = x_0. \end{cases}$$

3. The action functional

Hereafter, we denote by $h = h(x, p, u)$ a $\mathcal{C}^2$ Hamiltonian function such that $|\nabla^2 h| \le C$ for some positive constant C. In the following two Sections, we aim at looking for solutions $(x(\cdot), p(\cdot), u(\cdot))$ of system

$$\begin{pmatrix} \dot{x} \\ \dot{p} \\ 0 \end{pmatrix} = \mathbb{J}\nabla h(x, p, u), \qquad \begin{cases} p(T) = 0 \\ x(0) = x_0. \end{cases} \tag{5}$$

where $(x(\cdot), p(\cdot))$ belong to $W^{1,2}\big((0, T), \mathbb{R}^{2n}\big)$ and the function $u(\cdot)$ is chosen in the space $L^2\big((0, T), \mathbb{R}^m\big)$. We will call (simply) controls the elements of $L^2\big((0, T), \mathbb{R}^m\big)$. The strong assumption on the Hamiltonian function h to have bounded second derivatives provides that for any fixed $u(\cdot) \in L^2\big((0, T), \mathbb{R}^m\big)$, for any starting condition (x_0, p_0), the Cauchy problem

$$\begin{cases} \dot{x}(t) = \dfrac{\partial h}{\partial p}(x(t), p(t), u(t)) \\ \dot{p}(t) = -\dfrac{\partial h}{\partial x}(x(t), p(t), u(t)) \end{cases} \qquad \begin{cases} p(0) = p_0 \\ x(0) = x_0 \end{cases}$$

admits a unique Carathéodory solution. This condition guarantees also the existence of the Gâteaux derivatives of the below-defined functionals $\mathcal{A}$ and W. Finally, it plays a crucial role in order to obtain the existence and the regularity of a fixed point map that we will define later (see Lemma 2 and Remark 4 in Section 4).

REMARK 2. Notice that in this format the controls are not necessarily bounded. In fact, we are extending into L^2 controls the problem, we stated in previous Sections. By the way, once we are able to apply the Amann-Conley-Zehnder reduction, the u-component of the solution of (5) becomes a trigonometrical polynomial, which is bounded on $[0, T]$ and, hence, it is admissible control.

Let us introduce the Action Functional related to the Hamiltonian function h:

$$\begin{aligned} &\mathcal{A} : \Gamma \longrightarrow \mathbb{R} \\ &\gamma(\cdot) \longmapsto \mathcal{A}[\gamma(\cdot)] := \int_0^T \Big[p(t)\cdot \dot{x}(t) - h\big(x(t), p(t), u(t)\big)\Big]\, dt, \end{aligned} \tag{6}$$

where

$$\Gamma := \Big\{\gamma(\cdot) = \big(x(\cdot), p(\cdot), u(\cdot)\big) \in W^{1,2}\big((0, T), \mathbb{R}^{2n}\big) \times L^2\big((0, T), \mathbb{R}^m\big) \; : \; p(T) = 0\Big\}.$$

Thanks to the Sobolev Inequality Theorem (see for instance [8]), for any $\gamma(\cdot) = \big(x(\cdot), p(\cdot), u(\cdot)\big) \in \Gamma$ the (x, p)-components, namely $(x(\cdot), p(\cdot))$, provide a continuous curve in the cotangent fiber bundle $T^*\mathbb{R}^n$. Notice that the condition $p(T) = 0$ is justified by Remark 1. Moreover, we get a fibration

$$\begin{aligned} \pi : \Gamma &\longrightarrow \mathbb{R}^n \\ \gamma(\cdot) &\longmapsto \pi(\gamma(\cdot)) := x(0), \end{aligned}$$

where $x(0)$ is the starting point of the curve $x(\cdot)$. Indeed, a structure of vector space on the fibers $\pi^{-1}(x(0))$ with $x(0) \in \mathbb{R}^n$ is provided by the space of derivatives of the curves $(x(\cdot), p(\cdot))$ and the space of controls $u(\cdot)$; this is well expressed by means of the following bijection:

$$\begin{array}{llll} g: & \mathbb{R}^n \times L^2 & \longrightarrow & \Gamma \\ & (x(0);\phi) & \longmapsto & g\big(x(0);(\phi_x,\phi_p,\phi_u)\big)(\cdot), \end{array} \tag{7}$$

where

$$\begin{array}{llll} g\big(x(0);\phi\big): & [0,T] & \longrightarrow & \mathbb{R}^{2n+m} \\ & t & \longmapsto & \Big(x(0)+\int_0^t \phi_x(s)\,ds, -\int_t^T \phi_p(s)\,ds, C_N\phi_u(t)\Big), \end{array}$$

$L^2 := L^2\big((0,T),\mathbb{R}^{2n}\times\mathbb{R}^m\big)$, $C_N := \frac{T}{2\pi N}$ and $\phi = (\phi_x,\phi_p,\phi_u)$; then one can immediately prove that g is injective and surjective. Roughly speaking, once we fix the initial point $x(0)$, the (x,p)-components of $\gamma(\cdot)$, $(x(\cdot), p(\cdot))$, are given by integrating the velocities (ϕ_x,ϕ_p), obtaining a continuous curve ending at the zero-section of $T^*\mathbb{R}^n$ at time $t = T$; while we simply multiply the control by a suitable constant.

An important fact is that the solutions of the Hamiltonian system (5) are the stationary points of the function $\mathcal{A}$ defined above in (6). This connection is well explained by the following Lemma.

LEMMA 1. *A curve $\gamma(\cdot) \in \Gamma$ solves the Hamiltonian system (5) if and only if*

$$\delta\mathcal{A}[\gamma]\delta\gamma = 0, \qquad \forall\,\delta\gamma \in \Gamma \quad \textit{such that} \quad \delta x(0) = 0,$$

where by δ we denote the Gâteaux derivative.

Proof. We follow a classical scheme (cf. [1], [4] or [5]); notice that here we have one term more: the derivative of h with respect to u. For any $\delta\gamma \in T_\gamma\Gamma = \Gamma$ ($\gamma =$

(x, p, u)), let us consider

$$\begin{aligned}
\delta\mathcal{A}[\gamma]\delta\gamma &= \frac{d\mathcal{A}}{d\lambda}(\gamma+\lambda\delta\gamma)|_{\lambda=0} = \\
&= \int_0^T \left(\delta p\cdot\dot{x} + p\cdot\delta\dot{x} - \frac{\partial h}{\partial x}(\gamma)\cdot\delta x - \frac{\partial h}{\partial p}(\gamma)\cdot\delta p - \frac{\partial h}{\partial u}(\gamma)\cdot\delta u\right) dt = \\
&\quad \left(\text{integrating by parts } \int_0^T p\cdot\delta\dot{x}\,dt\right) \\
&= \int_0^T \left(\dot{x} - \frac{\partial h}{\partial p}(\gamma)\right)\cdot\delta p\,dt - \int_0^T \left(\dot{p} + \frac{\partial h}{\partial x}(\gamma)\right)\cdot\delta x\,dt + \\
&\quad - \int_0^T \frac{\partial h}{\partial u}(\gamma)\cdot\delta u\,dt + p(T)\cdot\delta x(T) - p(0)\cdot\delta x(0) = \\
&= -\int_0^T \left[\begin{pmatrix}\dot{x}\\ \dot{p}\\ 0\end{pmatrix} - \mathbb{J}\nabla h(\gamma)\right]\cdot\delta\gamma\;ds + p(0)\cdot\delta x(0),
\end{aligned}$$

which immediately proves the Lemma. □

Composing the bijection g defined above with the Action Functional, we obtain the functional $W = -\mathcal{A}\circ g$:

$$\begin{array}{llll}
W: & \mathbb{R}^n\times L^2 & \longrightarrow & \mathbb{R} \\
 & \big(x(0),\phi\big) & \longmapsto & W\big(x(0),\phi\big) := -\mathcal{A}\circ g\big(x(0),\phi\big) = -\mathcal{A}[g(x(0),\phi)].
\end{array}$$

Writing W explicitly, we have ($\gamma = (x, p, u) = g(x(0), \phi)$):

$$\begin{aligned}
&W\big(x(0),\phi\big) = \\
&\quad = -\int_0^T \big(p\cdot\dot{x} - h(x,p,u)\big)\,dt = \\
&\quad = -\int_0^T \left[\phi_x(t)\int_T^t \phi_p(s)\,ds + \right. \\
&\qquad \left. -h\left(x(0) + \int_0^t \phi_x(s)\,ds, -\int_t^T \phi_p(s)\,ds, C_N\phi_u(t)\right)\right] dt.
\end{aligned}$$

Let us compute the Gâteaux derivative of W with respect to ϕ:

$$\begin{aligned}
\frac{DW}{D\phi}\,\delta\phi = -\int_0^T &\left[\delta\phi_x(t)\int_T^t \phi_p(r)\,dr + \phi_x(t)\int_T^t \delta\phi_p(r)\,dr + \right. \\
&\left. -\frac{\partial h}{\partial x}(\gamma)\int_0^t \delta\phi_x(r)\,dr - \frac{\partial h}{\partial p}(\gamma)\int_T^t \delta\phi_p(r)\,dr - C_N\frac{\partial h}{\partial u}(\gamma)\delta\phi_u\right] dt.
\end{aligned}$$

By using the equality

$$\int_0^T \delta\phi_x(t)\int_t^T \phi_p(r)\,dr\,dt = \int_0^T \phi_p(t)\int_0^t \delta\phi_x(r)\,dr\,dt,$$

in the beginning of the expression of $DW/D\phi$, we obtain

$$
\begin{aligned}
&\frac{DW}{D\phi}\,\delta\phi = \\
&\quad = -\int_0^T \Big[-\phi_p(t)\int_0^t \delta\phi_x(r)\,dr - \phi_x(t)\int_t^T \delta\phi_p(r)\,dr + \\
&\quad\quad - \frac{\partial h}{\partial x}(\gamma)\Big(\int_0^t \delta\phi_x(r)\,dr\Big) - \frac{\partial h}{\partial p}(\gamma)\int_T^t \delta\phi_p(r)\,dr - C_N \frac{\partial h}{\partial u}(\gamma)\delta\phi_u \Big]\,dt = \\
&\quad = \int_0^T \left[\begin{pmatrix} \dot{x} \\ \dot{p} \\ 0 \end{pmatrix} - \mathbb{J}\nabla h(\gamma) \right] \delta\gamma\ dt,
\end{aligned}
$$

where $\delta\gamma(t) = \Big(-\int_t^T \delta\phi_p(r)\,dr, \int_0^t \delta\phi_x(r)\,dr, C_N\delta\phi_u(t)\Big) \in \Gamma$.
Therefore, we proved the following result.

PROPOSITION 1. . *Choose $x(0) \in \mathbb{R}^n$. An element $\phi \in L^2$ is a stationary point of $W(x(0),\cdot)$ if and only if $\gamma(\cdot) = g(x(0),\phi)(\cdot) \in \Gamma$ satisfies the Hamiltonian system (5).*

We conclude this Section considering the derivatives of W with respect to $x(0)$ and a consequent remark:

$$
\begin{aligned}
\frac{\partial W}{\partial x(0)}\Big|_{\frac{DW}{D\phi}=0}\delta x(0) &= -\int_0^T \Big(-\frac{\partial h}{\partial x}(\gamma)\Big)\cdot \delta x(0)\,dt = \\
&= -\int_0^T \dot{p}(t)\cdot\delta x(0)\,dt = \\
&= p(0)\cdot\delta x(0).
\end{aligned}
$$

REMARK 3. The functional W can be considered as a global generating function of $\Lambda \subset T^*\mathbb{R}^n$ with ∞-dimensional space of auxiliary parameters, where

$$
\Lambda := \left\{ (x(0), p(0)) :\ x(0) \in \mathbb{R}^n,\ p(0) = \frac{\partial W}{\partial x(0)}(x(0),\phi^*),\ \frac{DW}{D\phi}(x(0),\phi^*) = 0 \right\}.
$$

In fact, in the original scheme given by C. Viterbo (see [17], [18], [19] and cf. also [1], [5] and [4]) the Action Functional constitutes the main ingredient for constructing a global generating function for some Lagrangian submanifold related to a given Hamiltonian flow (see Appendix for basic definitions and properties on Lagrangian submanifolds). Notice that in singular linear quadratic problems the set $\Lambda \subset T^*\mathbb{R}^n$ might fail to be a Lagrangian submanifold; while Λ turns out to be a Lagrangian submanifold in regular L-Q problems (cf. [12] and, for regular cases, see also [4]).

4. The Amann-Conley-Zehnder reduction

The reduction method, introduced by H. Amann, C. Conley and E. Zehnder in [3] and [7], transforms an infinite dimensional variational problem involving the Action Functional into a finite dimensional one.

In the space L^2 we consider the orthonormal basis $\{e^{i\frac{2\pi k}{T}t}\}_{k\in\mathbb{Z}}$. Hence, denoting by $e_k(t) := e^{i\frac{2\pi k}{T}t}$, for all $\phi \in L^2$, we have the Fourier expansion

$$\phi(t) = \sum_{k\in\mathbb{Z}} \phi_k \, e_k(t).$$

For any $N \in \mathbb{N}$ fixed, we can define the projection operator $\mathbb{P}_N$ on the $K(n, m, N)$ central components of ϕ, where $K(n, m, N) := (2n + m)(2N + 1)$,

$$\mathbb{P}_N\phi(t) := \sum_{|k|\leq N} \phi_k \, e_k(t),$$

and the projection operator $\mathbb{Q}_N$ on the remaining infinite external components

$$\mathbb{Q}_N\phi(t) := \sum_{|k|>N} \phi_k \, e_k(t).$$

Take an element $\phi \in L^2 = \mathbb{P}_N L^2 \oplus \mathbb{Q}_N L^2$, we denote by $\mu := \mathbb{P}_N\phi$ (and by $\eta := \mathbb{Q}_N\phi$ respectively) the central (and the external respectively) components of ϕ.

We show that for a suitable N only the finite dimensional space $\mathbb{P}_N\phi$ is sufficient to find stationary points of W (and to construct a generating function of Λ); indeed, by a fixed point argument, we prove that $\mathbb{P}_N\phi$ alone uniquely determines $\mathbb{Q}_N\phi$.

LEMMA 2. *For a suitably large $N \in \mathbb{N}$ the map*

$$\begin{array}{rccc} \mathcal{G}: & \mathbb{Q}_N L^2 & \longrightarrow & \mathbb{Q}_N L^2 \\ & \eta & \longmapsto & \mathbb{Q}_N \mathbb{J}\nabla h\Big(g\big(x(0), \mu + \eta\big)\Big); \end{array} \tag{8}$$

is a contraction map, for any $x(0) \in \mathbb{R}^n$ and $\mu \in \mathbb{P}_N L^2$.

Proof. First, we recall that by the assumptions on the Hamiltonian h there exists $C > 0$

such that $|\nabla^2 h| \leq C$. For any $\eta_1, \eta_2 \in \mathbb{Q}_N L^2$, we obtain

$$\begin{aligned}
&\Big|\Big|\mathcal{G}(\eta_2) - \mathcal{G}(\eta_1)\Big|\Big|_{L^2} = \\
&= \Big|\Big|\mathbb{Q}_N \mathbb{J}\nabla h\Big(g\big(x(0), \mu + \eta_2\big)\Big) - \mathbb{Q}_N \mathbb{J}\nabla h\Big(g\big(x(0), \mu + \eta_1\big)\Big)\Big|\Big|_{L^2} \leq \\
&\leq C\Big|\Big|g\big(x(0), \mu + \eta_2\big) - g\big(x(0), \mu + \eta_1\big)\Big|\Big|_{L^2} = \\
&= C\Big|\Big|\Big(\int_0^t \sum_{|k|>N} x_k\, e^{i\frac{2\pi k}{T} r} dr,\ -\int_t^T \sum_{|k|>N} p_k\, e^{i\frac{2\pi k}{T} r} dr,\ C_N \sum_{|k|>N} u_k e^{i\frac{2\pi k}{T} t}\Big)\Big|\Big|_{L^2} = \\
&= C\Big|\Big|\Big(T \sum_{|k|>N} \frac{e^{i\frac{2\pi k}{T} t}}{i2\pi k} x_k,\ T \sum_{|k|>N} \frac{e^{i\frac{2\pi k}{T} t}}{i2\pi k} p_k,\ C_N \sum_{|k|>N} u_k e^{i\frac{2\pi k}{T} t}\Big) + \\
&\quad - \Big(\sum_{|k|>N} \frac{1}{i2\pi k} x_k,\ \sum_{|k|>N} \frac{1}{i2\pi k} p_k,\ 0\Big)\Big|\Big|_{L^2},
\end{aligned}$$

where $(x_k, p_k, u_k) = \eta_k$ is the k^{th} Fourier coefficient of $\eta = (x, p, u) = \eta_2 - \eta_1$.

$$\begin{aligned}
\Big|\Big|\mathcal{G}(\eta_2) - \mathcal{G}(\eta_1)\Big|\Big|_{L^2} &\leq C\Big(C_N\big|\big|\eta\big|\big|_{L^2} + T\Big|\Big| \sum_{|k|>N} \frac{(x_k, p_k, 0)}{i2\pi k} \Big|\Big|_{L^2}\Big) \leq \\
&\leq C\Big(C_N\big|\big|\eta\big|\big|_{L^2} + \big|\big|\langle (x, p, 0), \mathbb{Q}_N \mathrm{id}_{[0,T]}\rangle_{L^2}\big|\big|_{L^2}\Big) \leq \\
&\leq C\Big(C_N\big|\big|\eta\big|\big|_{L^2} + \big|\big|\eta\big|\big|_{L^2}\ \big|\big|\mathbb{Q}_N \mathrm{id}_{[0,T]}\big|\big|_{L^2}\Big) \leq \\
&\leq C\Big(C_N\big|\big|\eta\big|\big|_{L^2} + \frac{T}{2\pi N}\sqrt{2N}\big|\big|\eta\big|\big|_{L^2}\Big) \leq \\
&\leq CC_N(1 + \sqrt{2N})\big|\big|\eta\big|\big|_{L^2} = \\
&= CC_N(1 + \sqrt{2N})\big|\big|\eta_2 - \eta_1\big|\big|_{L^2}.
\end{aligned}$$

Hence, we get a contraction if we choose N such that

$$CC_N(1 + \sqrt{2N}) = \frac{TC}{2\pi N}(1 + \sqrt{2N}) < 1.$$

□

By the Banach-Caccioppoli contraction Lemma (see for example [15] or [10]) applied to $\mathcal{G}$ defined in (8), once we choose $x(0) \in \mathbb{R}^n$ and $\mu \in \mathbb{P}_N L^2$, we obtain one and only one fixed point of $\mathcal{G}$, denoted by $q(x(0), \mu)$, that satisfies

$$q(x(0), \mu) = \mathbb{Q}_N \mathbb{J}\nabla h(g(x(0), \mu + q(x(0), \mu))). \tag{9}$$

REMARK 4. Thanks to the fact that the Hamiltonian function $h \in \mathcal{C}^2$ and by using the implicit function (Dini) Theorem, the fixed point map

$$\begin{array}{rlcl} q: & \mathbb{R}^n \times \mathbb{P}_N L^2 & \longrightarrow & \mathbb{Q}_N L^2 \\ & (x(0), \mu) & \longmapsto & q(x(0), \mu), \end{array} \tag{10}$$

is continuously differentiable (see [1] or [4]).

Now, suppose that the u-component of the fixed point map $q(x(0), \mu)$ vanishes, namely $q_u(x(0), \mu) \equiv 0$. Then, this allows us to restrict our functional W to a finite-dimensional space of auxiliary parameters, $\mathbb{P}_N L^2 \cong \mathbb{R}^{K(n,m,N)}$. Indeed, if $x(0) \in \mathbb{R}^n$ and $\mu \in \mathbb{P}_N L^2$ are fixed, we can consider the curve

$$\gamma(\cdot) = (x(\cdot), p(\cdot), u(\cdot)) = g(x(0), \mu + q(x(0), \mu))(\cdot) \in \Gamma,$$

such that

$$\begin{pmatrix} \dot{x} \\ \dot{p} \\ \frac{u}{C_N} \end{pmatrix} = \mu + q(x(0), \mu) = \begin{pmatrix} \mu_x + q_x(x(0), \mu) \\ \mu_p + q_p(x(0), \mu) \\ \mu_u + q_u(x(0), \mu) \end{pmatrix},$$

where q_x, q_p and q_u are the x, p and u-components of $q(x(0), \mu)$ respectively. In particular, if $q_u(x(0), \mu) = 0$, notice that the equation

$$\mathbb{Q}_N \left[\begin{pmatrix} \dot{x} \\ \dot{p} \\ 0 \end{pmatrix} - \mathbb{J}\nabla h(\gamma) \right] = 0 \tag{11}$$

is satisfied by $\gamma(\cdot)$, because

$$\mathbb{Q}_N(\mathbb{J}\nabla h(\gamma)) = q(x(0), \mu) = \mathbb{Q}_N \begin{pmatrix} \dot{x} \\ \dot{p} \\ 0 \end{pmatrix}.$$

We summarize the result provided by the reduction machinery: let us consider a solution of system

$$\mu = \mathbb{P}_N \mathbb{J}\nabla h\Big(g\big(x(0), \mu + q(x(0), \mu)\big)\Big) \tag{12}$$

in the unknowns $\mu \in \mathbb{P}_N L^2 \cong \mathbb{R}^{k(n,m,N)}$; assume that the u-component of the fixed point map is zero, $q_u(x(0), \mu) = 0$, then we automatically obtain the solution of the projection of the Hamiltonian system (5) on $\mathbb{Q}_N L^2$ (thanks to (11)). Therefore, the curve $\gamma(\cdot) := g(x(0), \mu + q(x(0), \mu))(\cdot)$ solves the Hamiltonian system (5) (with the boundary conditions $x(0) = x_0$ and $p(T) = 0$).

For $K := K(n, m, N)$ where N is determined as in Lemma 2, we define the function

$$\begin{array}{rlcl} \mathcal{F}: & \mathbb{R}^n \times \mathbb{R}^K & \longrightarrow & \mathbb{R} \\ & \big(x(0), \mu\big) & \longmapsto & \mathcal{F}(x(0), \mu) := W\big(x(0), \mu + q(x(0), \mu)\big). \end{array}$$

The Amann-Conley-Zehender reduction permits us to find solutions of our original system (5) by studying critical points of function $\mathcal{F}$. We express also our main result in terms of generating functions; this well summarizes all richness of structure of the function $\mathcal{F}$.

THEOREM 1. *Let us suppose that $h \in \mathcal{C}^2$ and $|\nabla^2 h| \leq C$ for a positive constant C. If $\frac{\partial \mathcal{F}}{\partial \mu}(x(0), \mu) = 0$ and $q_u(x(0), \mu) = 0$, then we also obtain $\frac{DW}{D\phi}(x(0), \mu + q(x(0), \mu)) = 0$. Moreover, if the u-component of the fixed point q is identically zero, namely $q_u(x(0), \mu) \equiv 0$, then the function $\mathcal{F} = \mathcal{F}(x(0), \mu)$ is a (global) generating function of $\Lambda = \{(x(0), p(0))\} \subset T^*\mathbb{R}^n$ if and only if the functional $W(x(0), \phi)$ is a (global) generating function of Λ (with ∞-dimensional space of parameters).*

Proof. We use a classical argument based on the Amann-Conley-Zehender reduction. First of all let us compute the derivative with respect to μ:

(13)
$$\begin{aligned}\frac{\partial \mathcal{F}}{\partial \mu}(x(0), \mu) =& \frac{DW}{D\phi}\Big(\frac{D\phi}{D\mu} + \frac{D\phi}{D\eta}\frac{Dq}{D\mu}\Big) = \\ &= -\int_0^T \mathbb{P}_N \left[\begin{pmatrix} \dot{x} \\ \dot{p} \\ 0 \end{pmatrix} - \mathbb{J}\nabla h(\gamma) \right]\Bigg|_{\gamma = g(x(0), \mu + q(x(0), \mu))} dt + \\ &\quad - \int_0^T \mathbb{Q}_N \left[\begin{pmatrix} \dot{x} \\ \dot{p} \\ 0 \end{pmatrix} - \mathbb{J}\nabla h(\gamma) \right]\Bigg|_{\gamma = g(x(0), \mu + q(x(0), \mu))} \frac{Dq}{D\mu}\, dt.\end{aligned}$$

The second integral in (13) vanishes by the properties of the fixed point $q(x(0), \mu)$ (cf. (11)). Hence, we get

$$\frac{\partial \mathcal{F}}{\partial \mu}(x(0), \mu) = -\int_0^T \mathbb{P}_N \left[\begin{pmatrix} \dot{x} \\ \dot{p} \\ 0 \end{pmatrix} - \mathbb{J}\nabla h(\gamma) \right]\Bigg|_{\gamma = g(x(0), \mu + q(x(0), \mu))} dt$$

Similarly, deriving $\mathcal{F}$ with respect to $x(0)$, we obtain

$$\begin{aligned}&\frac{\partial \mathcal{F}}{\partial x(0)}(x(0), \mu) = \\ &\quad = \frac{\partial W}{\partial x(0)} + \frac{DW}{D\phi}\frac{D\phi}{D\eta}\frac{Dq}{Dx(0)} = \\ &\quad = \frac{\partial W}{\partial x(0)}(x(0), \phi)\Big|_{\gamma = g(x(0), \mu + q(x(0), \mu))} + \\ &\qquad - \int_0^T \mathbb{Q}_N \left[\begin{pmatrix} \dot{x} \\ \dot{p} \\ 0 \end{pmatrix} - \mathbb{J}\nabla h(\gamma) \right]\Bigg|_{\gamma = g(x(0), \mu + q(x(0), \mu))} \frac{Dq}{Dx(0)}\, dt = \\ &\quad = \frac{\partial W}{\partial x(0)}(x(0), \phi)\Big|_{\phi = \mu + q(x(0), \mu)}.\end{aligned}$$

We conclude observing that if $(x(0), \phi) \in \mathbb{R}^n \times L^2$ satisfies the system

$$
(14) \qquad \begin{cases} p(0) = \dfrac{\partial W}{\partial x(0)}(x(0), \phi) \\ \quad 0 = \dfrac{DW}{D\phi}(x(0), \phi)\ , \end{cases}
$$

then, considering the projection $\mu = \mathbb{P}_N \phi$, the couple $(x(0), \mu) \in \mathbb{R}^n \times \mathbb{R}^K$ satisfies

$$
\begin{cases} p(0) = \dfrac{\partial \mathcal{F}}{\partial x(0)}(x(0), \mu) \\ \quad 0 = \dfrac{\partial \mathcal{F}}{\partial \mu}(x(0), \mu)\ . \end{cases}
$$

Vice versa, if one computes the solution of (26), $(x(0), \mu) \in \mathbb{R}^n \times \mathbb{R}^K$, then, completing μ with $q(x(0), \mu)$ in $\phi = \mu + q(x(0), \mu)$, the couple $(x(0), \phi) \in \mathbb{R}^n \times L^2$ solves (14). □

REMARK 5. i) The condition $q_u(x(0), \mu) = 0$ in the hypothesis of Theorem 1 is strong, but it is satisfied in very simple examples, for instance some linear quadratic cases (see the next section). By using the Theorem above, we are able to study our Mayer Problem only considering a suitable truncation of Fourier series as far as it concerns the control parameters and also the derivatives of state variables (instead of whole L^2). Moreover, the controls turn out to be admissible because they are trigonometrical polynomials (cf. Remark 2).

ii) Suppose that $\mu \mapsto \mathcal{F}(x(0), \mu)$ is weakly quadratic at infinity, namely out of a compact set it is a quadratic form (even degenerate), then we can apply the Ljusternik-Schnirelman theory, which is a powerful tool to get lower bounds on the critical points of a given function (see for example the books [1] or [16] for general theory and [4] for the degenerate case).

5. Linear quadratic examples

In this Section, we apply our results to the well known linear quadratic (L-Q) optimal problem. Let us consider the linear control system

$$
\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ x(0) = x_0 \end{cases}
$$

and the running cost

$$
\ell(x, u) = \langle Px, u\rangle + \frac{1}{2}\langle Ru, u\rangle + \frac{1}{2}\langle Qx, x\rangle,
$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and A, B, P, Q, R are constant matrices; in particular, Q and R are symmetric. We underline the fact that here R can be any definite non-negative matrix, so also the *singular case* is included. Here, we have not a final target for the controlled trajectory $x(t, u(\cdot))$. We are going to minimize the functional

$$\int_0^T \ell(x(t,u), u(t))\, dt \ .$$

It is well known that the above Bolza Problem can be recast in a Mayer form, introducing the auxiliary variable

$$x_{n+1}(t) = \int_0^t \ell(x(s,u), u(s))\, ds \ ,$$

and defining $\Psi(x_1, \ldots, x_n, x_{n+1}) = -x_{n+1}$. Hence, we have a $(n+1)$-dimensional system

$$\begin{cases} (\dot{x}_i)_{i \in \{1,\ldots,n\}} = Ax + Bu \\ \dot{x}_{n+1} = \ell(x,u)\ . \end{cases}$$

In this case the Hamiltonian is

$$h(x, p, u) = \langle p, Ax\rangle + \langle p, Bu\rangle - (\langle Px, u\rangle + \frac{1}{2}\langle Ru, u\rangle + \frac{1}{2}\langle Qx, x\rangle),$$

hence

$$\nabla h(x, p, u) = \begin{pmatrix} \frac{\partial h}{\partial x}(x,p,u) \\ \frac{\partial h}{\partial p}(x,p,u) \\ \frac{\partial h}{\partial u}(x,p,u) \end{pmatrix} = \begin{pmatrix} pA - uP - Qx \\ Ax + Bu \\ pB - Px - Ru \end{pmatrix}.$$

Take $N \in \mathbb{N}$ as in Lemma 2, once $x(0)$ and μ are chosen, then the fixed point map defined in (9)-(10) has the property

$$q(x(0), \mu) = \mathbb{Q}_N \mathbb{J} \nabla h(g(x(0), \mu + q(x(0), \mu))).$$

Let us denote by η_x, η_p and η_u the components of the fixed point $q(x(0), \mu)$

$$q(x(0), \mu) = \begin{pmatrix} \eta_x \\ \eta_p \\ \eta_u \end{pmatrix} \in \mathbb{Q}_N L^2,$$

then we obtain a curve $\gamma(\cdot) = (x(\cdot), p(\cdot), u(\cdot)) \in \Gamma$ by means of the function g, defined in (7):

$$\gamma(\cdot) = g(x(0), \mu + q(x(0), \mu))(\cdot) \in \Gamma.$$

Moreover, we can explicitly write the components of $\gamma(\cdot)$ as follows

$$\begin{cases} x(t) = x(0) + \int_0^t (\mu + \eta)_x(s)\,ds \\ p(t) = -\int_t^T (\mu + \eta)_p(s)\,ds \\ u(t) = C_N(\mu + \eta)_u, \end{cases}$$

namely

$$\begin{cases} x(t) = \hat{x}_0 + T \sum_{0 \neq |k| \leq N} \frac{\mu_{x,k} + \mu_{x,0}}{i2\pi k} e_k(t) + T \sum_{|k|>N} \frac{\eta_{x,k} + \mu_{x,0}}{i2\pi k} e_k(t) \\ p(t) = \hat{p}_0 + T \sum_{0 \neq |k| \leq N} \frac{\mu_{p,k} + \mu_{p,0}}{i2\pi k} e_k(t) + T \sum_{|k|>N} \frac{\eta_{p,k} + \mu_{p,0}}{i2\pi k} e_k(t) \\ u(t) = C_N(\sum_{|k| \leq N} \mu_{u,k} e_k(t) + \sum_{|k|>N} \eta_{u,k} e_k(t)), \end{cases}$$

where $(\mu_{x,k}, \eta_{p,k}, \mu_{u,k}) = \mu_k$ and $(\eta_{x,k}, \eta_{p,k}, \eta_{u,k}) = \eta_k$ are the k^{th} Fourier coefficients of μ and η respectively (here we have to use a more detailed notation with respect to that we used in the proof of Lemma 2); $\hat{x}_0$ and $\hat{p}_0$ are suitable real numbers and recall that $e_k(t) = e^{i\frac{2\pi k}{T}t}$. By simple computations we get

$$\mathbb{Q}_N \mathbb{J} \nabla h(\gamma) =$$
$$\begin{pmatrix} A(T\sum_{|k|>N} \frac{\eta_{x,k}+\mu_{x,0}}{i2\pi k} e_k) + B(C_N \sum_{|k|>N} \eta_{u,k} e_k) \\ T\Big(Q(\sum_{|k|>N} \frac{\eta_{x,k}+\mu_{x,0}}{i2\pi k} e_k) - (\sum_{|k|>N} \frac{\eta_{p,k}+\mu_{p,0}}{i2\pi k} e_k)A\Big) + (C_N \sum_{|k|>N} \eta_{u,k} e_k)P \\ T\Big((\sum_{|k|>N} \frac{\eta_{p,k}+\mu_{p,0}}{i2\pi k} e_k)B - P(\sum_{|k|>N} \frac{\eta_{x,k}+\mu_{x,0}}{i2\pi k} e_k)\Big) - R(C_N \sum_{|k|>N} \eta_{u,k} e_k) \end{pmatrix}.$$

REMARK 6. Notice that in general $(\eta_x, \eta_p, \eta_u) = 0$ is not a fixed point for the map $\mathcal{G}$ defined in (8). But, if $\mu_{x,0} = 0$ and $\mu_{p,0} = 0$ (the so-called zero mean case), then, by the linearity, we obtain that the fixed point vanishes:

$$q(x_0, \mu) = 0.$$

EXAMPLE 1. In order to understand some properties on the fixed point map and on the critical points of W and $\mathcal{F}$, let us consider a very simple control system in $\mathbb{R}^2$ with $u = (u_1, u_2) \in \mathbb{R}^2$:

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = x_2 + u_2. \end{cases}$$

Once we choose a starting point $x(0) = x_0 \in \mathbb{R}^2$, we want to minimize the functional

$$\frac{1}{2} \int_0^T (u_1^2(t) - x_1^2(t))\,dt,$$

where $T = \pi$. The Hamiltonian function is

$$h(x, p, u) = p_2 x_2 + p_1 u_1 + p_2 u_2 - \frac{1}{2} u_1^2 + \frac{1}{2} x_1^2,$$

while the (controlled) Hamiltonian system related to h is given by

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = x_2 + u_2 \\ \dot{p}_1 = -x_1 \\ \dot{p}_2 = -p_2 \\ u_1 = p_1 \\ p_2 = 0, \end{cases}$$

with the boundary condition

$$\begin{cases} x(0) = x_0 \\ p(T) = 0. \end{cases}$$

It is straightforward to see that for any starting point $x(0) \in \mathbb{R}^2$, called $\mu \in P_N L^2$ the solution of (12), the u-component of the fixed point map vanishes: $q_u(x(0), \mu) \equiv 0$. Instead, the x_2-component of $q(x(0), \mu)$ is an infinite series, therefore $q \neq 0$. Notice that to solve the reduced problem, namely to find stationary points of $\mathcal{F}$, implies getting solutions of the ∞-dimensional problem (for W) or, equivalently, for the Hamiltonian system; but the opposite implication is not true: for any $u_2(\cdot) \in L^2((0, T), \mathbb{R})$ (not only for $u_2(\cdot) \in \mathbb{P}_N L^2((0, T), \mathbb{R})$) we have a solution of the Hamiltonian system. Finally, we underline the fact that for $x(0) = 0$ the function $\mu \mapsto \mathcal{F}(0, \mu)$ is a quadratic form degenerate with respect to the u_2-component (cf. ii) of Remark 5).

6. Appendix: Generating functions of Lagrangian submanifolds and symplectic structures

We recall some basic definitions and results which concern the Lagrangian submanifolds of the cotangent fiber bundle $T^*\mathbb{R}^n$ (cf. [20]). A differentiable manifold $\Lambda \subset T^*\mathbb{R}^n$ is Lagrangian if the following conditions hold true

1. $\dim \Lambda = n$
2. $\omega_{\mathbb{R}^n}|_\Lambda = 0$,

where $\omega_{\mathbb{R}^n} = d\theta_{\mathbb{R}^n}$ ($\theta_{\mathbb{R}^n}$ is the canonical 1-form of Liouville); in local coordinates we have $\theta_{\mathbb{R}^n} = \sum_{i=1}^n p_i dx^i$ and the 2-form $\omega_{\mathbb{R}^n} = dp \wedge dx = \sum_{i=1}^n dp_i \wedge dx^i$.
The Theorem of Maslov-Hörmander ([13], [11]) locally characterizes the Lagrangian manifolds $\Lambda \subset T^*\mathbb{R}^n$: a submanifold Λ is Lagrangian if and only if Λ is described by means of (local) functions $(x, v) \mapsto S(x, v)$, $S \in \mathcal{C}^2(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R})$ such that

$$(15) \qquad \Lambda = \left\{ (x,p) : \; p = \frac{\partial S}{\partial x}(x,\bar{v}), \quad 0 = \frac{\partial S}{\partial v}(x,\bar{v}) \quad \exists\, \bar{v} \in \mathbb{R}^k \right\},$$

with the rank condition

$$(16) \qquad \mathrm{rk}\left(\frac{\partial^2 S}{\partial x \partial v}, \frac{\partial^2 S}{\partial v \partial v} \right)\bigg|_{\left\{\frac{\partial S}{\partial v}=0\right\}} = \max = k.$$

Functions S satisfying (15)-(16) are called Morse Families for Λ; instead, we call S a *generating function* of Λ if (15) holds, but not necessarily (16).

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