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SOME NEW CONVERGENCE RESULTS AND APPLICATIONS OF A CLASS OF INTERPOLATING-DERIVATIVE SPLINES

Abstract. In this paper we construct quadrature rules for the numerical evaluation of some singular integrals by using the interpolating-derivative splines. Convergence properties and numerical results are given.

1. Introduction

A problem that arises in many physical applications is the evaluation of the integral

$$I\left(uf^{(p)}\right) := \int_a^b u(x)\, f^{(p)}(x)dx, \tag{1}$$

or the CPV integral ([11], [5])

$$J\left(\omega f^{(p)}; \lambda\right) := \,⨍_a^b \omega(x)\frac{f^{(p)}(x)}{x-\lambda}dx, \qquad \lambda \in (a,b), \tag{2}$$

where $u, \omega \in L_1[a,b]$ and $f \in C^k[a,b]$ (with k a positive integer and $p \leq k$) is a function such that $J(\omega f^{(p)}; \lambda)$ exists for the case (2).

In this paper, we first generalize the results obtained in [10] (where a uniform partition is considered) and we construct the spline of 4th degree minimizing the functional

$$F(s_f) := \int_I [s_f^{(3)}(x)]^2 dx, \qquad I = [a,b]. \tag{3}$$

After, we construct quadrature rules for numerical evaluation of the integral (1) and (2) by using the class of splines named interpolating-derivative splines. These splines, of 4th degree, have been constructed in [10] by minimizing the functional (3) and by considering a uniform partition. Now we shall consider a quasi-uniform partition on I giving, in a such way, more generality to the class of interpolating-derivative splines. Moreover, we shall prove the convergence of the interpolating-derivative splines towards f, $f \in C^k(I)$, $k = 1, 2$. These results will be useful in studying the convergence of quadrature rules here considered. We notice that the generalization for the quasi-uniform partition is not immediate.

The paper is organized as follows: in Section 2 we construct the 4th degree interpolating-derivative spline considering a quasi-uniform partition. In Section 3 we give some convergence results considering $f \in C^2(I)$. In Section 4 some applications on quadrature rules of Cauchy singular integrals with relative convergence results are given. Finally, in Section 5, some numerical results and comparisons with available rules are reported.

2. Construction and properties of interpolating-derivative splines

Firstly we recall the definition of interpolating-derivative splines.

Let n, m (with $m > 1$ and $n \geqslant m$) two given positive integers and consider the partition of $I \equiv [a, b]$

$$\Delta_n := \{a = x_0 < x_1 < \ldots < x_n < x_{n+1} = b\}$$

in $n+1$ subintervals $[x_k, x_{k+1})$, with $h_k = x_{k+1} - x_k$, $k = 0, 1, ..., n$.
We assume

$$h_{\max} = \max_{0 \le k \le n} h_k, \quad h_{\min} = \min_{0 \le k \le n} h_k.$$

We say that the sequence of partitions $\{\Delta_n,\ n = n_1, n_2, \ldots\}$ of I is quasi-uniform $(q.u.)$ if there exists a positive constant R such that

$$\frac{h_{\max}}{h_{\min}} \le R,\ \forall n.$$

From now on we shall consider quasi-uniform partitions and we shall assume that $h_{\max} \to 0$ as $n \to \infty$.

We denote by $I\!P_l$ the set of polynomials of degree $\le l$. The space of polynomial splines of degree $2m$ with simple knots $x_1, x_2, ..., x_n$ and $S_{2m}(\Delta_n) \subset C^{2m-1}(I)$ is defined by:

$$S_{2m}(\Delta_n) := \left\{ \begin{array}{l} s : s(x) = s_k(x) \in I\!P_{2m},\ x \in [x_k, x_{k+1}),\ k = 0, 1, ..., n; \\ D^j s_{k-1}(x_k) = D^j s_k(x_k),\ j = 0, 1, ..., 2m-1,\ k = 1, 2, ..., n \end{array} \right\}. \tag{4}$$

A function $s_f \in S_{2m}(\Delta_n)$ is called *interpolating-derivative* if considering

$$Y := \left\{ y^*, y_1', ..., y_n' \right\},\ Y \in I\!R^{n+1}, \tag{5}$$

a given vector such that $y^* = f(x^*)$, $y_k' = f'(x_k)$, $k = 1, \ldots, n$ there results:

$$\begin{array}{l} s_f(x^*) = y^*,\ x^* \in [a, b], \\ s_f'(x_k) = y_k',\ k = 1, 2, ..., n. \end{array} \tag{6}$$

We remark that now the point x^* of interpolation is an arbitrary point belonging to I, differently from the case considered in [10] where x^* is fixed and coincides with a, the left extreme of the interval I. The effect of such generalization shall be evident in the recurrence formula of c_k in (8) below. Assuming $m = 2$, if we set

$$M_k = s_f^{(3)}(x_k),\ k = 0, 1, ..., n+1,$$

by successive integrations, we obtain

$$\begin{aligned} s_f(x)|_{I_k} = \ & [M_{k+1}(x - x_k)^4 - M_k(x - x_{k+1})^4]/(4! h_k) \\ & + a_k(x - x_k)^2/2 + b_k(x - x_k) + c_k, \quad k = 0, 1, ..., n. \end{aligned} \tag{7}$$

By imposing the conditions (4) and (6), we obtain

$$(8)\quad \begin{cases} a_k = \frac{y'_{k+1}-y'_k}{h_k} - \frac{h_k}{6}(M_{k+1}-M_k), & k=1,...,n-1, \\ b_0 = y'_1 - \frac{h_0^2}{6}M_1 - a_0 h_0, & \\ b_k = y'_k - \frac{h_k^2}{6}M_k, & k=1,...,n, \\ c_i = y^* - [\frac{M_{i+1}(x^*-x_i)^4 - M_i(x^*-x_{i+1})^4}{4!h_i} + \frac{a_i(x^*-x_i)^2}{2} + b_i(x^*-x_i)], & i: x^* \in [x_i, x_{i+1}), \; i=n \text{ if } x^*=x_{n+1}, \\ c_k = \begin{cases} c_{k+1} - \left[b_k h_k + \frac{a_k h_k^2}{2} + \frac{M_{k+1}}{4!}\left(h_{k+1}^3 + h_k^3\right)\right], & k=i-1,...,0, \; i \neq 0, \\ c_{k-1} + \left[b_{k-1}h_{k-1} + \frac{a_{k-1}h_{k-1}^2}{2} + \frac{M_k}{4!}\left(h_k^3 + h_{k-1}^3\right)\right], & k=i+1,...,n, \; i \neq n, \end{cases} \end{cases}$$

and

$$(9)\qquad M_k\left(h_{k-1}+h_k\right)/2 = a_k - a_{k-1},\; k=1,...,n.$$

Following the procedure of [10] we obtain the linear system

$$(10)\qquad \widehat{A}\widetilde{M} = \widehat{\underline{b}}^*(a_0,a_n)$$

where

$$\widehat{A} = \begin{bmatrix} 3h_0+2h_1 & h_1 & & & & \\ \ddots & \ddots & \ddots & & & \\ & h_{i-1} & 2(h_{i-1}+h_i) & h_i & & \\ & & \ddots & \ddots & \ddots & \\ & & & h_{n-1} & 2h_{n-1}+3h_n \end{bmatrix},$$

$$\widetilde{M} = \begin{bmatrix} M_1 \\ M_2 \\ . \\ . \\ . \\ M_{n-1} \\ M_n \end{bmatrix}, \quad \widehat{\underline{b}}^*(a_0,a_n) = 6\begin{bmatrix} f'[x_1,x_2] - a_0 \\ f'[x_2,x_3] - f'[x_1,x_2] \\ \vdots \\ f'\left[x_i,x_{i+1}\right] - f'\left[x_{i-1},x_i\right] \\ \vdots \\ f'\left[x_{n-1},x_n\right] - f'\left[x_{n-2},x_{n-1}\right] \\ -f'\left[x_{n-1},x_n\right] + a_n \end{bmatrix}.$$

If we denote $r_i = h_i/(h_i + h_{i+1}),\ i = 1, ..., n-2$ and we consider

$$H = \begin{bmatrix} h_0 & & & & 0 \\ & h_1 + h_2 & & & \\ & & \ddots & & \\ & & & h_{n-2} + h_{n-1} & \\ 0 & & & & h_n \end{bmatrix},$$

the solution of (10) reduces to the solution of the system

$$\widetilde{A}\widetilde{M} = \underline{b}^*(a_0, a_n) \qquad (11)$$

since:

$$\begin{aligned} \widetilde{A} &= \begin{bmatrix} 3 + 2h_1/h_0 & h_1/h_0 & & & & & \\ r_1 & 2 & 1 - r_1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & r_{i-1} & 2 & 1 - r_{i-1} & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & r_{n-2} & 2 & 1 - r_{n-2} \\ & & & & & h_{n-1}/h_n & 3 + 2h_{n-1}/h_n \end{bmatrix} \\ &= H^{-1}\widehat{A}, \end{aligned}$$

$$\underline{b}^*(a_0, a_n) = 6 \begin{bmatrix} f'[x_1, x_2]/h_0 - a_0/h_0 \\ f'[x_1, x_2, x_3] \\ \vdots \\ f'\left[x_{i-1}, x_i, x_{i+1}\right] \\ \vdots \\ f'\left[x_{n-2}, x_{n-1}, x_n\right] \\ -f'\left[x_{n-1}, x_n\right]/h_n + a_n/h_n \end{bmatrix} = H^{-1}\widehat{\underline{b}}^*(a_0, a_n).$$

The spline function $s_f(x)$ will be determined by solving the following problem

$$\begin{cases} \min M^T \bar{A} M \\ \widetilde{A}\widetilde{M} = \underline{b}^*(a_0, a_n) \end{cases} \qquad (12)$$

with $M = [M_{0,...,}M_{n+1}]^T$,

$$\bar{A} = \left[\begin{array}{c|c|c} 2h_0 & h_0\underline{e}_1^T & 0 \\ \hline h_0\underline{e}_1 & A & h_n\underline{e}_n \\ \hline 0 & h_n\underline{e}_n^T & 2h_n \end{array}\right], \qquad (13)$$

where

$$(14) \qquad A = \begin{bmatrix} 2(h_0+h_1) & h_1 & & & & \\ \ddots & \ddots & \ddots & & & \\ & h_{i-1} & 2(h_{i-1}+h_i) & h_i & & \\ & & \ddots & \ddots & \ddots & \\ & & & h_{n-1} & 2(h_{n-1}+h_n) \end{bmatrix}$$

and $\underline{e}_1$, $\underline{e}_n$ are the first and the last basis vectors for IR^n, respectively.

If we denote $\widetilde{a}_0 = \frac{6}{h_0}a_0$, $\widetilde{a}_n = \frac{6}{h_n}a_n$ and

$$(15) \quad \underline{b}=6\left[\frac{f'[x_1,x_2]}{h_0},\ f'[x_1,x_2,x_3],\ldots,\ f'\left[x_{n-2},x_{n-1},x_n\right],-\frac{f'\left[x_{n-1},x_n\right]}{h_n}\right]^T$$

considering that $\widetilde{A}$ is a non singular matrix because is diagonally dominant with diagonal elements > 0, from (11) we get

$$(16) \qquad \widetilde{M} = \widetilde{A}^{-1}(\underline{b} - \underline{e}_1\widetilde{a}_0 + \underline{e}_n\widetilde{a}_n)$$

and

$$\min M^T \bar{A} M =$$

$$(17) \qquad \min\left\{\left[M_0 \widetilde{M}^T M_{n+1}\right]\left[\begin{array}{c|c|c} 2h_0 & h_0\underline{e}_1^T & 0 \\ \hline h_0\underline{e}_1 & A & h_n\underline{e}_n \\ \hline 0 & h_n\underline{e}_n^T & 2h_n \end{array}\right]\left[\begin{array}{c} M_0 \\ \widetilde{M}^T \\ M_{n+1} \end{array}\right]\right\}.$$

With some algebraic manipulations we can conclude that we need to determine the vector

$$N = [\widetilde{a}_0, -\widetilde{a}_n, -M_0, -M_{n+1}]^T,$$

solution of the linear system

$$(18) \qquad BN = P$$

where

$$(19) \qquad B=\begin{bmatrix} (I_2-B_2)B_2 & B_2 \\ B_2 & 2I_2 \end{bmatrix},\ B_2=\begin{bmatrix} \underline{e}_1^T\widetilde{A}^{-1}\underline{e}_1 & \underline{e}_1^T\widetilde{A}^{-1}\underline{e}_n \\ \underline{e}_n^T\widetilde{A}^{-1}\underline{e}_1 & \underline{e}_n^T\widetilde{A}^{-1}\underline{e}_n \end{bmatrix},$$

I_2 is the second order identity matrix and

$$(20) \qquad P = \begin{bmatrix} (I_2-B_2) \\ I_2 \end{bmatrix}\underline{q},\ \ \underline{q} = \begin{bmatrix} \underline{e}_1^T\widetilde{A}^{-1} \\ \underline{e}_n^T\widetilde{A}^{-1} \end{bmatrix}\underline{b}.$$

When N is determined, we can solve the system (11).

We report here some properties useful for proving the convergence of a sequence of interpolating-derivative splines.

PROPOSITION 1. *The infinitive norm of $\widetilde{A}^{-1}$ satisfies the following relation*

$$\frac{1}{3(1+R)} \leq \left\| \widetilde{A}^{-1} \right\|_\infty \leq 1. \tag{21}$$

Proof. It is easy to obtain the left inequality in (21) because $\|\widetilde{A}\|_\infty \leq 3(1+R)$ and $\|\widetilde{A}^{-1}\|_\infty \|\widetilde{A}\|_\infty \geq 1$. For proving the right inequality we consider that [1]

$$\left\| \widetilde{A}^{-1} \right\|_\infty \leq \left\{ \min_i \left[|\widetilde{a}_{ii}| - \sum_{j \neq i} |\widetilde{a}_{ij}| \right] \right\}^{-1} = \left[2 - (r_{i-1} + 1 - r_{i-1}) \right]^{-1} = 1.$$

□

COROLLARY 1. *For the condition number $K_\infty(\widetilde{A})$ the following inequality holds:*

$$K_\infty(\widetilde{A}) \leq 3(1+R).$$

PROPOSITION 2. *For the entries $\underline{e}_i^T \widetilde{A}^{-1} \underline{e}_j$, $i, j = 1, n$ of $\widetilde{A}^{-1}$, $n \geq 3$ the following inequalities:*

$$\begin{cases} \frac{1}{3(1+R)} \leq \underline{e}_i^T \widetilde{A}^{-1} \underline{e}_i < \frac{1}{3}, \quad \underline{e}_i^T \widetilde{A}^{-1} \underline{e}_i > 2^{n-1} \left| \underline{e}_j^T \widetilde{A}^{-1} \underline{e}_i \right|, & i = 1, n \\ \left| \underline{e}_i^T \widetilde{A}^{-1} \underline{e}_j \right| < \frac{1}{12} \quad \text{and } \underline{e}_i^T \widetilde{A}^{-1} \underline{e}_j \to 0 \text{ as } n \to \infty, & j = n - i + 1 \end{cases} \tag{22}$$

hold.

Proof. See the Appendix. □

Now we prove the following:

THEOREM 1. *The unique solution of system (18) is, for $n \geq 3$,*

$$N = \left[(B_2^{-1} q)^T, \ 0, \ 0 \right]^T. \tag{23}$$

Proof. We have $\det(B) = 2\det(B_2)\det(I_2 - \frac{3}{2}B_2)$ ([10]). In order to show that (23) holds, we only need to prove that $\det(B) \neq 0$. This relation follows immediately by using the propositions 1 and 2, from wich we deduce that $\det(B_2) \neq 0$ and that $\left\| \frac{3}{2} B_2 \right\|_\infty < 1$; but, if $\left\| \frac{3}{2} B_2 \right\|_\infty < 1$ then $\det(I_2 - \frac{3}{2}B_2) \neq 0$ [2]. □

COROLLARY 2 (See [10]). *For the spline $s_f(x)$ the following property holds*

$$M_1 = s_f^{(3)}(x_1) = M_n = s_f^{(3)}(x_n) = 0. \tag{24}$$

From Theorem 1 and Corollary 2 we can deduce that, the spline s_f reduces to a polynomial of second degree in the subintervals I_0 and I_n.

PROPOSITION 3. *For the condition number $K_\infty(B)$ the following inequality holds*

$$K_\infty(B) \le 29\left(1+\frac{8}{9}R\right),\ \ if\, n \ge 3. \tag{25}$$

Proof. The proof follows [10] with some modifications owing to the use of *q.u.* partition. □

We recall the property of polynomial reproducibility of the splines here considered.

PROPOSITION 4. *[Polynomial reproducibility] The 4th degree interpolating derivative spline s_f reproduces any $f \in I\!P_2$ for any sequences of partitions on I.*

Proof. See [10]. □

3. Convergence of interpolating-derivative splines

We give now some results necessary for proving the convergence of a quasi-uniform sequence of interpolating-derivative splines.
For all $g \in C^{(k)}(I)$, $k = 1, 2$, we denote by

$$\omega\left(g^{(p)}; h; I\right) = \max_{x, x+\delta \in I,\, 0<\delta\le h} \left|g^{(p)}(x+\delta) - g^{(p)}(x)\right|, \quad p = 0, 1, k$$

the modulus of continuity of $g^{(p)}$.

PROPOSITION 5. *Assume that Δ_n is a $q.u.$ partition of $[a, b]$. Then*

$$\|M\|_\infty = \left\|\widetilde{M}\right\|_\infty \le \begin{cases} 6R^2\omega\left(f'; h_{\max}; I\right)/h^2_{\max}, & if\ \ f \in C^1(I), \\ 6R\omega\left(f''; h_{\max}; I\right)/h_{\max}, & if\ \ f \in C^2(I). \end{cases} \tag{26}$$

Proof. We can consider the system (11) without the first and the last equation. Since $M_1 = M_n = 0$ in virtue of corollary 2, we can consider $\|M\|_\infty = \left\|\widetilde{M}\right\|_\infty = \|M^\circ\|_\infty \le \|A^\circ\|_\infty \left\|\underline{b}^\circ\right\|_\infty$ where M°, A°, $\underline{b}^\circ$ are obtained from $\widetilde{M}$, $\widetilde{A}$, $\underline{b}^*$ in (11) getting rid the first and the last component or the first and the last row and column. For all $\underline{x}$: $\left\|\underline{x}\right\|_\infty = 1$, there results

$$\left\|A^\circ\underline{x}\right\|_\infty \ge \left\|2\underline{x}\right\|_\infty - \left\|H^\circ\underline{x}\right\|_\infty \ge 2 - 1 = 1,$$

with $A^\circ = 2I + H^\circ$, where

$$H^\circ = \begin{bmatrix} 0 & 1-r_1 & & & \\ \ddots & \ddots & \ddots & & \\ & r_{i-1} & 0 & 1-r_{i-1} & \\ & & \ddots & \ddots & \ddots \\ & & & r_{n-2} & 0 \end{bmatrix}$$

and then [9], $\left\| A^{\circ -1} \right\|_{\infty} \leq 1$.

Considering that

$$\left\| \underline{b}^{\circ} \right\|_{\infty} \leq \begin{cases} 6 \cdot 2\omega \left(f^{'}; h_{\max}; I \right) / (2h_{\min}^{2}), & if \quad f \in C^{1}(I) \\ 6 \cdot 2\omega \left(f^{''}; h_{\max}; I \right) / (2h_{\min}), & if \quad f \in C^{2}(I) \end{cases}$$

the thesis follows. □

THEOREM 2. *Let $f \in C^{k}(I)$, $k = 1, 2$ and s_f the interpolating-derivative spline quoted in Section 2 for a given $q.u.$ partition Δ_n of I. Then*

$$\omega \left(s_f^{(p)}; h_{\max}; I \right) \leq C_p \omega \left(f^{(p)}; h_{\max}; I \right), \quad p = 1, k \tag{27}$$

where C_p, $p = 1, k$, are constants independent of the norm of partition.

Proof. It suffices to show that for $\forall u, v \in I$, $u < v$:

$$\left| s_f^{(p)}(v) - s_f^{(p)}(u) \right| \leq C_p \omega \left(f^{(p)}; v - u; I \right), \quad p = 1, k.$$

If $p = 1$ and $k = 1, 2$ the proof is similar to that one in [10] considering that now, from (8), (23) and (26), we have

$$\begin{aligned} \|a\|_{\infty} &= \|(a_1, ..., a_{n-1})\|_{\infty} \leq R(1 + 2R)\omega \left(f'; h_{\max}; I \right) / h_{\max} \\ \|(a_0, a_n)\|_{\infty} &\leq 4(1 + R)R^2 \omega \left(f'; h_{\max}; I \right) / h_{\max} \end{aligned} \tag{28}$$

and that $C_1 \leq 4\overline{C}_1 + 1$ where

$$\overline{C}_1 \leq \max \left\{ 4R^2(1 + R), \; R(1 + 8R) \right\}.$$

If $p = 2$ and $k = 2$, considering $u, v \in \left[x_i, x_{i+1} \right]$ for any $\xi \in (u, v)$, from (7) and (26), there results: $\left| s_f^{(3)}(\xi) \right| \leq \overline{C}_2 \omega \left(f^{(2)}; h_{\max}; I \right) / h_{\max}$, where $\overline{C}_2 \leq 12R$ and

$$\left| s_f^{(2)}(v) - s_f^{(2)}(u) \right| \leq C_2 \omega \left(f^{(2)}; |v - u|; I \right), \qquad C_2 = 2\overline{C}_2. \tag{29}$$

When $u \in \left[x_i, x_{i+1} \right]$, $v \in \left[x_j, x_{j+1} \right]$ and $j = i + 1$, we have $\left| s_f^{''}(v) - s_f^{''}(u) \right| \leq \left| s_f^{''}(v) - s_f^{''}(x_j) \right| + \left| s_f^{''}(x_{i+1}) - s_f^{''}(u) \right| \leq C_2 \omega \left(f^{''}; |v - u|; I \right)$ with $C_2 = 4\overline{C}_2$. If $j = i + l$, $l > 1$ and $n \geq 3$, recalling (6), for Rolle's theorem $\exists$ $n - 1$ points

$\xi_k \in [x_k, x_{k+1}]$, $k = 1, ..., n-1$, such that $f''(\xi_k) - s''_f(\xi_k) = 0$. Then

$$\begin{aligned}
&\left|s''_f(v) - s''_f(u)\right| \leq \\
\leq\ & \left|s''_f(v) - s''_f(x_{i+l})\right| + \left|s''_f(x_{i+l}) - s''_f\left(\xi_{i+(l-1)}\right)\right| + \left|s''_f\left(\xi_{i+(l-1)}\right) - s''_f(\xi_{i+1})\right| \\
& + \left|s''_f(\xi_{i+1}) - s''_f(x_{i+1})\right| + \left|s''_f(x_{i+1}) - s''_f(u)\right| = \\
=\ & \left|s''_f(v) - s''_f(x_{i+l})\right| + \left|s''_f(x_{i+l}) - s''_f\left(\xi_{i+(l-1)}\right)\right| + \left|f''\left(\xi_{i+(l-1)}\right) - f''(\xi_{i+1})\right| \\
& + \left|s''_f(\xi_{i+1}) - s''_f(x_{i+1})\right| + \left|s''_f(x_{i+1}) - s''_f(u)\right| \leq \\
\leq\ & C_2 \omega\left(f''; |v-u|; I\right),
\end{aligned}$$

where $C_2 \leq 8\overline{C}_2 + 1$. This proves the theorem when $p = 2$. □

Supposing $f \in C^k(I)$, $k = 1, 2$, we define $r^{(p)}(x) = f^{(p)}(x) - s_f^{(p)}(x)$, $p = 0, 1, k$ where s_f is the interpolating-derivative spline quoted in Section 2. We are ready to prove the following convergence result:

THEOREM 3. *Let $f \in C^k(I)$, $k = 1, 2$ and s_f the interpolating-derivative spline based on a given $q.u.$ partition Δ_n on I. There results*

$$\left\|r_n^{(p)}\right\|_\infty \leq \begin{cases} C_{ko}\omega\left(f^{(k)}; h_{\max}; I\right) h_{\max}^{k-1}, & if \quad p = 0, \\ C_{kp}\omega\left(f^{(k)}; h_{\max}; I\right) h_{\max}^{k-p}, & if \quad p = 1, k \end{cases} \tag{30}$$

where C_{k0} and C_{kp}, $k = 1, 2$, $p = 0, ..., k$ are constants independent of the norm of partition.

Proof. If $k = 1$, the proof is similar to [10] and we have:

$$\left|r'_n(x)\right|_{[x_i, x_{i+1}]} \leq (1 + C_1)\,\omega\left(f'; h_{\max}; I\right),\ i = 0, 1, \ldots, n,$$

$$|r_n(x)| \leq (b-a)(C_1+1)\,\omega\left(f'; h_{\max}; I\right)$$

with $C_{10} = (b-a)(C_1+1)$ and $C_{11} = (1 + C_1)$.

If $k = 2$ we have

$$\begin{aligned}
\left|r''_n(x)\right|_{[x_{i-1}, x_{i+1}]} &= \left|r''_n(x) - r''_n(\xi_i)\right| \leq \left|f''(x) - f''(\xi_i)\right| + \left|s''_f(x) - s''_f(\xi_i)\right| \leq \\
&\leq (1 + C_2)\,\omega\left(f''; 2h_{\max}; I\right) \leq 2(1 + C_2)\,\omega\left(f''; h_{\max}; I\right),
\end{aligned}$$

$i = 1, ..., n$, where $\xi_i \in [x_i, x_{i+1}]$, $i = 1, \ldots, n-1$ and ξ_i exists for Rolle's theorem because (6) holds.

$$\left|r'_n(x)\right| = \left|\int_{x_i}^{x} r''_n(t)\,dt\right| \leq \max_{x \in I}\left|r''_n(x)\right| h_{\max} \leq 2(C_2+1)\,\omega\left(f''; h_{\max}; I\right) h_{\max}$$

and so

$$|r_n(x)| \leq 2(b-a)(C_2+1)\,\omega\left(f''; h_{\max}; I\right) h_{\max}$$

and (30) is proved with $C_{21} = 2(b-a)(C_2+1)$, $C_{22} = 2(C_2+1)$. □

4. Quadrature rules based on interpolating-derivative splines

We make use of the quoted splines for constructing quadrature rules suitable for numerically evaluating the integrals:

$$I\left(uf^{(p)}\right) := \int_a^b u(x)\, f^{(p)}(x)dx \tag{31}$$

where $u \in L_1[a,b]$, $f \in C^k[a,b]$, $k = 1, 2$, $p = 0, ..., k$ and the CPV integrals

$$J\left(wf^{(p)}; \lambda\right) := \fint_a^b w(x) \frac{f^{(p)}(x)}{x-\lambda} dx, \qquad \lambda \in (a,b), \tag{32}$$

with $w \in L_1[a,b]$ such that $J(w;\lambda)$ exists, $f \in C^k[a,b]$, $k = 1, 2$, $p = 0, k-1$.

We consider product integration rules for (31) defined by

$$I\left(uf^{(p)}\right) = I\left(us_f^{(p)}\right) + E_n(uf^{(p)})$$

where

$$I\left(us_f^{(p)}\right) = \int_a^b u(x)s_f^{(p)}(x)dx = \sum_{i=0}^{n}\sum_{j=0}^{4-p} \eta_{ijp} I_{ijp} \tag{33}$$

and

$$E_n(uf^{(p)}) = \int_a^b u(x) r_n^{(p)}(x)dx$$

where, for $i = 0, ..., n$, $j = 0, ..., 4-p$, $p = 0, ..., k$, $I_{ijp} = \int_{x_i}^{x_{i+1}} u(x)x^{4-p-j}dx$,

$$\eta_{ijp} = t_i^{(4-j)}(0)/(4-p-j)!, \tag{34}$$

where $t_i(x) := s_f|_{[x_i,x_{i+1}]}(x)$, $x \in [a,b]$, is the $4th$ degree polynomial obtained by the polynomial piece $s_f|_{[x_i,x_{i+1}]}(x)$ defined on all $[a,b]$.
We consider for (32)

$$J\left(wf^{(p)}; \lambda\right) = J^*\left(ws_f^{(p)}; \lambda\right) + E_n^*(wf^{(p)}; \lambda)$$

where

$$\begin{aligned} J^*\left(ws_f^{(p)}; \lambda\right) &= \fint_a^b w(x) \frac{s_f^{(p)}(x) - s_f^{(p)}(\lambda)}{x-\lambda} dx + f^{(p)}(\lambda) \fint_a^b \frac{w(x)}{x-\lambda} dx \\ &= \sum_{i=0}^{n}\sum_{j=0}^{4-p-1} \nu_{ijp}(\lambda)\widetilde{I}_{ij(p-1)} + \sum_{i=0}^{n} \left(\nu_{i(4-p)p}(\lambda) + F(\lambda)\right) J_i(\lambda) \end{aligned} \tag{35}$$

and

$$E_n^*(\omega f^{(p)};\lambda) = \int_a^b \omega(x) \frac{r_n^{(p)}(x) - r_n^{(p)}(\lambda)}{x-\lambda} dx$$

where, for i=0, ..., n, j=0, ..., $4-p$, p=0, ..., k, $\widetilde{I}_{ij(p-1)} = \int_{x_i}^{x_{i+1}} \omega(x) x^{4-p-1-j} dx$,

$$\nu_{ijp}(\lambda) = \begin{cases} \eta_{i0p}, & j=0, \\ \lambda \nu_{i(j-1)p}(\lambda) + \eta_{ijp}, & j=1, ...4-p, \end{cases} \tag{36}$$

$J_i(\lambda) = \int_{x_i}^{x_{i+1}} \frac{\omega(x)}{x-\lambda} dx$ and $F(\lambda) = f^{(p)}(\lambda) - s_f^{(p)}(\lambda)$.
We observe that from (36), we can write $\eta_{ijp} = \nu_{ijp}(0)$ and by induction on j and by (34),the following equalities hold:

$$\nu_{ijp}(\lambda) = \sum_{k=0}^{j} \lambda^{j-k} \eta_{ikp} = \sum_{k=0}^{j} \lambda^{j-k} t_i^{(p)(4-p-k)}(0)/(4-p-k)!. \tag{37}$$

In particular, when $j = 4-p$, (37) represents the Taylor expansion of $t_i^{(p)}(\lambda)$ at zero and so we have: $\nu_{i(4-p)p}(\lambda) = t_i^{(p)}(\lambda)$.

Now, it is easy to prove the following convergence results:

THEOREM 4. *Let Δ_n a q.u. partition of $[a,b]$, $u \in L_1[a,b]$, $f \in C^k[a,b]$, $k = 1,2$, $p = 0,...,k$ and s_f the interpolating-derivative spline. Then*

$$I\left(us_f^{(p)}\right) \to I\left(uf^{(p)}\right) \quad uniformly \quad as\ n \to \infty$$

with convergence order coinciding with the convergence order of $s_f^{(p)} \to f^{(p)}$.

Proof. $\left|I\left(uf^{(p)}\right) - I\left(us_f^{(p)}\right)\right| = \left|E_n(uf^{(p)})\right| = \left|\int_a^b u(x) r_n^{(p)}(x) dx\right|$
$\leq \left\|r_n^{(p)}\right\|_\infty \int_a^b |u(x)|\, dx$. The thesis follows because $u \in L_1[a,b]$. □

THEOREM 5. *Let Δ_n a q.u. partition of $[a,b]$, $\omega \in L_1[a,b]$ such that $J(\omega;\lambda)$ exists, $f \in C^k[a,b]$, $k = 1,2$, $p = 0,...,k-1$ and s_f the interpolating-derivative spline. Then*

$$J^*\left(\omega s_f^{(p)}\right) \to J\left(\omega f^{(p)}\right) \quad uniformly \quad as\ n \to \infty$$

with convergence order coinciding with the convergence order of $s_f^{(p+1)} \to f^{(p+1)}$.

Proof. We have

$$\begin{aligned}\left|J\left(wf^{(p)};\lambda\right)-J^*\left(ws_f^{(p)};\lambda\right)\right| &= \left|E_n^*(wf^{(p)};\lambda)\right| \\ &= \left|\fint_a^b w(x)\frac{r_n^{(p)}(x)-r_n^{(p)}(\lambda)}{x-\lambda}dx\right| \\ &= \left|\fint_a^b w(x)r_n^{(p+1)}(\xi(x))dx\right| \\ &\leq \left\|r_n^{(p+1)}\right\|_\infty \fint_a^b |w(x)|\,dx,\end{aligned}$$

where $\xi(x) \in (x,\lambda)$. The thesis follows because $w \in L_1[a,b]$ and $J(w;\lambda)$ exists. □

5. Numerical results

Numerical results obtained by approximating some test functions and some CPV integrals by interpolating-derivative splines on a non uniform partition are considered. We consider a non uniform partition of the interval $[-1,1]$ in $n+1$ subintervals.

The results in tables 1 and 2 are relative to the convergence of the interpolating-derivative splines to the functions:
$f(x) = 1/(x^2+25),\ f(x) \in C^\infty[-1,1]$,
$f(x) = sign(x)x^2/2 + e^x,\ f(x) \in C^1[-1,1]$
equal to the functions considered in [10] for comparing the results: analyzing the respectives tables we can see as a non uniform partition produces better results. We consider $n = 2, 17, 32$. In both of the functions, the interpolation point x^* coincides with the left bound of the interval.

Table 1

$f(x) = 1/(x^2+25)$

x	$\lvert\mathbf{r}_2(x)\rvert$	$\lvert\mathbf{r}_{17}(x)\rvert$	$\lvert\mathbf{r}_{32}(x)\rvert$
-1	0.0 (0)	0.0 (0)	0.0 (0)
-0.6	3.3 (-5)	9.7 (-10)	1.6 (-11)
-0.2	3.1 (-5)	5.5 (-10)	3.6 (-12)
0.2	3.1 (-5)	5.5 (-10)	3.6 (-12)
0.6	3.3 (-5)	9.7 (-10)	1.6 (-11)
1	0.0 (0)	2.1 (-17)	6.9 (-18)

Table 2

$f(x) = sign(x)x^2/2 + e^x$

x	$\lvert\mathbf{r}_2(x)\rvert$	$\lvert\mathbf{r}_{17}(x)\rvert$	$\lvert\mathbf{r}_{32}(x)\rvert$
-1	0.0 (0)	0.0 (0)	0.0 (0)
-0.6	1.8 (-1)	1.2 (-6)	5.1 (-8)
-0.2	1.3 (-1)	5.6 (-4)	2.4 (-6)
0.2	7.9 (-2)	5.6 (-3)	7.5 (-4)
0.6	1.4 (-1)	5.1 (-3)	7.6 (-4)
1	9.5 (-2)	5.1 (-3)	7.6 (-4)

Tables 3, 4, 5 and 6 report the absolute errors, evaluated for different values of n and λ, $\left|E_n^*\right| = \left|E_n^*(wf;\lambda)\right|$ in evaluating the CPV integrals:
$J(wf;\lambda) = \fint_{-1}^1 1/[\sqrt{25-x^2}(x-\lambda)]dx,\ \lambda = 0.25,\ 0.99$,
$J(wf;\lambda) = \fint_{-1}^1 1/[\sqrt{1-x^2}(25+x^2)(x-\lambda)]dx,\ \lambda = 0.25,\ 0.99$.

Table 3

$J(wf;\lambda) = -0.1002688603$

$\lambda = 0.25$	
n	$\lvert E_n^* \rvert$
2	6.2 (-5)
8	1.4 (-7)
14	2.2 (-9)
20	4.0 (-10)
32	8.9 (-11)

Table 4

$J(wf;\lambda) = -1.0717993352$

$\lambda = 0.99$	
n	$\lvert E_n^* \rvert$
2	2.0 (-4)
8	6.2 (-7)
14	1.8 (-8)
20	9.5 (-9)
32	3.9 (-10)

Table 5

$J(wf;\lambda) = -0.0012291611$

$\lambda = 0.25$	
n	$\lvert E_n^* \rvert$
2	2.7 (-6)
8	7.9 (-8)
14	9.4 (-10)
26	8.9 (-11)

Table 6

$J(wf;\lambda) = -0.0046955619$

$\lambda = 0.99$	
n	$\lvert E_n^* \rvert$
2	1.8 (-4)
8	1.5 (-6)
14	6.3 (-8)
20	9.1 (-10)

By observing this tables we deduce that we can obtain good error bound by using only very few knots.

Tables 7 and 8 report some comparisons with the errors E_n^{QI} obtained by utilizing 4th degree quasi-interpolating splines in evaluating the CPV integrals: $J(wf;\lambda) = \fint_{-1}^{1} \frac{e^x}{x-\lambda}dx$, $\lambda = 0.1,\ 0.9$. The errors obtained are comparable with the errors of quasi-interpolating splines.

Table 7

$J(wf;\lambda) = 1.99903605021$

$\lambda = 0.1$		
n	$\lvert E_n^* \rvert$	$\lvert E_n^{QI} \rvert$
20	1.2 (-6)	7.6 (-6)
36	1.0 (-7)	1.1 (-7)

Table 8

$J(wf;\lambda) = -3.85323498264$

$\lambda = 0.9$		
n	$\lvert E_n^* \rvert$	$\lvert E_n^{QI} \rvert$
20	1.8 (-6)	4.6 (-6)
36	6.0 (-8)	4.4 (-7)

6. Appendix.

We now prove the following result:

PROPOSITION 2. *For the entries $\underline{e}_i^T \widetilde{A}^{-1} \underline{e}_j, i, j = 1, n$ of $\widetilde{A}^{-1}, n \geq 3$ the following inequalities:*

$$(38)\quad \begin{cases} \frac{1}{3(1+R)} \leq \underline{e}_i^T \widetilde{A}^{-1} \underline{e}_i < \frac{1}{3}, \quad \underline{e}_i^T \widetilde{A}^{-1} \underline{e}_i > 2^{n-1} \left| \underline{e}_j^T \widetilde{A}^{-1} \underline{e}_i \right|, & i = 1, n \\ \left| \underline{e}_i^T \widetilde{A}^{-1} \underline{e}_j \right| < \frac{1}{12} \quad \text{and } \underline{e}_i^T \widetilde{A}^{-1} \underline{e}_j \to 0 \text{ as } n \to \infty, & j = n - i + 1 \end{cases}$$

hold.

Proof. We can write $\widehat{A} := H\widetilde{A}$ where $\widehat{A} = A + h_0\underline{e}_1\underline{e}_1^T + h_n\underline{e}_n\underline{e}_n^T$ that is a tridiagonal symmetric matrix. Using the results in [3], for the evaluation of the inverse matrix of a tridiagonal symmetric matrix, we can write $\widehat{A}^{-1} = L + \underline{u}\underline{v}^T$ and then, $\underline{e}_i^T\widehat{A}^{-1}\underline{e}_j = l_{ij} + u_i v_j$ where $l_{ij} = 0$ for $i \leq j$.

In our case there results

$$u_1 = 1, \quad u_2 = -\left(2 + 3\tfrac{h_0}{h_1}\right), \tag{39}$$
$$u_{i+1} = -\left[2\left(1 + \tfrac{h_{i-1}}{h_i}\right)u_i + \tfrac{h_{i-1}}{h_i}u_{i-1}\right], \quad i = 2, ..., n-1$$

$$\overline{v}_n = 1, \quad \overline{v}_{n-1} = -(2 + 3\tfrac{h_n}{h_{n-1}}) \tag{40}$$
$$\overline{v}_{i-1} = -\left[2\left(1 + \tfrac{h_i}{h_{i-1}}\right)\overline{v}_i + \tfrac{h_i}{h_{i-1}}\overline{v}_{i+1}\right], \quad i = n-1, ..., 2$$

with $\overline{v} = \alpha v$ and $\alpha = (2h_{n-1}+3h_n)u_n + h_{n-1}u_{n-1} = (2h_1+3h_0)\overline{v}_1 + h_1\overline{v}_2$. Moreover, after some logic consideration on (39) and (40) we deduce that $\underline{e}_1^T\widehat{A}^{-1}\underline{e}_1 = \alpha^{-1}\overline{v}_1 > 0$ and $\underline{e}_n^T\widehat{A}^{-1}\underline{e}_n = \alpha^{-1}u_n > 0$ because α has the same sign of $\overline{v}_1$ and of u_n. Moreover $\underline{e}_1^T\widehat{A}^{-1}\underline{e}_n = \underline{e}_n^T\widehat{A}^{-1}\underline{e}_1 = u_1 v_n = \alpha^{-1}$. Then, considering that $\widehat{A} := H\widetilde{A}$, we have

$$\underline{e}_n^T\widetilde{A}^{-1}\underline{e}_n = \left|\alpha^{-1}u_n\right| h_n = \frac{h_n}{(2h_{n-1} + 3h_n) - h_{n-1}\frac{|u_{n-1}|}{|u_n|}}$$

and

$$\left|\underline{e}_1^T\widetilde{A}^{-1}\underline{e}_n\right| = \left|\underline{e}_1^T\widehat{A}^{-1}\underline{e}_n\right| h_n = \left|\alpha^{-1}\right| h_n = \frac{h_n}{(2h_{n-1} + 3h_n)\,|u_n| - h_{n-1}\,|u_{n-1}|}.$$

It is not difficult to prove that

$$\frac{1}{3(1+R)} < \frac{h_n}{(2h_{n-1} + 3h_n) - h_{n-1}\frac{|u_{n-1}|}{|u_n|}} < \frac{1}{3}$$

and that

$$\frac{h_n}{(2h_{n-1} + 3h_n)\,|u_n| - h_{n-1}\,|u_{n-1}|} < \frac{1}{12}.$$

Moreover

$$\underline{e}_n^T\widetilde{A}^{-1}\underline{e}_n = \left|\alpha^{-1}u_n\right| h_n \geq 2^{n-1}\left|\alpha^{-1}\right| h_n = 2^{n-1}\left|\underline{e}_1^T\widetilde{A}^{-1}\underline{e}_n\right|.$$

In similar way we can prove the thesis for

$$\underline{e}_1^T\widetilde{A}^{-1}\underline{e}_1 \quad \text{and} \quad \underline{e}_n^T\widetilde{A}^{-1}\underline{e}_1.$$

□

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