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NONSTANDARD DISCRETE DERIVATIVES AND EXISTENCE THEOREMS FOR ODE

Abstract. We present nonstandard generalizations of Peano's and Carathéodory's Existence Theorems, which avoid Ascoli's Theorem as well as Lebesgue's Dominated Convergence Theorem.

1. Introduction

Nonstandard Analysis is a mathematical theory discovered by Abraham Robinson in the early 1960's ([9]) which among other things provides a logical foundation for the concept of infinitesimal number.

We begin with a brief and informal presentation of the main tools necessary for the understanding of this paper. Background on foundations of Nonstandard Analysis may be found in either [2], [4] or [11] and nonstandard integration theory is treated in [5], [6], [10] or [1].

${}^\star\mathbb{R}$ is a proper ordered field extension of $\mathbb{R}$, the set of real numbers. The elements of ${}^\star\mathbb{R}$ are said **hyperreals** numbers. A hyperreal number x is

1. **infinitesimal** if $|x| < \frac{1}{n}$ for all $n \in \mathbb{N}$;
2. **finite** if $|x| < n$ for some $n \in \mathbb{N}$;
3. **infinite** if it is not finite.

We denote by ${}^\star\mathbb{R}_b$ the set of all finite hyperreal numbers and by ${}^\star\mathbb{R}_\infty$ the set of all infinite hyperreal numbers. For $x, y \in {}^\star\mathbb{R}$, $x \approx y$ means that $x - y$ is infinitesimal or, in other words, x is **infinitely close** to y.

THEOREM 1. **(Standard Part Theorem)** *If $x \in {}^\star\mathbb{R}_b$, there exists one and only one real number $r \in \mathbb{R}$, called the **standard part** of x and denoted by* **st(x)** *or* $^\circ$**x**, *such that $x \approx r$. Moreover, $st(x + y) = st(x) + st(y)$ and $st(xy) = st(x)st(y)$ whenever $x, y \in {}^\star\mathbb{R}_b$.*

The **nonstandard universe** consists of a pair of structures, $V(\mathbb{R})$ and $V({}^\star\mathbb{R})$, and a mapping

$$^\star(.) : V(\mathbb{R}) \to V({}^\star\mathbb{R})$$

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which associates to each element $a \in V(\mathbb{R})$ its **nonstandard extension** ${}^\star a \in V({}^\star\mathbb{R})$.

All elements of $V(\mathbb{R})$ and their nonstandard extensions are called **standard**, that is, for all $a \in V(\mathbb{R})$, a and ${}^\star a$ are standard. Elements of standard sets will be called **internal**; in particular, ${}^\star a$ is also internal. All elements of $V({}^\star\mathbb{R})$ which are not internal, are called **external**.

The nonstandard extension of $\mathbb{N}$, ${}^\star\mathbb{N}$, is the set of **hypernatural** numbers and we denote by ${}^\star\mathbb{N}_\infty$ the set of infinite hypernatural numbers.

A family $\mathcal{C}$ of sets satisfies the **finite intersection property (f.i.p.)** if intersections of finite subfamilies of $\mathcal{C}$ are non empty. In the following, if E is a set, $\mathcal{P}(E)$ denotes the set of subsets of E, $\mathcal{P}_{fin}(E)$ the set of all finite subsets of E and $card(E)$ the cardinality of E. If $E \in V(\mathbb{R})$, the elements of ${}^\star\mathcal{P}_{fin}(E)$ are called **hyperfinite** subsets of ${}^\star E$.

A **bounded formula** is a first order formula that can be written in such a way that all quantifiers range over a fixed set. A **sentence** is a formula without free variables.

The mapping ${}^\star(.)$ satisfies the following principles.

THEOREM 2. **(Polysaturation Principle)** *Given a set $E \in V(\mathbb{R})$ and $\mathcal{C} \subseteq {}^\star\mathcal{P}(E)$, if $\mathcal{C}$ verifies the f.i.p. and $card(\mathcal{C}) < card(V(\mathbb{R}))$, then $\mathcal{C}$ has non empty intersection.*

THEOREM 3. **(Transfer Principle)** *Suppose $\varphi(a_1, \cdots, a_n)$ is a bounded sentence whose only constants are the a_i. Then $\varphi(a_1, \cdots, a_n)$ is true in $V(\mathbb{R})$ if and only if $\varphi({}^\star a_1, \cdots, {}^\star a_n)$ is true in $V({}^\star\mathbb{R})$.*

The Transfer Principle shows that hyperfinite sets have the same formal properties of finite sets; any hyperfinite set of hyperreals has a minimum and a maximum. The Transfer Principle also shows that a set $B \in V({}^\star\mathbb{R})$ is hyperfinite iff there exists $N \in {}^\star\mathbb{N}$ and an internal bijective map $f : B \to \{n \in {}^\star\mathbb{N} \mid n \leq N\}$. Another consequence of the Transfer Principle is that the map ${}^\star(.)$ respects boolean operations. Polysaturation Principle creates new nonstandard elements.

We say that a bounded formula φ is standard (resp. internal) if all of its constants denote standard (resp. internal) elements of $V({}^\star\mathbb{R})$.

THEOREM 4. *A set $b \in V({}^\star\mathbb{R})$ is standard (resp. internal) iff there exits a set $a \in V(\mathbb{R})$ and a bounded standard (resp. internal) formula φ such that $b = \{x \in {}^\star a | \varphi(x)\}$.*

THEOREM 5. *For any set $A \in V(\mathbb{R})$ there exists a hyperfinite set H such that*

$$A \subseteq H \subseteq {}^\star A.$$

A is infinite iff both inclusions are strict. In particular, the set A is infinite iff ${}^\star A$ contains nonstandard elements.

DEFINITION 1. *Let $Y \in {}^\star\mathcal{P}(\mathbb{R})$ and $F : Y \to {}^\star\mathbb{R}$ be an internal function. Then*

*F is said to be S-**continuous** if for all $x, y \in Y$ we have*

$$x \approx y \Rightarrow F(x) \approx F(y).$$

Some relations between this notion and the usual continuity are the following results.

THEOREM 6. *A function $f : \mathbb{R} \to \mathbb{R}$ is*

1. *continuous on $c \in \mathbb{R}$ iff for all $x \in {}^\star\mathbb{R}$ such that $x \approx c$ then ${}^\star f(x) \approx f(c)$;*
2. *continuous iff ${}^\star f$ is S-continuous in ${}^\star\mathbb{R}_b$;*
3. *uniformily continuous iff ${}^\star f$ is S-continuous in ${}^\star\mathbb{R}$.*

THEOREM 7. *If $[a, b] \subseteq \mathbb{R}$, $F : {}^\star[a, b] \to {}^\star\mathbb{R}$ is internal and S-continuous and there exists $z \in {}^\star[a, b]$ such that $F(z)$ is finite, then*

1. *$F(x)$ is finite for all $x \in {}^\star[a, b]$;*
2. *the standard function $f : [a, b] \to \mathbb{R}$, defined by $f(t) = {}^\circ F(t)$ is continuous and for all $x \in {}^\star[a, b]$, ${}^\star f(x) \approx F(x)$.*

DEFINITION 2. *Let $Y \in {}^\star\mathcal{P}(\mathbb{R})$ and $F : Y \to {}^\star\mathbb{R}$ an internal function. Then F is **S-absolutely continuous** if*

$$\sum_{i=1}^{N} |F(b_i) - F(a_i)| \approx 0$$

for every hyperfinite collection

$$\{[a_1, b_1[, [a_2, b_2[, \cdots, [a_N, b_N[\}$$

(where $[a, b[$ denotes the set $\{t \in {}^\star\mathbb{R} : a \le t < b\} \cap Y$) of non overlapping subintervals of Y such that $\sum_{i=1}^{N}(b_i - a_i) \approx 0$.

THEOREM 8. *If $[a, b] \subseteq \mathbb{R}$, $F : {}^\star[a, b] \to {}^\star\mathbb{R}_b$ is internal and S-absolutely continuous function, then there exists a standard absolutely continuous function $f : [a, b] \to \mathbb{R}$ such that $f(t) = {}^\circ F(t)$.*

REMARK 1.

1. It is clear that each S-absolutely continuous function is S-continuous.
2. Theorems 7 and 8 remain true if we substitute ${}^\star[a, b]$ by a hyperfinite set $\mathbb{X}$ such that $st(\mathbb{X}) = [a, b]$ with $a, b \in \mathbb{R}$.
3. Often we avoid * on nonstandard extensions of functions.

2. Loeb integration theory

Loeb measures were discovered by Peter Loeb in 1975 ([8]). These measures are obtained from an internal measure in the following way.

Suppose that $(\Omega, \mathcal{A}, \mu)$ is an **internal measure space**, that is, Ω is an internal non empty set, $\mathcal{A}$ an internal algebra on Ω and $\mu : \mathcal{A} \to {}^\star\mathbb{R}$ an internal finitely additive measure. In general, this is not a measure space because $\mathcal{A}$ is not a σ-algebra except in the trivial case where $\mathcal{A}$ is finite. The Loeb measure generated by μ will be denoted by μ_L and is a measure defined in a family of subsets of Ω that contains the internal algebra $\mathcal{A}$ and that coincide with ${}^\circ\mu = st(\mu)$ on $\mathcal{A}$.

DEFINITION 3. *Let $B \subseteq \Omega$ (B not necessarily internal). We say that*

1. *B is a **Loeb null set** if for each real $\epsilon > 0$ there exists an internal set $A \in \mathcal{A}$ such that $B \subseteq A$ and $\mu(A) < \epsilon$;*
2. *B is **Loeb measurable** if there exists a set $A \in \mathcal{A}$ such that $A \triangle B := (A \setminus B) \cup (B \setminus A)$ is Loeb null. Denote the collection of all Loeb measurable sets by $L(\mathcal{A})$;*
3. *For $B \in L(\mathcal{A})$ define*
$$\mu_L(B) = {}^\circ\mu(A)$$
*for all $A \in \mathcal{A}$ such that $A \triangle B$ is Loeb null (where ${}^\circ x = st(x) = +\infty$ if $0 < x \in {}^\star\mathbb{R}_\infty$); $\mu_L(B)$ is called the **Loeb measure** of B.*

Note that $\mu_L : L(\mathcal{A}) \to [0, +\infty]$ and

$$\forall A \in \mathcal{A} \ \ \mu_L(A) = {}^\circ\mu(A).$$

THEOREM 9. *$L(\mathcal{A})$ is a σ-algebra, called **Loeb σ-algebra**, and μ_L is a complete σ-additive measure on $L(\mathcal{A})$.*

$(\Omega, L(\mathcal{A}), \mu_L)$ is a measure space, called the **Loeb space**, generated by $(\Omega, \mathcal{A}, \mu)$. Note that μ_L acts on sets which may not be standard.

An important example of a Loeb space is the Loeb counting measure space. Fix $N \in {}^\star\mathbb{N}_\infty$, define $\triangle = \frac{1}{N}$ and make

$$\mathbb{T} = \{k\triangle : k = 0, 1, 2, \cdots, N-1\} = \{0, \frac{1}{N}, \frac{2}{N}, \cdots, 1 - \frac{1}{N}\}. \tag{1}$$

$\mathbb{T}$ is usually called **hyperfinite time line** with increment $\triangle$. Denoting the set of all internal subsets of $\mathbb{T}$ by $\mathcal{A}$ and defining $\nu : \mathcal{A} \to {}^\star[0, 1]$ by

$$\nu(A) = \frac{card(A)}{card(\mathbb{T})} = \frac{card(A)}{N}$$

we obtain an internal measure space $(\mathbb{T}, \mathcal{A}, \nu)$ called **internal counting measure space**. The Loeb space $(\mathbb{T}, L(\mathcal{A}), \nu_L)$ generated by $(\mathbb{T}, \mathcal{A}, \nu)$ is called the **Loeb counting measure space**. This hyperfinite space can be used to represent the Lebesgue space $([0, 1], \mathcal{L}, \lambda)$:

THEOREM 10. *Let* $(\mathbb{T}, L(\mathcal{A}), \nu_L)$ *be the Loeb counting measure space. A set* $A \subseteq [0, 1]$ *is Lebesgue measurable iff*

$$st_{\mathbb{T}}^{-1}(A) = \{t \in \mathbb{T} : \,^{\circ}t \in A\}$$

is Loeb measurable and

$$\lambda(A) = \nu_L(st_{\mathbb{T}}^{-1}(A)).$$

We deal now with measurable functions.

DEFINITION 4. *A function* $f : \Omega \to \mathbb{R}$ *is* ***Loeb measurable*** *if* f *is* μ_L*-measurable in the conventional sense, that is, for every open set* $B \subseteq \mathbb{R}$, $f^{-1}(B) \in L(\mathcal{A})$.

DEFINITION 5. *An internal function* $F : \Omega \to {}^{\star}\mathbb{R}$ *is* ***⋆-measurable*** *if* $F^{-1}(A) \in \mathcal{A}$*, for any* **open set* $A \subseteq {}^{\star}\mathbb{R}$.

Some connections between these notions are given in the following theorem.

THEOREM 11. *If* $F : \Omega \to {}^{\star}\mathbb{R}$ *is internal and ⋆-measurable, then* ${}^{\circ}F$ *is Loeb measurable.*

DEFINITION 6. *Let* $(X, \mathcal{L}, \lambda)$ *be a standard measure space. An internal ⋆-measurable function* $F : {}^{\star}X \to {}^{\star}\mathbb{R}$ *is a (two legged)* ***lifting*** *of* $f : X \to \mathbb{R}$ *if*

$$^{\circ}F(x) = f(^{\circ}x) \qquad {}^{\star}\lambda_L\text{- a.a. } x \in {}^{\star}X$$

(a.a. means almost all).

THEOREM 12. **(Anderson's Theorem)** *Let* $(X, \mathcal{L}, \lambda)$ *be a Lebesgue measure space,* (Y, Γ) *a Hausdorff space with a countable base of open sets and* $f : X \to Y$ *a Lebesgue measurable function. Then* ${}^{\star}f$ *is a lifting of* f.

This may be considered the main lemma for Carathéodory's Existence Theorem 19, the basic ideas of its proof being that nearness in ${}^{\star}Y$ is "measured countably" — $u \approx y \in Y$ if and only if $u \in {}^{\star}B_n$, for whatever basic open set B_n such that $y \in B_n \quad (n \in \mathbb{N})$ —, countable unions of null sets are also null as well as sets of the form ${}^{\star}C \Delta st^{-1}(C)$ ([5, page 158]).

REMARK 2. Anderson proves this result in the case where $(X, \mathcal{L}, \lambda)$ is a complete Radon space. A proof of a version of Anderson's Theorem is given in [5, page 167].

Loeb measures are classical measures over σ-algebras (with possibly nonstandard elements), thus Loeb integration theory is simply the classical theory of integration with respect to Loeb measure: in particular, a Loeb measurable function $f : \Omega \to$

$\mathbb{R}$ is Loeb integrable if f is integrable in the classical sense with respect to the Loeb measure μ_L, in which case the Loeb integral $\int_\Omega f d\mu_L$ is a real number.

The **$\star$-integral** or **internal integral** of a $\star$-measurable function $F : \Omega \to {}^\star\mathbb{R}$ is obtained by Transfer of the definition of the standard integral.

Although Theorem 11 says that if $F : \Omega \to {}^\star\mathbb{R}$ is internal and $\star$-measurable then ${}^\circ F$ is Loeb measurable and for all $x \in \Omega$

$$F(x) \approx {}^\circ F(x),$$

the equation

$$(2) \qquad {}^\circ\left(\int_\Omega F d\mu\right) = \int_\Omega {}^\circ F d\mu_L$$

is, in general, false:

EXAMPLE 1. Let $(\mathbb{T}, L(\mathcal{A}), \nu_L)$ be the Loeb counting measure space. Define the internal $\star$-measurable function

$$F(\tau) = \begin{cases} N^2 & \text{if} \quad \tau = 0 \\ 0 & \text{if} \quad \tau \in \mathbb{T} \setminus \{0\} \end{cases}$$

where $N \in {}^\star\mathbb{N}_\infty$ is the same used in the construction of $\mathbb{T}$. Then $\int_\mathbb{T} {}^\circ F d\nu_L = 0$ (since ${}^\circ F(\tau) = 0$ for ν_L- almost all $\tau \in \mathbb{T}$) and

$$\int_\mathbb{T} F d\nu = \sum_{\tau \in \mathbb{T}} F(\tau)\frac{1}{N} = N.$$

To obtain equality of ${}^\circ\left(\int_\Omega F d\mu\right)$ and $\int_\Omega {}^\circ F d\mu_L$ we must restrict the class of $\star$-integrable functions.

DEFINITION 7. *An internal $\star$-measurable function $F : \Omega \to {}^\star\mathbb{R}$ is S-**integrable** if*

1. $\int_\Omega \mid F \mid d\mu$ *is finite;*
2. *for all $A \in \mathcal{A}$ such that $\mu(A) \approx 0$, then $\int_A \mid F \mid d\mu \approx 0$;*
3. *if $A \in \mathcal{A}$ and $F \approx 0$ on A, then $\int_A \mid F \mid d\mu \approx 0$.*

Condition 1 is necessary to guarantee that all S-integrable function are $\star$-integrable. Condition 2 is needed for equality (2), because $\int_A {}^\circ \mid F \mid d\mu_L = 0$, so $\int_A \mid F \mid d\mu$ must be infinitesimal. The last condition is also necessary to obtain equality (2) because in this case, $\int_A {}^\circ \mid F \mid d\mu_L = 0$.

Note that if the internal measure μ is finite, the last condition is always satisfied, since $F \approx 0$ on A means that for every $\epsilon \in \mathbb{R}^+$, $\int_A \mid F \mid d\mu \leq \epsilon\mu(A)$.

For $f : [0,1] \to \mathbb{R}$ we define $\widehat{f} : \mathbb{T} \to \mathbb{R}$ by

$$\widehat{f}(\tau) = f(^\circ\tau).$$

THEOREM 13. *Let* $(\mathbb{T}, L(\mathcal{A}), \nu_L)$ *be the Loeb counting measure space and* $([0,1], \mathcal{L}, \lambda)$ *the Lebesgue measure space on* $[0,1]$. *The following conditions are equivalent:*

1. $f : [0,1] \to \mathbb{R}$ *is Lebesgue integrable;*
2. $\widehat{f} : \mathbb{T} \to \mathbb{R}$ *is Loeb integrable;*
3. *there exists an internal S-integrable function* $F : \mathbb{T} \to {}^\star\mathbb{R}$ *that is a lifting of* f.

In this case

$$\int_{[0,1]} f(t)d\lambda(t) = \int_{\mathbb{T}} \widehat{f}(\tau)d\nu_L(\tau) = {}^\circ\left(\int_{\mathbb{T}} F d\nu\right) = {}^\circ\left(\sum_{\tau\in\mathbb{T}} F(\tau)\frac{1}{N}\right)$$

REMARK 3. Note that the last theorem defines the Lebesgue integral on [0, 1] as the standard part of some hyperfinite sum. This is also true for the Lebesgue integral on $\mathbb{R}$ (see [10] for details).

The next theorem characterizes nonstandard extensions of Lebesgue integrable functions.

THEOREM 14. *Let* $(Z, \mathcal{L}, \lambda)$ *be a Lebesgue measure space and suppose that* $f : Z \to \mathbb{R}$ *is Lebesgue integrable. Then* ${}^\star f : {}^\star Z \to {}^\star\mathbb{R}$ *is S-integrable.*

3. Nonstandard discrete derivative

Let $\mathbb{T}$ be the hyperfinite time line with respect to the increment $\triangle = \frac{1}{N}$ and $N \in \mathbb{N}_\infty$ (see (1)). The **nonstandard discrete derivative** ([12]) of an internal function $X : \mathbb{T} \to {}^\star\mathbb{R}$ is the function $X' : \mathbb{T} \setminus \{1 - \triangle\} \to {}^\star\mathbb{R}$ defined by

$$X'(t) := \frac{X(t+\triangle) - X(t)}{\triangle}.$$

THEOREM 15. *Let* $(\mathbb{T}, \mathcal{A}, \nu)$ *be the internal counting measure space and suppose* $X : \mathbb{T} \to {}^\star\mathbb{R}_b$ *is an internal function. The following conditions are equivalent:*

1. X *is S-absolutely continuous;*
2. X' *is S-integrable;*
3. $\int_A \mid X' \mid d\nu = \sum_{\tau\in A} \mid X'(\tau) \mid \triangle \approx 0$ *for all* $A \in \mathcal{A}$ *such that* $\nu(A) \approx 0$.

4. Nonstandard Peano's Existence Theorem

THEOREM 16. **(Nonstandard Peano's Existence Theorem)** * *Suppose* $F : \mathbb{T} \times {}^\star\mathbb{R} \to {}^\star\mathbb{R}_b$ *is internal and* $\alpha \in {}^\star\mathbb{R}$. *Then there exists one and only one internal S-absolutely continuous function* $X : \mathbb{T} \to {}^\star\mathbb{R}$ *such that*

$$\begin{cases} X(0) &= \alpha \\ X'(t) &= F(t, X(t)) \quad (t \in \mathbb{T} \setminus \{1 - \triangle\}) \end{cases} \tag{3}$$

Moreover, if α *is finite, then* $X(\mathbb{T}) \subseteq {}^\star\mathbb{R}_b$.

Proof. Define $X : \mathbb{T} \to {}^\star\mathbb{R}$ recursively by

$$\begin{aligned} X(0) &= \alpha \\ X(t + \triangle) &:= X(t) + F(t, X(t))\triangle \qquad (t \in \mathbb{T} \setminus \{1 - \triangle\}). \end{aligned}$$

X is internal, by construction

$$X(t) = \alpha + \sum_{i=0}^{k-1} F(i\triangle, X(i\triangle))\triangle \quad (t = k\triangle \in \mathbb{T}) \tag{4}$$

and

$$X'(t) = \frac{X(t + \triangle) - X(t)}{\triangle} = F(t, X(t)) \quad (t \in \mathbb{T} \setminus \{1 - \triangle\}). \tag{5}$$

X is actually S-Lipschitz, that is,

$$|X(t) - X(s)| \leq M|t - s| \qquad (s, t \in \mathbb{T}) \tag{6}$$

where $M \in \mathbb{R}$ is such that $|F(t, z)| \leq M$ for all $(t, z) \in \mathbb{T} \times {}^\star\mathbb{R}$:

Suppose that $s = k_1\triangle < k_2\triangle = t$, for certain $k_1, k_2 \in \{0, 1, \cdots, N - 1\}$. Then

$$\begin{aligned} |X(t) - X(s)| &= \Big|\sum_{i=k_1}^{k_2-1} F(i\triangle, X(i\triangle))\triangle\Big| \\ &\leq \sum_{i=k_1}^{k_2-1} |F(i\triangle, X(i\triangle))|\triangle \\ &\leq \sum_{i=k_1}^{k_2-1} M\triangle \\ &= M(t - s). \end{aligned}$$

*The reader might wish to consult chapter 8 of [7], where this subject is also treated from another viewpoint.

Then X satisfies (6) and is S-absolutely continuous.

Using the definition of the discrete derivative, it is clear that there exists only one internal function $X : \mathbb{T} \to {}^{\star}\mathbb{R}$ such that (3) holds.

Using (4) we can prove that, for each $k = 0, 1, \cdots, N-1$

$$|X(k\triangle) - \alpha| \leq \sum_{i=0}^{k-1} |F(i\triangle, X(i\triangle))|\triangle \leq \sum_{i=0}^{k-1} M\triangle \leq M$$

hence, $X(\mathbb{T}) \subseteq [\alpha - M, \alpha + M]$. If α is finite, we conclude that $X(\mathbb{T}) \subseteq {}^{\star}\mathbb{R}_b$. □

5. Peano's Existence Theorem

Using Nonstandard Peano's Existence Theorem we can prove

THEOREM 17. **(Peano's Existence Theorem)** *Suppose* $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ *is bounded and continuous and* $x_0 \in \mathbb{R}$. *Then there exists* $x : [0, 1] \to \mathbb{R}$ *such that*

$$\begin{cases} x(0) &= x_0 \\ x'(t) &= f(t, x(t)) \end{cases}$$

Proof. Suppose $F = {}^{\star}f_{|\mathbb{T}\times{}^{\star}\mathbb{R}} : \mathbb{T} \times {}^{\star}\mathbb{R} \to {}^{\star}\mathbb{R}$. F is internal, $F(\mathbb{T} \times {}^{\star}\mathbb{R}) \subseteq {}^{\star}\mathbb{R}_b$ and for each $\tau \in \mathbb{T}$ and $y \in {}^{\star}\mathbb{R}_b$,

$$^{\circ}F(\tau, y) = f(^{\circ}\tau, ^{\circ}y);$$

note that

$$F(\tau, y) = {}^{\star}f(\tau, y) \approx f(^{\circ}\tau, ^{\circ}y)$$

because $\tau \approx {}^{\circ}\tau \in [0, 1]$, $y \approx {}^{\circ}y \in \mathbb{R}$ and f is continuous.

By Nonstandard Peano's Existence Theorem, there exists an internal S-absolutely continuous function

$$X : \mathbb{T} \to {}^{\star}\mathbb{R}_b$$

such that

$$\begin{cases} X(0) &= x_0 \\ X'(\tau) &= F(\tau, X(\tau)) \quad (\tau \in \mathbb{T} \setminus \{1 - \triangle\}) \end{cases}$$

Theorem 8 says there exists a standard absolutely continuous function $x : [0, 1] \to \mathbb{R}$ such that

$$x(^{\circ}\tau) = {}^{\circ}X(\tau)$$

for all $\tau \in \mathbb{T}$. Hence

$$x(0) = {}^{\circ}X(0) = x_0$$

so that x satisfies the initial condition.

Using the definition and continuity of x we have that

$$X(\tau) \approx x({}^\circ\tau) \approx x(\tau) \quad (\tau \in \mathbb{T})$$

and since f is continuous,

$${}^\star f(\tau, X(\tau)) \approx {}^\star f(\tau, x(\tau)) \approx f({}^\circ\tau, x({}^\circ\tau)) \quad (\tau \in \mathbb{T}).$$

Hence $G : \mathbb{T} \to {}^\star\mathbb{R}$ such that $G(\tau) = {}^\star f(\tau, X(\tau))$ is a lifting of the Lebesgue integrable function $g : [0, 1] \to \mathbb{R}$, $g(t) = f(t, x(t))$.

Moreover, G is S-integrable since for all $A \in \mathcal{A}$

$$\int_A |\, G \,|\, d\nu \leq \int_A M d\nu = M\nu(A)$$

where $M \in \mathbb{R}$ is an upper bound of f, and then

$$\int_{\mathbb{T}} |\, G \,|\, d\nu \leq M$$

and

$$\int_A |\, G \,|\, d\nu \approx 0$$

whenever $\nu(A) \approx 0$.

Next, we will prove that x is a solution to the initial value problem.

Fix $z \in [0, 1]$ and $\tau = k\triangle \in \mathbb{T}$ such that $\tau \approx z$. Observe that

$$\begin{aligned} x(z) &= {}^\circ X(\tau) \\ &= x_0 + {}^\circ\left(\textstyle\sum_{i=0}^{k-1} F(i\triangle, X(i\triangle))\triangle\right) \\ &= x_0 + {}^\circ\left(\textstyle\sum_{i=0}^{k-1} G(i\triangle)\triangle\right) \\ &= x_0 + \textstyle\int_{[0,z]} f(t, x(t)) d\lambda(t) \qquad \text{(Theorem 13)} \end{aligned} \tag{7}$$

□

6. Nonstandard Carathéodory's Existence Theorem

THEOREM 18. **(Nonstandard Carathéodory's Existence Theorem)** *Let $F : \mathbb{T} \times {}^\star\mathbb{R} \to {}^\star\mathbb{R}$ be an internal $\star$-measurable function. Suppose there exists an internal S-integrable function $M : \mathbb{T} \to {}^\star\mathbb{R}$ such that*

$$\forall (\tau, x) \in \mathbb{T} \times^\star \mathbb{R} \;\; |\, F(\tau, x) \,| \leq M(\tau).$$

Then, for each $\alpha \in {}^\star\mathbb{R}$ there exists one and only one internal S-absolutely continuous function $X : \mathbb{T} \to {}^\star\mathbb{R}$ such that

$$\begin{cases} X(0) = \alpha \\ X'(\tau) = F(\tau, X(\tau)) \quad (\tau \in \mathbb{T} \setminus \{1 - \triangle\}) \end{cases} \tag{8}$$

If α is finite, then $X(\mathbb{T}) \subseteq {}^\star\mathbb{R}_b$.

Proof. Define $X : \mathbb{T} \to {}^\star\mathbb{R}$ as in the proof of Nonstandard Peano's Existence Theorem. In this case, for each $\tau = k\triangle \in \mathbb{T}$ we have

$$| X(\tau) - \alpha | = | \sum_{i=0}^{k-1} F(i\triangle, X(i\triangle))\triangle | \leq \sum_{i=0}^{k-1} M(i\triangle)\triangle \leq \int_{\mathbb{T}} M d\nu$$

and $\int_{\mathbb{T}} M d\nu$ is finite since M is S-integrable. Hence, if α is finite, $X(\mathbb{T}) \subseteq {}^\star\mathbb{R}_b$. It remains to be proven that X is S-absolutely continuous. We will use Theorem 15. Take A an internal subset of $\mathbb{T}$ such that $\nu(A) \approx 0$. Note that

$$\sum_{\tau \in A} | X'(\tau) | \triangle = \sum_{\tau \in A} | F(\tau, X(\tau)) | \triangle \leq \sum_{\tau \in A} M(\tau)\triangle = \int_A M d\nu.$$

Since M is S-integrable, $\int_A M d\nu \approx 0$ and therefore $\int_A | X' | d\nu \approx 0$ which proves that X is S-absolutely continuous. □

7. Carathéodory's Existence Theorem

Using Nonstandard Carathéodory's Existence Theorem we can prove

THEOREM 19. **(Carathéodory's Existence Theorem)** *Suppose that the function $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, continuous in the second variable and let $x_0 \in \mathbb{R}$. If there exists a Lebesgue integrable function $m : [0, 1] \to \mathbb{R}$ such that*

$$\forall (t, x) \in [0, 1] \times \mathbb{R} \quad | f(t, x) | \leq m(t)$$

then there exists a solution x to the problem

$$\left\{ \begin{array}{rcl} x(0) & = & x_0 \\ x'(t) & = & f(t, x(t)) \quad a.a.\ t \in [0, 1] \end{array} \right. \tag{9}$$

Proof. Since $f : [0, 1]\times\mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function then $F = {}^\star f\,|_{\mathbb{T}\times{}^\star\mathbb{R}} : \mathbb{T} \times {}^\star\mathbb{R} \to {}^\star\mathbb{R}$ is $\star$-measurable. Theorem 14 says ${}^\star m : {}^\star[0, 1] \to {}^\star\mathbb{R}$ is S-integrable and therefore $M = {}^\star m\,|_{\mathbb{T}}$ is also S-integrable. Using the Transfer Principle we conclude that

$$\forall (t, x) \in {}^\star[0, 1] \times {}^\star\mathbb{R} \quad | {}^\star f(t, x) | \leq {}^\star m(t)$$

and then

$$\forall (\tau, x) \in \mathbb{T} \times {}^\star\mathbb{R} \quad | F(\tau, x) | \leq M(\tau).$$

Nonstandard Carathéodory's Existence Theorem shows that there exists an internal S-absolutely continuous $X : \mathbb{T} \to {}^\star\mathbb{R}_b$ such that

$$\left\{ \begin{array}{rcl} X(0) & = & x_0 \\ X'(\tau) & = & F(\tau, X(\tau)) \quad (\tau \in \mathbb{T} \setminus \{1 - \triangle\}) \end{array} \right.$$

and for all $\tau = k\triangle \in \mathbb{T}$

$$X(\tau) = x_0 + \sum_{i=0}^{k-1} F(i\triangle, X(i\triangle))\triangle \in {}^\star\mathbb{R}_b.$$

Since $X(\mathbb{T}) \subseteq {}^\star\mathbb{R}_b$, we can choose $r \in \mathbb{R}^+$ such that

$$\forall \tau \in \mathbb{T} \ \mid X(\tau) \mid \leq r.$$

Defining $x : [0, 1] \to \mathbb{R}$ by

$$x({}^\circ\tau) = {}^\circ X(\tau) \quad (\tau \in \mathbb{T})$$

we conclude, by Theorem 8, that x is absolutely continuous. We will prove that this function is a solution to problem (9).

By hypothesis f is Lebesgue measurable, then the function

$$\widetilde{f} : [0, 1] \to \mathbf{C}([-r, r])$$

where $\mathbf{C}([-r, r])$ denotes the Banach space of real continuous functions on $[-r, r]$, defined by

$$\widetilde{f}(t)(z) = f(t, z) \quad ((t, z) \in [0, 1] \times [-r, r])$$

is also Lebesgue measurable. Taking the uniform topology in $\mathbf{C}([-r, r])$ and using Anderson's Theorem we can conclude that

$$^\star\widetilde{f} : {}^\star[0, 1] \to {}^\star\mathbf{C}([-r, r])$$

is a lifting of $\widetilde{f}$ we respect of the Loeb measure ${}^\star\lambda_L$, that is

$$^\star\widetilde{f}(\tau) \approx \widetilde{f}({}^\circ\tau) \quad {}^\star\lambda_L - \text{a.a. } \tau \in {}^\star[0, 1].$$

Using the definition of the uniform norm in $\mathbf{C}([-r, r])$ we conclude that

$$\big(\forall z \in {}^\star[-r, r]\ {}^\star f(\tau, z) \approx {}^\star f({}^\circ\tau, z)\big) \quad {}^\star\lambda_L - \text{a.a. } \tau \in {}^\star[0, 1].$$

Since f is continuous in the second variable, we obtain that

$$\big(\forall z \in {}^\star[-r, r]\ {}^\star f(\tau, z) \approx f({}^\circ\tau, {}^\circ z)\big) \quad {}^\star\lambda_L - \text{a.a. } \tau \in {}^\star[0, 1].$$

Therefore

$$^\star f(\tau, X(\tau)) \approx f({}^\circ\tau, {}^\circ(X(\tau))) = f({}^\circ\tau, x({}^\circ\tau)) \quad \nu_L - \text{a.a. } \tau \in \mathbb{T}$$

because $\nu_L(\mathbb{T}) = 1$.

Finally we may now prove that for all $t \in [0, 1]$,

$$x(t) = x_0 + \int_{[0,t]} f(s, x(s))d\lambda(s)$$

as we did in the proof of Peano's Existence Theorem. □

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