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MIXED PROBLEMS AND EDGE CALCULUS SYMBOL STRUCTURES

Abstract. Mixed problems, i.e., boundary value problems with conditions that have a jump along a submanifold Z of the boundary of codimension 1, may be interpreted as boundary value problems on a manifold with edge Z. We investigate the symbol hierarchy of general mixed problems under the aspect of the edge operator calculus and discuss, in particular, the role of additional conditions on Z that depend on weights and satisfy an analogue of the Shapiro - Lopatinskij condition.

Introduction

Boundary value problems on configurations with (geometric) singularities and with discontinuous coefficients are motivated by models of applied sciences and engineering, for instance, mechanics, elasticity, crack theory, scattering theory and numerical mathematics. Precise and satisfying solutions in terms of parametrix constructions or the characterisation of regularity and asymptotics in suitable weighted Sobolev spaces belong to the program of a corresponding pseudo-differential calculus. In fact, parametrices to elliptic boundary value problems for differential operators in smooth domains are pseudo-differential operators (more precisely, pseudo-differential boundary value problems with the transmission property).

The present paper studies mixed elliptic problems in a smooth domain, where the boundary conditions are admitted to be discontinuous along a smooth submanifold of the boundary of codimension 1. A classical example is the Zaremba problem for the Laplacian with a jump from Dirichlet to Neumann conditions.

The idea is to interpret the jump of conditions as an edge and to apply the pseudo-differential calculus of boundary value problems on a manifold with edges. The model cone of wedges in our case is a half-plane, regarded as a cone with base $[0, \pi]$ and axial variable $r \in \mathbb{R}_+$ from polar coordinates in $\mathbb{R}^2$.

Numerous authors have contributed results to mixed problems under different aspects, see, for instance, Eskin [9], Rempel and Schulze [25], [27], and the references there.

The purpose of this paper is to make the relations between mixed problems and edge operators as transparent as possible, starting from problems for differential operators with mixed differential boundary conditions. For convenience we mainly consider scalar operators, though all methods and results have an evident generalisation to systems (or operators on a manifold acting between spaces of distributional sections of vector bundles). Mixed and crack problems are formally close to each other, see, for instance, Kapanadze and Schulze [18], and one may treat them to some extent in a unified way. Nevertheless, if one is interested in concrete questions from applications, it seems advisable to investigate them separately. Another point is that mixed problems in the present form are more regular than operators in the larger algebra of edge-degenerate problems; the latter one also admits the jump to be an edge of the configuration. In that sense mixed problems are specialisations of boundary value problems on manifolds with edges (with the transmission property along the smooth parts of the boundary). The case of mixed problems is characterised by additional regularity properties of the coefficients in the operators which are not typical for the general case.

The idea of the present paper is to give the background for a specific pseudo-differential algebra of mixed problems that contains all problems for differential operators (with differential conditions) as well as the parametrices of elliptic elements. In a forthcoming paper we continue this program and study, in particular, regularity of solutions with asymptotics in weighted edge spaces. Note that when the boundary conditions are smooth (i.e., without jumps) we have the standard situation of pseudo-differential boundary value problems with the transmission property, see, Boutet de Monvel [4] or Rempel and Schulze [24]. Regularity with asymptotics in this case corresponds to regularity in standard Sobolev spaces (with "Taylor asymptotics" up to the boundary). In our case we will have such asymptotics outside the jump Z of the conditions, while we get typical edge asymptotics in a neighbourhood of Z.

In smooth boundary value problems it is customary to reduce orders to get the same orders in operators and boundary conditions. Order reducing operators induce isomorphisms between Sobolev spaces; in the smooth case they do not disturb results. Order reductions can also be constructed for mixed (and more general edge boundary value) problems; they represent, in fact, very nice elements in the edge pseudo-differential calculus. Unfortunately, their construction requires a separate paper, see, for instance, Behm [3] for the analogous simpler situation of operators on manifolds with edges without boundary. In addition, if one is not careful, the meromorphic Mellin symbolic ingredients of order reductions may affect asymptotic data; this is highly undesirable in concrete situations. For that reason we avoid reductions of orders here and formulate operators in analogy to Douglis-Nirenberg systems.

We characterise the symbolic hierarchy of mixed problems, construct scales of spaces and operator conventions that yield continuous operators in these spaces (especially, a Mellin operator convention), and we discuss additional conditions along the jump Z of the mixed conditions that complete a given mixed problem to a Fredholm operator in weighted Sobolev spaces. For simplicity, we content ourselves with constant discrete asymptotics. The material in Section 2.4 - 2.5 and 3.1 - 3.3 prepares the structures (symbols as well as spaces) that reflect the structure of parametrices and elliptic regularity with asymptotics. Our results have analogues for the case of continuous asymptotics. In a forthcoming paper we construct parametrices of elliptic elements. Parametrices will be elements of a corresponding version of edge algebra. This algebra consists of 3×3 block matrix operators, where the lower right 2×2 corners belong to the (pseudo-differential) algebra of transmission problems on the boundary Y with the interface Z. Restricting that algebra to, say, the $+$-side Y_+ of the boundary, the transmission algebra may be regarded as a generalisation of the algebra of pseudo-differential boundary value problems on Y_+ (where Z is the boundary), cf.

Harutjunjan, Schulze, Witt [16], or Schulze and Seiler [41].

1. Mixed problems for differential operators

1.1. Basic constructions

Mixed boundary value problems for differential operators are formulated as follows:

Let X be a compact C^∞ manifold with boundary Y, and suppose that Y is subdivided into C^∞ manifolds $Y_\pm$ with common boundary Z, i.e., $Y = Y_+ \cup Y_-$, $Z = Y_+ \cap Y_-$. On X we consider an equation

$$Au = f \tag{1}$$

with an elliptic differential operator A of order m and elliptic boundary conditions

$$T_\pm u = g_\pm \quad \text{on} \quad \operatorname{int} Y_\pm, \tag{2}$$

where $T_\pm$ are assumed to be of the form $\mathrm{r}^\pm B_\pm$ with differential operators $B_\pm$ with smooth coefficients in a neighbourhood of $Y_\pm$, where $\mathrm{r}^\pm$ denotes the operator of restriction to $\operatorname{int} Y_\pm = Y_\pm \setminus Z$. More precisely, $B_\pm$ are vectors

$$B_\pm = (B_\pm^1, \ldots, B_\pm^N) \tag{3}$$

of differential operators of order $m_\pm^j$, $j = 1, \ldots, N$. The manifold X can be regarded as a manifold with boundary that has an edge Z. According to the general ideas from the edge operator calculus, cf. [40] or [18], we then pass to the associated stretched manifold $\mathbb{X}$ and to corresponding weighted Sobolev spaces $\mathcal{W}^{s,\gamma}(\mathbb{X})$, cf. the constructions in Section 2.3 below. Similarly, the manifolds $Y_\pm$ with smooth boundary Z will be regarded as manifolds with edge Z (these are the same as their stretched versions). We then also have the spaces $\mathcal{W}^{s,\gamma}(\operatorname{int} Y_\pm)$. Our mixed boundary value problem then represents an operator

$$\mathcal{A} = \begin{pmatrix} A \\ T_+ \\ T_- \end{pmatrix} : \mathcal{W}^{s,\gamma}(\mathbb{X}) \longrightarrow \begin{array}{c} \mathcal{W}^{s-m,\gamma-m}(\mathbb{X}) \\ \oplus \\ \oplus_{j=1}^N \mathcal{W}^{s-m_+^j-\frac{1}{2},\gamma-m_+^j-\frac{1}{2}}(\operatorname{int} Y_+) \\ \oplus \\ \oplus_{j=1}^N \mathcal{W}^{s-m_-^j-\frac{1}{2},\gamma-m_-^j-\frac{1}{2}}(\operatorname{int} Y_-) \end{array}, \tag{4}$$

that is continuous for all $s \in \mathbb{R}$ (here, in the case of differential operators, also for all $\gamma \in \mathbb{R}$).

Let us now pass to the local description in polar coordinates and define the symbol hierarchy of the operator $\mathcal{A}$. Let us represent X in a neighbourhood U of a point of Z in local coordinates as

$$\overline{\mathbb{R}}_+^2 \times \Omega \ni \{(z, x_{n-1}, x_n) \in \mathbb{R}^n : x_n \geq 0,\ x_{n-1} \in \mathbb{R},\ z \in \Omega\},$$

where $\Omega \subseteq \mathbb{R}^{n-2}$ is an open set. More precisely, U is chosen in such a way that there is a chart $\chi : U \to \overline{\mathbb{R}}^2_+ \times \Omega$, where $\chi : U \cap Z \to \Omega$ is a chart on Z, and $\overline{\mathbb{R}}^2_+$ represents locally the normal half-plane to Z, generated by the inner normal $\overline{\mathbb{R}}_+$ and the normal to Z tangential to Y. Let $(r, \varphi) \in \mathbb{R}_+ \times [0, \pi]$ be polar coordinates in $\overline{\mathbb{R}}^2_+ \setminus \{0\}$. Then, the given differential operator

$$A = \sum_{|\alpha| \le m} a_\alpha(x) D_x^\alpha$$

with smooth coefficients in a neighbourhood of $\overline{\mathbb{R}}^2_+ \times \Omega$ takes the form

$$A = r^{-m} \sum_{k+|\beta| \le m} a_{k\beta}(r, z)(-r\frac{\partial}{\partial r})^k (rD_z)^\beta \tag{5}$$

with operator-valued coefficients $a_{k\beta}(r, z) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{m-(k+|\beta|)}([0, \pi]))$. Similarly, the operators $B^j_\pm$ that are in local coordinates given by

$$B^j_\pm = \sum_{|\alpha| \le m^j_\pm} b^j_{\pm,\alpha}(x) D_x^\alpha$$

with smooth coefficients in a neighbourhood of $\overline{\mathbb{R}}_\pm \times \Omega$ take the form

$$B^j_\pm = r^{-m^j_\pm} \sum_{k+|\beta| \le m^j_\pm} b^j_{\pm,k\beta}(r, z)(-r\frac{\partial}{\partial r})^k (rD_z)^\beta \tag{6}$$

with operator-valued coefficients $b^j_{\pm,k\beta} \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{m^j_\pm-(k+|\beta|)}([0, \pi]))$, and then

$$T^j_\pm = \mathrm{r}^\pm r^{-m^j_\pm} \sum_{k+|\beta| \le m^j_\pm} b^j_{\pm,k\beta}(r, z)(-r\frac{\partial}{\partial r})^k (rD_z)^\beta. \tag{7}$$

We now establish some symbol structures that are connected with the operators $(A,\ T_+,\ T_-)$. First, in coordinates $x \in \mathbb{R}^n$ with covariables ξ we have the respective homogeneous principal symbols of A and $B^j_\pm$ of orders m and $m^j_\pm$, respectively, namely

$$\sigma^m_\psi(A)(x, \xi) \quad \text{and} \quad \sigma^{m^j_\pm}_\psi(B^j_\pm)(x, \xi), \tag{8}$$

$\xi \ne 0$. These induce corresponding boundary symbols

$$\sigma^m_\partial(A)(y, \eta) := \sigma^m_\psi(A)(0, y, D_t, \eta) \tag{9}$$

for $\eta \ne 0$, where $x := (t, y)$ with $t := x_n,\ y := (x_{n-1}, z)$, and the covariable ξ splits into (τ, η). Similarly, we set

$$\sigma^{m^j_\pm+\frac{1}{2}}_\partial(T^j_\pm)(y, \eta) := \mathrm{r}^\pm \sigma^{m^j_\pm}_\psi(B^j_\pm)(0, y, D_t, \eta)$$

for $\eta \neq 0$ and $y = (z, x_{n-1})$, $x_{n-1} > 0$, $x_{n-1} < 0$, respectively (according to $\pm$ at the operators). This gives us the homogeneous principal boundary symbol $\sigma_\partial(\mathcal{A}) := (\sigma_{\partial,+}(\mathcal{A}), \sigma_{\partial,-}(\mathcal{A}))$ of $\mathcal{A}$ in local coordinates, namely

$$\sigma_{\partial,\pm}(\mathcal{A})(y,\eta) := \begin{pmatrix} \sigma_\partial^m(A)(y,\eta) \\ (\sigma_\partial^{m_\pm^j+\frac{1}{2}}(T_\pm^j)(y,\eta))_{j=1,\dots,N} \end{pmatrix} \tag{10}$$

for $x_{n-1} > 0$, $x_{n-1} < 0$, respectively, $z \in \Omega$ and $\eta \neq 0$. These are families of continuous operators

$$\sigma_{\partial,\pm}(\mathcal{A})(y,\eta) : H^s(\mathbb{R}_+) \longrightarrow \begin{array}{c} H^{s-m}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^N \end{array} \tag{11}$$

for all $s \in \mathbb{R}$, where $s - m_\pm^j - \frac{1}{2} > 0$ for all j. The specific choice of s will be unessential, in fact, it suffices to take s sufficiently large. Instead of (11) we also may consider the families continuous operators

$$\sigma_{\partial,\pm}(\mathcal{A})(y,\eta) : \mathcal{S}(\overline{\mathbb{R}}_+) \longrightarrow \begin{array}{c} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^N \end{array} . \tag{12}$$

Writing $(\kappa_\lambda u)(t) := \lambda^{\frac{1}{2}} u(\lambda t),\ \lambda \in \mathbb{R}_+$, we have

$$\sigma_\partial^m(A)(y,\lambda\eta) = \lambda^m \kappa_\lambda \sigma_\partial^m(A)(y,\eta)\kappa_\lambda^{-1} \tag{13}$$

and

$$\sigma_\partial^{m_\pm^j+\frac{1}{2}}(T_\pm^j)(y,\lambda\eta) = \lambda^{m_\pm^j+\frac{1}{2}} \sigma_\partial^{m_\pm^j+\frac{1}{2}}(T_\pm^j)(y,\eta)\kappa_\lambda^{-1} \tag{14}$$

for all $\lambda \in \mathbb{R}_+$. Our basic assumption in mixed problems is the ellipticity of A in the standard sense, i.e., $\sigma_\psi^m(A)(x,\xi) \neq 0$ for all x and all $\xi \neq 0$, together with the ellipticity of the boundary conditions on $Y_\pm$, i.e., that the operators (11) (or, equivalently, (12)) define isomorphisms for all y on the respective side of Y and for all $\eta \neq 0$. Note that although we want to control symbols and weighted distributions on $\operatorname{int} Y_\pm$, the coefficients in the boundary conditions are (by assumption) smooth up to Z and the isomorphisms (11) (or (12)) are required including $z \in Z$.

EXAMPLE 1. If $A = \Delta$ is the Laplace operator, an example for mixed elliptic boundary conditions is the case Dirichlet conditions on Y_+, Neumann conditions on Y_-. We also may impose oblique derivative conditions on both sides, where the coefficients have a jump on Z (with smoothness from the respective sides up to Z).

We now formulate so-called *edge symbols*, associated with the operators (5), (6), where Z is regarded as an edge. To this end we set $I := [0, \pi]$ and form the open stretched cone $I^\wedge := \mathbb{R}_+ \times I$ with base I. We then have the weighted Sobolev spaces $\mathcal{K}^{s,\gamma}(I^\wedge)$ as well as $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$, $s, \gamma \in \mathbb{R}$, that are defined as follows:

We first have the weighted Sobolev spaces $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ and $\mathcal{H}^{s,\gamma}(X^\wedge)$, respectively, where $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ and $\mathcal{H}^{s,\gamma}(X^\wedge)$ are as usual, cf. [39],[40]. Here, X is a closed compact C^∞- manifold and $X^\wedge = \mathbb{R}_+ \times X$. In particular, for $X = S^1$ (the unit circle in $\mathbb{R}^2$) we identify $\mathbb{R}^2 \setminus \{0\}$ with $\mathbb{R}_+ \times S^1$ and write $\mathcal{H}^{s,\gamma}(\mathbb{R}^2 \setminus \{0\}) = \mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times S^1)$. Identifying $\operatorname{int} I = (0, \pi)$ with $S^1_+ = S^1 \cap \mathbb{R}^2_+$, $\mathbb{R}^2_+ = \{(x_{n-1}, x_n) : x_n > 0\}$, we then get

$$\mathcal{H}^{s,\gamma}(I^\wedge) := \mathcal{H}^{s,\gamma}(\mathbb{R}^2_+) = \{u\,|_{\mathbb{R}^2_+} : u \in \mathcal{H}^{s,\gamma}(\mathbb{R}^2 \setminus \{0\})\}.$$

Moreover,

$$\mathcal{K}^{s,\gamma}(\mathbb{R}_+) := \{\omega u + (1-\omega)\upsilon : u \in \mathcal{H}^{s,\gamma}(\mathbb{R}_+),\ \upsilon \in H^s(\mathbb{R}_+)\}, \tag{15}$$

where $\omega(r)$ is any cut-off function (thoughout this paper a *cut-off function* is a non-negative function $\omega(r) \in C_0^\infty(\overline{\mathbb{R}}_+)$ such that $\omega(r) \equiv 1$ in a neighbourhood of $r = 0$), and

$$\begin{aligned}\mathcal{K}^{s,\gamma}(I^\wedge) &:= \mathcal{K}^{s,\gamma}(\mathbb{R}^2_+) := \\ &\{\omega u + (1-\omega)\upsilon : u \in \mathcal{H}^{s,\gamma}(\mathbb{R}^2_+),\ \upsilon \in H^s(\mathbb{R}^2_+)\}.\end{aligned} \tag{16}$$

Similarly, we can form the space

$$\mathcal{K}^{s,\gamma}(\mathbb{R}_+ \times S^1) := \{\omega u + (1-\omega)\upsilon : u \in \mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times S^1), \upsilon \in H^s(\mathbb{R}^2)\}. \tag{17}$$

Notice that then $u \to u\,|_{\mathbb{R}^2_+}$ defines a continuous operator $\mathcal{K}^{s,\gamma}(\mathbb{R}_+ \times S^1) \to \mathcal{K}^{s,\gamma}(I^\wedge)$ and $u(r,\varphi) \to u(r,\varphi_0)$ for fixed $\varphi_0 \in S^1$ a continuous operator $\mathcal{K}^{s,\gamma}(\mathbb{R}_+ \times S^1) \to \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+)$ for $s > \frac{1}{2}$, $\gamma \in \mathbb{R}$.

On the spaces $\mathcal{K}^{s,\gamma}(I^\wedge)$ and on $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ we consider groups of isomorphisms $\{\kappa^\wedge_\lambda\}_{\lambda \in \mathbb{R}_+}$ and $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$, respectively, namely

$$(\kappa^\wedge_\lambda u)(r,\varphi) := \lambda\, u(\lambda r, \varphi), \quad u \in \mathcal{K}^{s,\gamma}(I^\wedge), \tag{18}$$

$$(\kappa_\lambda \upsilon)(r) := \lambda^{\frac{1}{2}} \upsilon(\lambda r), \quad \upsilon \in \mathcal{K}^{s,\gamma}(\mathbb{R}_+). \tag{19}$$

We set

$$\sigma^m_\wedge(A)(z,\zeta) := r^{-m} \sum_{k+|\beta| \le m} a_{k\beta}(0,z)\Big(-r\frac{\partial}{\partial r}\Big)^k (r\zeta)^\beta, \tag{20}$$

$(z,\zeta) \in \Omega \times (\mathbb{R}^{n-2} \setminus \{0\})$, regarded as a family of continuous operators

$$\sigma^m_\wedge(A)(z,\zeta) : \mathcal{K}^{s,\gamma}(I^\wedge) \longrightarrow \mathcal{K}^{s-m,\gamma-m}(I^\wedge),$$

for any fixed $\gamma \in \mathbb{R}$. Also here $s \in \mathbb{R}$ is taken sufficiently large (precise conditions will be given below). Moreover, we set

$$\sigma^{m^j_\pm + \frac{1}{2}}_\wedge(T^j_\pm)(z,\zeta) := \mathrm{r}^\pm r^{-m^j_\pm} \sum_{k+|\beta| \le m^j_\pm} b^j_{\pm,k\beta}(0,z)\Big(-r\frac{\partial}{\partial r}\Big)^k (r\zeta)^\beta,$$

(here, $\mathrm{r}^{\pm}$ denotes the restriction operator to $\mathbb{R}_+$, the corresponding boundary component of $I^\wedge$, where $\{0\} \times \mathbb{R}_+$ corresponds to the $+$, $\{\pi\} \times \mathbb{R}_+$ to the $-$ sign) $(z, \zeta) \in \Omega \times (\mathbb{R}^{n-2} \setminus \{0\})$, regarded as families of continuous operators

$$\sigma_\wedge^{m_\pm^j + \frac{1}{2}}(T_\pm^j)(z, \zeta) : \mathcal{K}^{s,\gamma}(I^\wedge) \longrightarrow \mathcal{K}^{s - m_\pm^j - \frac{1}{2}, \gamma - m_\pm^j - \frac{1}{2}}(\mathbb{R}_+),$$

for $s - m_\pm^j - \frac{1}{2} > 0$. Similarly to (13), (14) we have

$$\sigma_\wedge^m(A)(z, \lambda\zeta) = \lambda^m \kappa_\lambda^\wedge \sigma_\wedge^m(A)(z, \zeta)(\kappa_\lambda^\wedge)^{-1} \tag{21}$$

and

$$\sigma_\wedge^{m_\pm^j + \frac{1}{2}}(T_\pm^j)(z, \lambda\zeta) = \lambda^{m_\pm^j + \frac{1}{2}} \kappa_\lambda \sigma_\wedge^{m_\pm^j + \frac{1}{2}}(T_\pm^j)(z, \zeta)(\kappa_\lambda^\wedge)^{-1} \tag{22}$$

for all $\lambda \in \mathbb{R}_+$. The operator family

$$\sigma_\wedge(\mathcal{A})(z, \zeta) := \begin{pmatrix} \sigma_\wedge^m(A)(z, \zeta) \\ (\sigma_\wedge^{m_\pm^j + \frac{1}{2}}(T_\pm^j)(z, \zeta))_{j=1,\dots,N} \end{pmatrix} \tag{23}$$

represents a parameter-dependent boundary problem on the infinite (stretched) cone $I^\wedge$, where $\zeta \in \mathbb{R}^{n-2} \setminus \{0\}$ is the parameter and $z \in \Omega$ an additional variable. Writing

$$\sigma_\wedge(\mathcal{A})(z, \zeta) : \mathcal{K}^{s,\gamma}(I^\wedge) \longrightarrow \begin{matrix} \mathcal{K}^{s-m, \gamma-m}(I^\wedge) \\ \oplus \\ \oplus_{j=1}^N \mathcal{K}^{s - m_+^j - \frac{1}{2}, \gamma - m_+^j - \frac{1}{2}}(\mathbb{R}_+) \\ \oplus \\ \oplus_{j=1}^N \mathcal{K}^{s - m_-^j - \frac{1}{2}, \gamma - m_-^j - \frac{1}{2}}(\mathbb{R}_+) \end{matrix} \tag{24}$$

we shall choose γ in such a way that (24) is a family of Fredholm operators. To express homogeneity of the operator function (24) in the sense of (23) we can also write

$$\sigma_\wedge(\mathcal{A})(z, \lambda\zeta) = \lambda^m \tilde{\kappa}_\lambda^\wedge \sigma_\wedge(\mathcal{A})(z, \zeta)(\kappa_\lambda^\wedge)^{-1}, \tag{25}$$

where $\{\kappa_\lambda^\wedge\}_{\lambda \in \mathbb{R}_+}$ is as before, while $\{\tilde{\kappa}_\lambda^\wedge\}_{\lambda \in \mathbb{R}_+}$ is a diagonal block matrix of isomorphisms, acting on corresponding direct sums of spaces (as they occur on the right hand side of (24)), namely,

$$\tilde{\kappa}_\lambda^\wedge := \begin{pmatrix} \kappa_\lambda^\wedge & & 0 \\ & \mathrm{diag}(\lambda^{m_+^j + \frac{1}{2} - m} \kappa_\lambda)_{j=1,\dots,N} & \\ 0 & & \mathrm{diag}(\lambda^{m_-^j + \frac{1}{2} - m} \kappa_\lambda)_{j=1,\dots,N} \end{pmatrix}. \tag{26}$$

1.2. Conormal symbols

The choice of γ in (24) depends on the so-called *conormal symbol*, namely, the family of maps

$$\sigma_M \sigma_\wedge(\mathcal{A})(z, w) := \begin{pmatrix} \sigma_M \sigma_\wedge^m(A)(z, w) \\ (\sigma_M \sigma_\wedge^{m_\pm^j + \frac{1}{2}}(T_\pm^j)(z, w))_{j=1,\dots,N} \end{pmatrix}, \tag{27}$$

where

$$\sigma_M \sigma_\wedge^m(A)(z,w) := \sum_{k=0}^{m} a_{k0}(0,z)w^k, \tag{28}$$

$$\sigma_M \sigma_\wedge^{m_\pm^j+\frac{1}{2}}(T_\pm^j)(z,w) := \mathrm{r}^\pm \sum_{k=0}^{m_\pm^j} b_{\pm,k0}^j(0,z)w^k, \tag{29}$$

$w \in \mathbb{C}$. We then have

$$\sigma_M \sigma_\wedge(\mathcal{A})(z,w) : H^s(I) \longrightarrow \begin{array}{c} H^{s-m}(I) \\ \oplus \\ \mathbb{C}^N \oplus \mathbb{C}^N \end{array}. \tag{30}$$

Note that (30) is a holomorphic operator function in w, smoothly dependent on $z \in \Omega$.

PROPOSITION 1. *Under the above-mentioned ellipticity conditions on the mixed boundary value problem, given by the operators* (A, T_-, T_+), *the operators* (30) *represent* (*for every fixed* $z \in \Omega$) *a holomorphic* (*in* $w \in \mathbb{C}$) *family of Fredholm operators for every* $s \in \mathbb{R}$, *where* $s - m > -\frac{1}{2}$ *and* $s - m_\pm^j - \frac{1}{2} > 0$ *for all* j. *Moreover, for every compact subset* $K \subset \Omega$ *and every* $c \le c'$ *there is an* $M > 0$ *such that the operators* (30) *are isomorphisms for all* $z \in K$, $c \le \mathrm{Re}\, w \le c'$ *and* $|\mathrm{Im}\, w| \ge M$.

Proof. First, the operators (30) are C^∞ in $(z,w) \in \Omega \times \mathbb{C}$ and holomorphic in w. They are elliptic as boundary value problems on the interval I and parameter-dependent elliptic with $\mathrm{Im}\, w$ as parameter, for every $\beta = \mathrm{Re}\, w$. Ellipticity entails the Fredholm property of (30), while parameter-dependent ellipticity gives rise to isomorphisms between the respective spaces for $|\mathrm{Im}\, w|$ sufficiently large. Because of the smoothness of coefficients in $z \in \Omega$ and $\beta \in \mathbb{R}$, for every compact set $\tilde{K} \subset \Omega \times \mathbb{R}$ the operators (30) are isomorphisms for all $|\mathrm{Im}\, w| \ge M$ for a suitable choice of M. This is a consequence of parameter-dependent ellipticity, cf. [18]. □

PROPOSITION 2. *For every fixed* $z \in \Omega$ *there exists a countable set* $D(z) \subset \mathbb{C}$, *where* $D(z) \cap \{w : c \le \mathrm{Re}\, w \le c'\}$ *is finite for every* $c \le c'$, *such that the operators* (30) *are isomorphisms for all* $w \in \mathbb{C} \setminus D(z)$ *and for* $s \in \mathbb{R}$, *where* $s - m > -\frac{1}{2}$ *and* $s - m_\pm^j - \frac{1}{2} > 0$.

This is a well-known result on holomorphic families of Fredholm operators, cf. [40, Theorem 1.2.33].

THEOREM 1. *Let* $\mathcal{A} = \begin{pmatrix} A \\ T_+ \\ T_- \end{pmatrix}$ *be a mixed problem in* $\overline{\mathbb{R}}_+^2 \times \Omega$ *that is elliptic with respect to* σ_ψ^m *and* $\sigma_{\partial,\pm}$ (*i.e.,* $\sigma_\psi^m(\mathcal{A})(x,\xi) := \sigma_\psi^m(A)(x,\xi) \neq 0$ *for all* x *and* $\xi \neq 0$ *and* $\sigma_{\partial,\pm}(\mathcal{A})(y,\eta)$ *defines bijective operators* (11) *or* (12) *for all* y *and* $\eta \neq 0$).

Then the edge symbol (24) *is a Fredholm operator for a point* $z \in \Omega$ *and arbitrary* $\zeta \neq 0$ *if and only if* $1-\gamma \notin \{\mathrm{Re}\, w : w \in D(z)\}$ (*with the set* $D(z)$ *from Proposition* 2) *for all* $s-m > -\frac{1}{2}$, $s-m_{\pm}^{j}-\frac{1}{2} > 0$.

Proof. We shall show the Fredholm property under the required ellipticity conditions on the symbols which is the essential point here; the converse will be dropped. The operator family (24) belongs to a Douglis-Nirenberg analogue of the cone algebra of boundary value problems on $I^{\wedge}$, cf. [29],[30], where the ellipticity with respect to σ_{ψ}^{m}, $\sigma_{\partial,\pm}$ and $\sigma_M\sigma_{\wedge}$ guarantees the existence of a parametrix. Moreover, $I^{\wedge}$ is a manifold with conical exit to infinity ($r \to \infty$) and for $\zeta \neq 0$ the exit symbols are also elliptic. This is a similar effect as in the boundaryless case, cf. [40, Theorem 3.5.1]. Near infinity we can apply a Douglis-Nirenberg analogue of the parametrix construction from [18, Chapter 3]. This gives us altogether a parametrix $\sigma_{\wedge}(\mathcal{P})(z,\zeta)$ of (24) globally on $I^{\wedge}$, where $\sigma_{\wedge}(\mathcal{A})\sigma_{\wedge}(\mathcal{P}) - \mathrm{id}$ as well as $\sigma_{\wedge}(\mathcal{P})\sigma_{\wedge}(\mathcal{A}) - \mathrm{id}$ are compact in the respective Sobolev spaces. This entails the Fredholm property. □

REMARK 1. If (24) is a Fredholm operator, we have

$$\dim \ker \sigma_{\wedge}(\mathcal{A})(z,\zeta) \;=\; \dim \ker \sigma_{\wedge}(\mathcal{A})(z, \frac{\zeta}{|\zeta|}),$$

$$\dim \operatorname{coker} \sigma_{\wedge}(\mathcal{A})(z,\zeta) \;=\; \dim \operatorname{coker} \sigma_{\wedge}(\mathcal{A})(z, \frac{\zeta}{|\zeta|}).$$

This is a direct consequence of the relation (25). In fact, we have

$$\sigma_{\wedge}(\mathcal{A})(z, \frac{\zeta}{|\zeta|}) \;=\; |\zeta|^{-m}(\tilde{\kappa}^{\wedge}_{|\zeta|})^{-1}\sigma_{\wedge}(\mathcal{A})(z,\zeta)\kappa^{\wedge}_{|\zeta|}$$

for all $\zeta \neq 0$.

1.3. Examples

EXAMPLE 2. Let $X = \overline{\mathbb{R}}^{n}_{+} = \{x : x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n; x_n \geq 0\}$, $Y_+ = \{x : x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n; x_n = 0, x_{n-1} \geq 0\}$, $Y_- = \{x : x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n; x_n = 0, x_{n-1} \leq 0\}$, $Z = Y_+ \cap Y_- = \mathbb{R}^{n-2} = \{x : x = (x_1, x_2, \ldots, x_{n-2}, 0, 0)\}$. Let us consider the Zaremba problem $\mathcal{A} = \begin{pmatrix} \Delta \\ T_+ \\ T_- \end{pmatrix}$ for the Laplacian Δ where $T_+u = \mathrm{r}^+u$, $T_-u = -\mathrm{r}^-\frac{\partial u}{\partial x_n}$.In polar coordinates $(r,\varphi) \in \mathbb{R}_+ \times [0,\pi]$ (with respect to x_{n-1}, x_n) the entries of $\mathcal{A}$ take the form

$$\begin{aligned} \Delta &= r^{-2}((-r\frac{\partial}{\partial r})^2 + \frac{\partial^2}{\partial \varphi^2} - r^2(D^2_{x_1} + \ldots + D^2_{x_{n-2}})), \\ T_+u &= u\,|_{\varphi=0},\;\; T_-u = \frac{1}{r}\frac{\partial u}{\partial \varphi}\,|_{\varphi=\pi}\;. \end{aligned}$$

We then have

$$\sigma_M \sigma_\wedge^2(\Delta)(w) = \frac{\partial^2}{\partial\varphi^2} + w^2 \ : \ H^s(I) \to \ H^{s-2}(I)$$

and

$$(31) \qquad \sigma_M \sigma_\wedge(\mathcal{A})(w) = \begin{pmatrix} \sigma_M \sigma_\wedge^2(\Delta)(w) \\ \sigma_M \sigma_\wedge^{\frac{1}{2}}(T_+) \\ \sigma_M \sigma_\wedge^{\frac{3}{2}}(T_-) \end{pmatrix} : H^s(I) \to \begin{matrix} H^{s-2}(I) \\ \oplus \\ \mathbb{C} \oplus \mathbb{C} \end{matrix} ,$$

where $\sigma_M \sigma_\wedge^{\frac{1}{2}}(T_+)u = u \mid_{\varphi=0}, \ \sigma_M \sigma_\wedge^{\frac{3}{2}}(T_-)u = \frac{\partial u}{\partial \varphi} \mid_{\varphi=\pi}$. A simple argument gives us

$$\ker(\sigma_M \sigma_\wedge^2(\Delta)(w)) = \{c_1 e^{iw\varphi} + c_2 e^{-iw\varphi} \ : \ c_1, c_2 \in \mathbb{C}\}.$$

Now (31) is an isomorphism if and only if $w \in \mathbb{C}$ satisfies the condition $w \notin \{n + \frac{1}{2} : n \in \mathbb{Z}\}$. Hence, Theorem 1 tells us that

$$(32) \qquad \sigma_\wedge(\mathcal{A})(\zeta) = \begin{pmatrix} \sigma_\wedge^2(\Delta)(\zeta) \\ \sigma_\wedge^{\frac{1}{2}}(T_+)(\zeta) \\ \sigma_\wedge^{\frac{3}{2}}(T_-)(\zeta) \end{pmatrix} : \mathcal{K}^{s,\gamma}(I^\wedge) \to \begin{matrix} \mathcal{K}^{s-2,\gamma-2}(I^\wedge) \\ \oplus \\ \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+) \\ \oplus \\ \mathcal{K}^{s-\frac{3}{2},\gamma-\frac{3}{2}}(\mathbb{R}_+) \end{matrix} ,$$

where $\sigma_\wedge^2(\Delta)(\zeta) = r^{-2}(\frac{\partial^2}{\partial\varphi^2} + (-r\frac{\partial}{\partial r})^2 - r^2|\zeta|^2), \sigma_\wedge^{\frac{1}{2}}(T_+)u = u \mid_{\varphi=0}, \sigma_\wedge^{\frac{3}{2}}(T_-)u = \frac{1}{r}\frac{\partial u}{\partial \varphi} \mid_{\varphi=\pi}$, is a Fredholm operator for any $s \in \mathbb{R}$, $s > \frac{3}{2}$, if and only if $\gamma \notin \{n + \frac{1}{2} : n \in \mathbb{Z}\}$.

EXAMPLE 3. Let X, Y_+, Y_-, Z be as in Example *2*. For the Laplacian Δ we consider a mixed problem $\mathcal{A} = \begin{pmatrix} \Delta \\ T_+ \\ T_- \end{pmatrix}$ with $T_+ = r^+ B_+, \ T_- = r^- B_-$, where

$$B_+ = \sum_{i=1}^{n-2} \alpha_i D_{x_i} + \alpha \ D_{x_{n-1}} + \gamma \ D_{x_n},$$

$$B_- = \sum_{i=1}^{n-2} \beta_i D_{x_i} + \beta \ D_{x_{n-1}} + \delta \ D_{x_n}.$$

The coefficients $\alpha, \beta, \gamma, \delta, \alpha_i, \beta_i$ are functions of $z = (x_1, \ldots, x_{n-2}, 0, 0) \in Z$, and we assume that γ, δ are nowhere vanishing (the operators $T_\pm$ satisfy the Shapiro-

Lopatinskij condition). Similarly to Example 2 we get

$$\begin{aligned}\Delta &= r^{-2}((-r\frac{\partial}{\partial r})^2 + \frac{\partial^2}{\partial\varphi^2} - r^2(D_{x_1}^2 + \ldots + D_{x_{n-2}}^2)),\\ T_+u &= r^{-1}(\sum_{i=1}^{n-2}\alpha_i r D_{x_i} + \frac{\alpha}{i} r\frac{\partial}{\partial r} + \frac{\gamma}{i}\frac{\partial}{\partial\varphi})u\,|_{\varphi=0},\\ T_-u &= r^{-1}(\sum_{i=1}^{n-2}\beta_i r D_{x_i} - \frac{\beta}{i} r\frac{\partial}{\partial r} - \frac{\delta}{i}\frac{\partial}{\partial\varphi})u\,|_{\varphi=\pi}\ .\end{aligned}$$

Then

$$\sigma_\wedge(\mathcal{A})(\zeta) = \begin{pmatrix}\sigma_\wedge^2(\Delta)(\zeta)\\ \sigma_\wedge^{\frac{3}{2}}(T_+)(\zeta)\\ \sigma_\wedge^{\frac{3}{2}}(T_-)(\zeta)\end{pmatrix} : \mathcal{K}^{s,\gamma}(I^\wedge) \to \begin{array}{c}\mathcal{K}^{s-2,\gamma-2}(I^\wedge)\\ \oplus\\ \mathcal{K}^{s-\frac{3}{2},\gamma-\frac{3}{2}}(\mathbb{R}_+)\\ \oplus\\ \mathcal{K}^{s-\frac{3}{2},\gamma-\frac{3}{2}}(\mathbb{R}_+)\end{array}, \tag{33}$$

where $\sigma_\wedge^2(\Delta)(\zeta) = r^{-2}(\frac{\partial^2}{\partial\varphi^2} + (-r\frac{\partial}{\partial r})^2 - r^2|\zeta|^2)$, $\sigma_\wedge^{\frac{3}{2}}(T_+)(\zeta) = r^{-1}(\frac{1}{i}(\gamma\ \cos\varphi - \alpha\ \sin\varphi)\frac{\partial}{\partial\varphi} - \frac{1}{i}(\gamma\ \sin\varphi + \alpha\ \cos\varphi)(-r\frac{\partial}{\partial r}) + r(\alpha_1\zeta_1 + \alpha_2\zeta_2 + \ldots + \alpha_{n-2}\zeta_{n-2}))$, $\sigma_\wedge^{\frac{3}{2}}(T_-)(\zeta) = r^{-1}(\frac{1}{i}(\delta\cos\varphi - \beta\ \sin\varphi)\frac{\partial}{\partial\varphi} - \frac{1}{i}(\delta\ \sin\varphi + \beta\ \cos\varphi)(-r\frac{\partial}{\partial r}) + r(\beta_1\zeta_1 + \beta_2\zeta_2 + \ldots + \beta_{n-2}\zeta_{n-2}))$. For the conormal symbol of (33) we have

$$\sigma_M\sigma_\wedge(\mathcal{A})(z,w) = \begin{pmatrix}\frac{\partial^2}{\partial\varphi^2} + w^2\\ \frac{1}{i}(\gamma\frac{\partial}{\partial\varphi} - \alpha\, w)\,|_{\varphi=0}\\ \frac{1}{i}(\beta\, w - \delta\frac{\partial}{\partial\varphi})\,|_{\varphi=\pi}\end{pmatrix} : H^s(I) \to \begin{array}{c}H^{s-2}(I)\\ \oplus\\ \mathbb{C}\oplus\mathbb{C}\end{array}. \tag{34}$$

The operator *(33)* is Fredholm for all $s \in \mathbb{R}, s > \frac{3}{2}$ and all $\zeta \neq 0$ if and only if $1-\gamma \notin \{\mathrm{Re}\, w : 2iw^2(\cos w\pi(\alpha\delta - \gamma\beta) - \sin w\pi(\alpha\beta + \gamma\delta)) = 0\}$.

2. Calculus in weighted Sobolev spaces

2.1. Operator-valued symbols and abstract edge Sobolev spaces

This section contains some necessary material on operator-valued symbols and associated Sobolev spaces based on spaces with strongly continuous group actions.

If a Hilbert space E is equipped with a strongly continuous group of isomorphisms $\kappa_\lambda : E \to E, \lambda \in \mathbb{R}_+$, where $\kappa_\lambda\kappa_{\lambda'} = \kappa_{\lambda\lambda'}$ for all $\lambda, \lambda' \in \mathbb{R}_+$, we say that E is endowed with a group action. More generally, if E is a Fréchet space, written as a projective limit $\varprojlim_{j\in\mathbb{N}} E_j$ of Hilbert spaces E_j, $j \in \mathbb{N}$, with continuous embeddings $E_{j+1} \hookrightarrow E_j$ for all j, and if E_0 is endowed with a group action $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ that restricts to a group action on E_j for every $j \in \mathbb{N}$, then E is said to be equipped with a group action. If E and $\tilde{E}$ are Hilbert spaces endowed with group actions $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively, $S^\mu(U\times\mathbb{R}^q;E,\tilde{E})$ for an open set $U\subseteq\mathbb{R}^p$ denotes the set of all $a(z,\zeta)\in C^\infty(U\times\mathbb{R}^q,\mathcal{L}(E,\tilde{E}))$ such that

$$(35)\qquad ||\tilde{\kappa}^{-1}_{\langle\zeta\rangle}\{D_z^\alpha D_\zeta^\beta a(z,\zeta)\}\kappa_{\langle\zeta\rangle}||_{\mathcal{L}(E,\tilde{E})}\le c\langle\zeta\rangle^{\mu-|\beta|}$$

for all $\alpha\in\mathbb{N}^p,\beta\in\mathbb{N}^q$ and $z\in K$ for arbitrary $K\subset\subset U$, $\zeta\in\mathbb{R}^q$, with constants $c=c(\alpha,\beta,K)>0$. The space $S^\mu(U\times\mathbb{R}^q;E,\tilde{E})$ is Fréchet in the semi-norm system, given by the best constants c in the symbol estimates (35). Let $S^{(\mu)}(U\times(\mathbb{R}^q\setminus\{0\});E,\tilde{E})$ be the space of all $f(z,\zeta)\in C^\infty(U\times(\mathbb{R}^q\setminus\{0\}),\mathcal{L}(E,\tilde{E}))$ such that $f(z,\lambda\zeta)=\lambda^\mu\tilde{\kappa}_\lambda f(z,\zeta)\kappa_\lambda^{-1}$ for all $\lambda\in\mathbb{R}_+$, $(z,\zeta)\in U\times(\mathbb{R}^q\setminus\{0\})$. Then, if $\chi(\zeta)$ is any excision function in $\mathbb{R}^q$, we have

$$\chi(\zeta)S^{(\mu)}(U\times(\mathbb{R}^q\setminus\{0\});E,\tilde{E})\subset S^\mu(U\times\mathbb{R}^q;E,\tilde{E}).$$

Now $S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^q;E,\tilde{E})$ (the space of *classical symbols*) is defined to be the subspace of all $a(z,\zeta)\in S^\mu(U\times\mathbb{R}^q;E,\tilde{E})$ such that there are elements $a_{(\mu-j)}(z,\zeta)\in S^{(\mu-j)}(U\times(\mathbb{R}^q\setminus\{0\});E,\tilde{E})$, $j\in\mathbb{N}$, where

$$(36)\qquad a(z,\zeta)-\sum_{j=0}^N\chi(\zeta)a_{(\mu-j)}(z,\zeta)\in S^{\mu-(N+1)}(U\times\mathbb{R}^q;E,\tilde{E})$$

for all $N\in\mathbb{N}$. The semi-norms in $S^{(\mu-j)}(U\times(\mathbb{R}^q\setminus\{0\});E,\tilde{E})$ from the unique $a_{(\mu-j)}(z,\zeta)$, $j\in\mathbb{N}$, as well as those from the remainders (36) in $S^{\mu-(N+1)}(U\times\mathbb{R}^q;E,\tilde{E})$, $N\in\mathbb{N}$, turn $S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^q;E,\tilde{E})$ into a Fréchet space. If relations are valid both for general and classical symbols we write "(cl)" as subscript.

It is obvious that

$$(37)\qquad S^{\tilde{\mu}}_{(\mathrm{cl})}(U\times\mathbb{R}^q;E,\tilde{E})\subset S^\mu_{(\mathrm{cl})}(U\times\mathbb{R}^q;E,\tilde{E})$$

for $\mu\ge\tilde{\mu}$ ($\mu-\tilde{\mu}\in\mathbb{N}$ in the classical case).

Let $\tilde{E}$ be a Fréchet space written as a projective limit of Hilbert spaces $\{\tilde{E}^j\}_{j\in\mathbb{N}}$ and endowed with a group action $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$, we have the symbol spaces $S^\mu_{(\mathrm{cl})}(U\times\mathbb{R}^q;E,\tilde{E}^j)$ for all j and then define

$$S^\mu_{(\mathrm{cl})}(U\times\mathbb{R}^q;E,\tilde{E})=\varprojlim_{j\in\mathbb{N}}S^\mu_{(\mathrm{cl})}(U\times\mathbb{R}^q;E,\tilde{E}^j).$$

Also when both E and $\tilde{E}$ are Fréchet spaces there is a notion of symbol spaces $S^\mu_{(\mathrm{cl})}(U\times\mathbb{R}^q;E,\tilde{E})$ that we tacitly use here; details may be found in [38] or [40]. Parallel to the symbol spaces we have "abstract" wedge Sobolev spaces $\mathcal{W}^s(\mathbb{R}^q,E)$ of smoothness $s\in\mathbb{R}$. First, for a Hilbert space E with group action $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ we define $\mathcal{W}^s(\mathbb{R}^q,E)$ to be the completion of $\mathcal{S}(\mathbb{R}^q,E)$ with respect to the norm

$$\left\{\int\langle\zeta\rangle^{2s}||\kappa^{-1}_{\langle\zeta\rangle}\hat{u}(\zeta)||^2_E\,d\zeta\right\}^{\frac{1}{2}}.$$

Here, $\hat{u}(\zeta)$ is the Fourier transform of $u(z)$. More generally, if $E = \varprojlim_{j\in\mathbb{N}} E_j$ is Fréchet with a group action $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$, we set $\mathcal{W}^s(\mathbb{R}^q, E) = \varprojlim_{j\in\mathbb{N}} \mathcal{W}^s(\mathbb{R}^q, E_j)$. Finally, for an open set $\Omega \subseteq \mathbb{R}^q$ we have adequate analogues of the standard "comp" and "loc" spaces, here, denoted by $\mathcal{W}^s_{\mathrm{comp}}(\Omega, E)$ and $\mathcal{W}^s_{\mathrm{loc}}(\Omega, E)$, respectively.

Recall from [38] or [40] that when $U = \Omega \times \Omega$ with $(z, z') \in \Omega \times \Omega$ we have spaces of pseudo-differential operators

$$L^\mu_{(\mathrm{cl})}(\Omega; E, \tilde{E}) := \{\mathrm{Op}(a) : a(z, z', \zeta) \in S^\mu_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^q; E, \tilde{E})\}$$

and a corresponding calculus that extends the known scalar calculus in an adequate way. In all these notations we did not indicate the group actions $\kappa = \{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$, $\tilde{\kappa} = \{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$ though the various spaces depend of them. Usually, κ and $\tilde{\kappa}$ are known by the context and fixed, otherwise we occasionally write

$$S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \tilde{E})_{\kappa,\tilde{\kappa}}, \ \mathcal{W}^s(\mathbb{R}^q, E)_\kappa, \quad \text{etc.} \tag{38}$$

Let us finally note that $E = \mathbb{C}^N$ is admitted, too. In most cases the corresponding group action is taken to be trivial in this case, i.e., $\kappa_\lambda = \mathrm{id}_{\mathbb{C}^N}$, $\lambda \in \mathbb{R}_+$. The basic properties of (abstract) pseudo-differential operators with operator-valued symbols are similar to those with scalar symbols (i.e., where $E = \tilde{E} = \mathbb{C}$ and the group actions are trivial, i.e., the identity operators for all λ).

THEOREM 2. *Given $a(z, z', \zeta) \in S^\mu(\Omega\times\Omega\times\mathbb{R}^q; E, \tilde{E})$ the associated pseudo-differential operator* $\mathrm{Op}(a) : C_0^\infty(\Omega, E) \to C^\infty(\Omega, \tilde{E})$ *extends to continuous operators*

$$\mathrm{Op}(a) : \mathcal{W}^s_{\mathrm{comp}}(\Omega, E) \to \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, \tilde{E})$$

for all $s \in \mathbb{R}$. In particular if $a = a(\zeta)$ has constant coefficients (*i.e., a is independent of z and z'*), $\mathrm{Op}(a)$ *induces continuous operators*

$$\mathrm{Op}(a) : \mathcal{W}^s(\mathbb{R}^q, E) \to \mathcal{W}^{s-\mu}(\mathbb{R}^q, \tilde{E})$$

for all $s \in \mathbb{R}$.

REMARK 2. If the coefficients $a_{k\beta}(r, z)$ in (5) and $b^j_{\pm,k\beta}(r, z)$ in (6) are independent of r for large r (that can be done without loss of generality), and if we set

$$a(z, \zeta) := r^{-m} \sum_{k+|\beta|\le m} a_{k\beta}(r, z)(-r\frac{\partial}{\partial r})^k (r\zeta)^\beta$$

and

$$t^j_\pm(z, \zeta) := \mathrm{r}^\pm r^{-m^j_\pm} \sum_{k+|\beta|\le m^j_\pm} b^j_{\pm,k\beta}(r, z)(-r\frac{\partial}{\partial r})^k (r\zeta)^\beta,$$

we get families of operators

$$a(z, \zeta) : \mathcal{K}^{s,\gamma}(I^\wedge) \to \mathcal{K}^{s-m,\gamma-m}(I^\wedge) \tag{39}$$

and

$$t_{\pm}^{j}(z,\zeta) : \mathcal{K}^{s,\gamma}(I^{\wedge}) \to \mathcal{K}^{s-m_{\pm}^{j}-\frac{1}{2},\gamma-m_{\pm}^{j}-\frac{1}{2}}(\mathbb{R}_{+}) \tag{40}$$

for $s \in \mathbb{R},\ s - m_{\pm}^{j} - \frac{1}{2} > 0$. Then,

$$a(z,\zeta) \in S^{m}(\Omega \times \mathbb{R}^{n-2}; \mathcal{K}^{s,\gamma}(I^{\wedge}), \mathcal{K}^{s-m,\gamma-m}(I^{\wedge})) \tag{41}$$

and

$$t_{\pm}^{j}(z,\zeta) \in S^{m_{\pm}^{j}}(\Omega \times \mathbb{R}^{n-2}; \mathcal{K}^{s,\gamma}(I^{\wedge}), \mathcal{K}^{s-m_{\pm}^{j}-\frac{1}{2},\gamma-m_{\pm}^{j}-\frac{1}{2}}(\mathbb{R}_{+})) \tag{42}$$

for all j.

The latter relations in connection with Theorem 2 suggest the following wedge Sobolev spaces: We set

$$\mathcal{W}^{s,\gamma}(I^{\wedge} \times \mathbb{R}^{q}) := \mathcal{W}^{s}(\mathbb{R}^{q}, \mathcal{K}^{s,\gamma}(I^{\wedge}))$$

and

$$\mathcal{W}^{s,\gamma}(\mathbb{R}_{+} \times \mathbb{R}^{q}) := \mathcal{W}^{s}(\mathbb{R}^{q}, \mathcal{K}^{s,\gamma}(\mathbb{R}_{+})),$$

where $\mathcal{K}^{s,\gamma}(I^{\wedge})$ and $\mathcal{K}^{s,\gamma}(\mathbb{R}_{+})$ play the role of E, equipped with the respective group actions (18) and (19), respectively. More generally, we have corresponding "comp" and "loc" spaces for an open set $\Omega \subseteq \mathbb{R}^{q}$ that we denote by

$$\mathcal{W}^{s,\gamma}_{\mathrm{comp}}(I^{\wedge} \times \Omega),\ \ \mathcal{W}^{s,\gamma}_{\mathrm{loc}}(I^{\wedge} \times \Omega),\quad \text{etc.},$$

where we keep in mind that "comp" and "loc" only refer to z-variables in Ω. Applying Theorem 2 to the operator-valued symbol consisting of a column vector with the components (41) and (42) we get continuous operators

$$\mathrm{Op}_{z}\begin{pmatrix} a \\ (t_{+}^{j})_{j=1,\dots,N} \\ (t_{-}^{j})_{j=1,\dots,N} \end{pmatrix} : \mathcal{W}^{s,\gamma}_{\mathrm{comp}}(I^{\wedge} \times \Omega) \to \begin{array}{c} \mathcal{W}^{s-m,\gamma-m}_{\mathrm{loc}}(I^{\wedge} \times \Omega) \\ \oplus \\ \oplus_{j=1}^{N}\mathcal{W}^{s-m_{+}^{j}-\frac{1}{2},\gamma-m_{+}^{j}-\frac{1}{2}}_{\mathrm{loc}}(\mathbb{R}_{+} \times \Omega) \\ \oplus \\ \oplus_{j=1}^{N}\mathcal{W}^{s-m_{-}^{j}-\frac{1}{2},\gamma-m_{-}^{j}-\frac{1}{2}}_{\mathrm{loc}}(\mathbb{R}_{+} \times \Omega) \end{array} \tag{43}$$

for all real s such that $s - m_{\pm}^{j} - \frac{1}{2} > 0$ for all j. Clearly, we may write "comp" or "loc" in the spaces on both sides, since we discuss here differential operators that are also local in z.

2.2. Notation for Douglis-Nirenberg orders

We now fix some notation that is well-known in connection with systems of Douglis-Nirenberg type, here, adapted to our specific context. The continuity property (43) (that also holds in analogous form on operators globally on our configuration with mixed elliptic conditions, cf. Section 2.3 below) suggests to generalise our symbol spaces as follows:

Let $E := \oplus_{m=1}^{M} E^m$, $\tilde{E} := \oplus_{n=1}^{N} \tilde{E}^n$ be direct sums of (say, Hilbert) spaces with group actions $\{\kappa_\lambda^m\}_{\lambda\in\mathbb{R}_+, m=1,\dots,M}$ and $\{\tilde{\kappa}_\lambda^n\}_{\lambda\in\mathbb{R}_+, n=1,\dots,N}$, respectively. Moreover, consider matrices of symbols

$$f(z,z',\zeta) = (f_{nm}(z,z',\zeta))_{n=1,\dots,N, m=1,\dots,M},$$

where $f_{nm}(z,z',\zeta) \in S_{(\mathrm{cl})}^{\mu_{nm}}(\Omega\times\Omega\times\mathbb{R}^q; E^m, \tilde{E}^n)$ with orders μ_{nm} of the form

$$\mu_{nm} := \mu - \alpha_m + \beta_n$$

for given $(\alpha_1,\dots,\alpha_M), (\beta_1,\dots,\beta_N)$. The numbers μ_{nm} will also be referred to as *DN-orders* (Douglis-Nirenberg orders). Then $\mathrm{Op}(f)$ induces continuous operators

$$\mathrm{Op}(f) : \oplus_{m=1}^{M} \mathcal{W}_{\mathrm{comp}}^{s-\alpha_m}(\Omega, E^m)_{\kappa^m} \to \oplus_{n=1}^{N} \mathcal{W}_{\mathrm{loc}}^{s-\beta_n-\mu}(\Omega, \tilde{E}^n)_{\tilde{\kappa}^n} \tag{44}$$

for all s, cf. Theorem 2; here we used subscripts $\kappa^m := \{\kappa_\lambda^m\}_{\lambda\in\mathbb{R}_+}$ and $\tilde{\kappa}^n := \{\tilde{\kappa}_\lambda^n\}_{\lambda\in\mathbb{R}_+}$ in the sense of notation (38). It may be advantegeous to unify orders by a formal change of the underlying group actions. In fact, instead of

$$\kappa := \mathrm{diag}(\{\kappa_\lambda^m\}_{\lambda\in\mathbb{R}_+})_{m=1,\dots,M} \quad \text{and} \quad \tilde{\kappa} := \mathrm{diag}(\{\tilde{\kappa}_\lambda^n\}_{\lambda\in\mathbb{R}_+})_{n=1,\dots,N}$$

we may take

$$\chi := \mathrm{diag}(\{\lambda^{\alpha_m}\kappa_\lambda^m\}_{\lambda\in\mathbb{R}_+})_{m=1,\dots,M} \quad \text{and} \quad \tilde{\chi} := \mathrm{diag}(\{\lambda^{\beta_n}\tilde{\kappa}_\lambda^n\}_{\lambda\in\mathbb{R}_+})_{n=1,\dots,N}.$$

It is then easy to verify that

$$f(z,z',\zeta) \in S_{(\mathrm{cl})}^{\mu}(\Omega\times\Omega\times\mathbb{R}^q; E, \tilde{E})_{\chi,\tilde{\chi}}.$$

If we pass from the Sobolev spaces with subscripts κ^m and $\tilde{\kappa}^n$ to those with subscript $\chi^m := \{\lambda^{\alpha_m}\kappa_\lambda^m\}_{\lambda\in\mathbb{R}_+}$ and $\tilde{\chi}^n := \{\lambda^{\beta_n}\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively, we get the spaces

$$\begin{aligned} \mathcal{W}_{\mathrm{comp}}^{s}(\Omega, E)_{\chi} &= \oplus_{m=1}^{M} \mathcal{W}_{\mathrm{comp}}^{s-\alpha_m}(\Omega, E^m)_{\chi^m}, \\ \mathcal{W}_{\mathrm{loc}}^{s}(\Omega, \tilde{E})_{\tilde{\chi}} &= \oplus_{n=1}^{N} \mathcal{W}_{\mathrm{loc}}^{s-\beta_n}(\Omega, \tilde{E}^n)_{\tilde{\chi}^n}. \end{aligned}$$

Then (44) takes the form

$$\mathrm{Op}(f) : \mathcal{W}_{\mathrm{comp}}^{s}(\Omega, E)_{\chi} \longrightarrow \mathcal{W}_{\mathrm{loc}}^{s-\mu}(\Omega, \tilde{E})_{\tilde{\chi}}. \tag{45}$$

2.3. Weighted Sobolev spaces for mixed problems

We now introduce the global weighted Sobolev spaces as they are announced in the formula (4). To this end we fix a system of charts on X, associated with a system of coordinate neighbourhoods

$$\{U_1,\dots,U_L, U_{L+1},\dots,U_M, U_{M+1},\dots,U_N\}, \tag{46}$$

where $U_j \cap Z \neq \emptyset$ for $1 \leq j \leq L$, $U_j \cap Z = \emptyset$ and $U_j \cap Y \neq \emptyset$ for $L+1 \leq j \leq M$, $U_j \cap Y = \emptyset$ for $M+1 \leq j \leq N$. Choose a partition of unity $\{\varphi_1, \dots, \varphi_N\}$ subordinate to (46). We then have charts $\chi_j : U_j \to \overline{\mathbb{R}}_+ \times \mathbb{R}^{n-1}$ for $1 \leq j \leq M$, where for $1 \leq j \leq L$ we use the splitting $\mathbb{R}^{n-1} = \mathbb{R} \times \mathbb{R}^{n-2}$ in the sense of the above notation, where $\chi_j(U_j \cap Z) \to \mathbb{R}^{n-2}$. For simplicity we assume the transition diffeomorphisms only dependent on the variable of $\mathbb{R}^{n-1}$ in a neighbourhood of $(x_{n-1}, x_n) = 0$.

First we have the spaces $\mathcal{W}^s(\mathbb{R}^{n-2}, \mathcal{K}^{s,\gamma}(I^\wedge))$, where $I^\wedge$ is identified with $\overline{\mathbb{R}}^2_+ \setminus \{0\}$ and we now define

(47) $$\mathcal{W}^{s,\gamma}(\mathbb{X}) := \{u \in H^s_{\mathrm{loc}}(2X \setminus Z)\,|_{\mathrm{int}X} : (\chi_j)_*(\varphi_j u) \in \mathcal{W}^s(\mathbb{R}^{n-2}, \mathcal{K}^{s,\gamma}(I^\wedge)) \text{ for all } j, 1 \leq j \leq L\}.$$

Here, $2X$ denotes the double of X, that is a closed compact C^∞ manifold, obtained by gluing together two copies $X_\pm$ of X along their common boundary Y (we then identify X_+ with X).

Moreover, let M be any compact C^∞ manifold with boundary N, $m = \dim M$. Let $\{V_1, \dots, V_I, V_{I+1}, \dots, V_J\}$ be an open covering of M by coordinate neighbourhoods, where $V_i \cap N \neq \emptyset$ for $1 \leq i \leq I$, $V_i \cap N = \emptyset$ for $I+1 \leq i \leq J$; fix a partition of unity $\{\psi_1, \dots, \psi_J\}$ on M subordinate to the covering and charts $\kappa_i : V_i \to \overline{\mathbb{R}}_+ \times \mathbb{R}^{n-1}, i = 1, \dots, I$, where transition diffeomorphisms are independent of the normal variable x_m for small x_m. Then we have the spaces

$$\mathcal{W}^{s,\gamma}(\mathrm{int}\,M) = \{v \in H^s_{\mathrm{loc}}(\mathrm{int}\,M) : (\kappa_i)_*(\psi_i v) \in \mathcal{W}^s(\mathbb{R}^{m-1}, \mathcal{K}^{s,\gamma}(\mathbb{R}_+)) \text{ for } 1 \leq i \leq I\}.$$

Applying the latter notation to $M = Y_\pm$ we get the spaces

(48) $$\mathcal{W}^{s,\gamma}(\mathrm{int}\,Y_\pm) \quad \text{for all} \quad s, \gamma \in \mathbb{R}.$$

The spaces $\mathcal{W}^{s,\gamma}(\mathbb{X})$ and $\mathcal{W}^{s,\gamma}(\mathrm{int}\,Y_\pm)$ will be considered with Hilbert space norms. In particular, we identify $\mathcal{W}^{0,0}(\mathbb{X})$ with a weighted L^2 -space. More precisely, let U_ε denote a neighbourhood of Z that is locally described by $\mathbf{\Omega}_\varepsilon := \{x \in \overline{\mathbb{R}}^2_+ \times \mathbb{R}^{n-2} : |x_1, x_2| < \varepsilon\}$ and $\chi : U_\varepsilon \to \mathbf{\Omega}_\varepsilon$ a corresponding diffeomorphism. Further, let $\omega \in C^\infty(X)$ be a function that is equal to 1 in U_{ε_1} and 0 outside U_{ε_0} for certain $0 < \varepsilon_1 < \varepsilon_0$. Then

$$\omega\mathcal{W}^{0,0}(\mathbb{X}) = \chi^*\varphi\, r^{-\frac{1}{2}} L^2(I^\wedge \times \mathbb{R}^{n-2}),$$

where φ is defined by $\omega = \chi^*\varphi$, and

$$\mathcal{W}^{0,0}(\mathbb{X}) = \omega\mathcal{W}^{0,0}(\mathbb{X}) + (1-\omega)L^2(X)$$

yields a scalar product in $\mathcal{W}^{0,0}(\mathbb{X})$. In a similar way we proceed with $\mathcal{W}^{0,0}(\mathrm{int}Y_\pm)$ and fix scalar products in these spaces.

REMARK 3. The space $C_0^\infty(X \setminus Z) := \{u \in C^\infty(X) : \operatorname{supp} u \cap Z = \emptyset\}$ is dense in $\mathcal{W}^{s,\gamma}(\mathbb{X})$ for every $s, \gamma \in \mathbb{R}$. Moreover, $C_0^\infty(Y_\pm \setminus Z) := \{v \in C^\infty(Y_\pm) : \operatorname{supp} v \cap Z = \emptyset\}$ is dense in $\mathcal{W}^{s,\gamma}(Y_\pm)$ for every $s, \gamma \in \mathbb{R}$. The restriction operators

$$\mathrm{r}^\pm : C_0^\infty(X \setminus Z) \longrightarrow C_0^\infty(Y_\pm \setminus Z), \;\; \mathrm{r}^\pm : u \to u\,|_{\mathrm{int} Y_\pm},$$

extend as continuous operators

$$\mathrm{r}^\pm : \mathcal{W}^{s,\gamma}(\mathbb{X}) \longrightarrow \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathrm{int}\, Y_\pm)$$

for all $s > \frac{1}{2}, \gamma \in \mathbb{R}$.

2.4. Subspaces with asymptotics

In this section we analyse subspaces with so-called *discrete asymptotics*. We employ notation from [38] that we briefly recall here, both for the case $\mathbb{R}_+$ and then for the (stretched) plane cone $I^\wedge$ with boundary. In addition, we formulate some useful new information.

Let us fix a weight $\gamma \in \mathbb{R}$ and an associated weight strip $\{z \in \mathbb{C} : \frac{n+1}{2} - \gamma + \vartheta < \operatorname{Re} z < \frac{n+1}{2} - \gamma\}$ for some $-\infty \leq \vartheta < 0$, where we set $n = 0$ for $\mathbb{R}_+$ and $n = 1$ for $I^\wedge$. Let $\mathbf{g} = (\gamma, \Theta)$ for $\Theta = (\vartheta, 0]$. For the case $n = 0$ we define $\mathrm{As}(\mathbf{g})$ to be the set of all sequences $P = \{(p_j, m_j)\}_{j=0,\dots,N} \subset \mathbb{C} \times \mathbb{N}$ for some $N = N(P)$, where $N(P) < \infty$ for finite ϑ, such that $\pi_{\mathbb{C}} P := \{p_j\}_{j=0,\dots,N} \subset \{z \in \mathbb{C} : \frac{1}{2} - \gamma + \vartheta < \operatorname{Re} z < \frac{1}{2} - \gamma\}$, and $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$ for the case $N(P) = \infty$. Similarly, for $n = 1$ we define $\mathrm{As}([0, \pi], \mathbf{g})$ to be the set of all sequences $P = \{(p_j, m_j, L_j)\}_{j=0,\dots,N}$, $N = N(P)$, where $(p_j, m_j) \in \mathbb{C} \times \mathbb{N}$, and $L_j \subset C^\infty([0, \pi])$ is a subspace of finite dimension. Concerning $\pi_{\mathbb{C}} P = \{p_j\}_{j=0,\dots,N}$ we require $\pi_{\mathbb{C}} P \subset \{z \in \mathbb{C} : 1 - \gamma + \vartheta < \operatorname{Re} z < 1 - \gamma\}$ and again $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$ when $N(P) = \infty$. The elements $P \in \mathrm{As}(\cdot, \mathbf{g})$ are called *discrete asymptotic types* for the cones $\mathbb{R}_+$ and $I^\wedge$, respectively, associated with *weight data* $\mathbf{g} = (\gamma, \Theta)$.

REMARK 4. We can formally unify the notation for $n = 0$ and $n = 1$ by writing $\mathrm{As}(\mathbf{g}) = \mathrm{As}(\{a\}, \mathbf{g})$ for a point a (that may be regarded as the base of the cone $\mathbb{R}_+$, e.g., $a = 1$) and write $P \in \mathrm{As}(\mathbf{g})$ in the form $P = \{(p_j, m_j, \mathbb{C})\}_{j=0,\dots,N}$. Then, for $P = \{(p_j, m_j, L_j\}_{j=0,\dots,N} \in \mathrm{As}([0, \pi], \mathbf{g})$ we have restriction maps to the end points 0 and π induced by $L_j \to \mathbb{C}$, $f(\varphi) \to f(0)$ or $f(\varphi) \to f(\pi)$. This gives us corresponding restriction maps

$$\mathrm{As}([0, \pi], \mathbf{g}) \longrightarrow \mathrm{As}(\mathbf{g}). \tag{49}$$

Let us now introduce spaces with discrete asymptotics, say, for the case $I^\wedge$; the case $\mathbb{R}_+$ is easier and may be found, e.g., in [38]. Concerning asymptotic types with a non-trivial cone base we may formally replace $[0, \pi]$ by the circle S^1 and talk about coefficient spaces $L_j \subset C^\infty(S^1)$ instead of $L_j \subset C^\infty([0, \pi])$. In other words, we also have a notion of $\mathrm{As}(S^1, \mathbf{g})$ of evident meaning and then a restriction map $\mathrm{As}(S^1, \mathbf{g}) \to \mathrm{As}([0, \pi], \mathbf{g})$ by restricting $L_j \to L_j\,|_{[0,\pi]}$.

For $S := S^1$ and $S^\wedge := \mathbb{R}_+ \times S$ we first have spaces of flat functions

$$\mathcal{K}^{s,\gamma}_{\Theta}(S^\wedge) := \varprojlim_{\varepsilon>0} \mathcal{K}^{s,\gamma-\vartheta-\varepsilon}(S^\wedge).$$

We consider $\mathcal{K}^{s,\gamma}_{\Theta}(S^\wedge)$ in its natural Fréchet topology. Now let Θ be finite, choose $P \in \mathrm{As}(S, \mathbf{g})$, $\mathbf{g} = (\gamma, \Theta)$, written as $P = \{(p_j, m_j, L_j)\}_{j=0,\dots,N}$, and set

$$\mathcal{E}_P(S^\wedge) := \{\omega(r)\sum_{j=0}^{N}\sum_{k=0}^{m_j} c_{jk}(\varphi) r^{-p_j}\log^k r : c_{jk} \in L_j,\ 0 \le k \le m_j,\ 0 \le j \le N\}. \tag{50}$$

Here, $\omega(r)$ is any fixed cut-off function. The space $\mathcal{E}_P(S^\wedge)$ is finite-dimensional, and we obviously have

$$\mathcal{K}^{s,\gamma}_{\Theta}(S^\wedge) \cap \mathcal{E}_P(S^\wedge) = \{0\}.$$

Moreover, we have

$$\mathcal{E}_P(S^\wedge) \subset \mathcal{K}^{\infty,\gamma}(S^\wedge).$$

We now define

$$\mathcal{K}^{s,\gamma}_{P}(S^\wedge) := \mathcal{K}^{s,\gamma}_{\Theta}(S^\wedge) + \mathcal{E}_P(S^\wedge)$$

in the Fréchet topology of the direct sum. For $P \in \mathrm{As}(S, \mathbf{g})$, $\mathbf{g} = (\gamma, (-\infty, 0])$ we choose an arbitrary sequence $\vartheta_k < 0$, $\vartheta_k \to -\infty$ as $k \to \infty$, set $\Theta_k = (\vartheta_k, 0]$, form $P_k := \{(p, m, L) \in P : 1-\gamma+\vartheta_k < \mathrm{Re}\, p < 1-\gamma\} \in \mathrm{As}(S, \mathbf{g}_k)$ for $\mathbf{g}_k = (\gamma, \Theta_k)$, and consider the associated spaces $\mathcal{K}^{s,\gamma}_{P_k}(S^\wedge)$. Then we set

$$\mathcal{K}^{s,\gamma}_{P}(S^\wedge) := \varprojlim_{k\in\mathbb{N}} \mathcal{K}^{s,\gamma}_{P_k}(S^\wedge)$$

in the topology of the projective limit. This space is independent of the choice of the sequence $(\vartheta_k)_{k\in\mathbb{N}}$. Moreover, if $R \in \mathrm{As}([0,\pi], \mathbf{g})$ is the restriction of $P \in \mathrm{As}(S, \mathbf{g})$ we set

$$\mathcal{K}^{s,\gamma}_{R}(I^\wedge) := \{u|_{I^\wedge} : u \in \mathcal{K}^{s,\gamma}_{P}(S^\wedge)\}.$$

In particular, for $\pi_{\mathbb{C}} R = \emptyset$ we have

$$\mathcal{K}^{s,\gamma}_{\Theta}(I^\wedge) = \{u|_{I^\wedge} : u \in \mathcal{K}^{s,\gamma}_{\Theta}(S^\wedge)\}.$$

In a similar (but simpler) way we can introduce the spaces $\mathcal{K}^{s,\gamma}_{Q}(\mathbb{R}_+)$ for $Q \in \mathrm{As}(\mathbf{g})$, $\mathbf{g} = (\gamma, \Theta)$, see, for instance, [38].

REMARK 5. The operator of restriction $u(r,\varphi) \to u(r,\varphi_0)$ for arbitrary fixed $\varphi_0 \in S$ which is continuous as $\mathcal{K}^{s,\gamma}(S^\wedge) \to \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+)$ for $s > \frac{1}{2}$ induces continuous operators

$$\mathcal{K}^{s,\gamma}_{P}(S^\wedge) \longrightarrow \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}_{Q}(\mathbb{R}_+)$$

for every $P \in \mathrm{As}(S, \mathbf{g})$, where Q is the restriction of P in a similar sense as (49). In particular, we have restrictions of $\mathcal{K}_P^{s,\gamma}(I^\wedge),\ P \in \mathrm{As}([0,\pi], \mathbf{g})$, namely

$$\mathcal{K}_P^{s,\gamma}(I^\wedge) \longrightarrow \mathcal{K}_P^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{R}_+)$$

induced by $u(r,\varphi) \to u(r,\varphi_0)$, for every fixed $0 \le \varphi_0 \le \pi,\ s > \frac{1}{2}$.

For purposes below we now set

$$\mathcal{S}_P^\gamma(S^\wedge) := \{\omega u + (1-\omega)v : u \in \mathcal{K}_P^{\infty,\gamma}(S^\wedge), v \in \mathcal{S}(\mathbb{R} \times S)\},$$

where $P \in \mathrm{As}(S, \mathbf{g}),\ \mathcal{S}(\mathbb{R} \times S) = \mathcal{S}(\mathbb{R}, C^\infty(S))$, and, similarly,

$$\mathcal{S}_P^\gamma(I^\wedge) \quad \text{and} \quad \mathcal{S}_P^\gamma(\mathbb{R}_+)$$

by replacing the base S by $[0,\pi]$ and a single point, respectively, for an asymptotic type P belonging to the respective class.

To define wedge spaces with asymptotics we employ the fact that the spaces $\mathcal{K}_P^{s,\gamma}(\ldots), \mathcal{S}_P^\gamma(\ldots)$ may be written as projective limits of Hilbert spaces $E_j,\ j \in \mathbb{N}$, where the group action that is fixed on $\mathcal{K}^{s,\gamma}(\ldots)$ restricts to a group action on E_j for every j.

Let us express E_j, for instance, for the case of spaces on $I^\wedge$. For finite $\Theta = (\vartheta, 0]$ it suffices to set

$$\text{(51)} \qquad \begin{aligned} E_j &= \mathcal{K}^{s,\gamma-\vartheta-(1+j)^{-1}}(I^\wedge) + \mathcal{E}_{P_j}(I^\wedge), \\ P_j &= \{(p,m,L) \in P : 1-\gamma+\vartheta+(1+j)^{-1} < \mathrm{Re}\, p\}, \end{aligned}$$

for $\mathcal{K}_P^{s,\gamma}(I^\wedge)$ (the space $\mathcal{E}_P(I^\wedge)$ for $P \in \mathrm{As}([0,\pi], \mathbf{g}),\ P = \{(p_j, m_j, L_j)\}_{j=0,\ldots,N}$, $L_j \in C^\infty([0,\pi])$ is defined as in (50)) and

$$\text{(52)} \qquad \begin{aligned} E_j &= \langle r\rangle^{-j}\mathcal{K}^{j,\gamma-\vartheta-(1+j)^{-1}}(I^\wedge) + \mathcal{E}_{P_j}(I^\wedge), \\ P_j &= \{(p,m,L) \in P : 1-\gamma+\vartheta+(1+j)^{-1} < \mathrm{Re}\, p\}, \end{aligned}$$

for $\mathcal{S}_P^\gamma(I^\wedge)$ while for $\vartheta = -\infty$ we replace ϑ in formula (50) by $-(1+j)$ and P by $P_j = \{(p,m,L) \in P : 2+j-\gamma+(1+j)^{-1} < \mathrm{Re}\, p\}$.

We now define weighted edges Sobolev spaces with asymptotics

$$\text{(53)} \qquad \mathcal{W}_P^{s,\gamma}(\mathbb{X}) \quad \text{and} \quad \mathcal{W}_Q^{s,\gamma}(\mathrm{int}\, M)$$

of types $P \in \mathrm{As}([0,\pi], \mathbf{g})$ and $Q \in \mathrm{As}(\mathbf{g})$, respectively, for $\mathbf{g} = (\gamma, \Theta)$, by inserting the spaces $\mathcal{K}_P^{s,\gamma}(I^\wedge)$ instead of $\mathcal{K}^{s,\gamma}(I^\wedge)$ in (47) and $\mathcal{K}_Q^{s,\gamma}(\mathbb{R}_+)$ instead of $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ in (48). To make this more explicit, for instance, for the spaces on $\mathbb{X}$, we first write $\mathcal{K}_P^{s,\gamma}(I^\wedge) = \varprojlim_{k\in\mathbb{N}} E_k$ with the scale of spaces E_k in formula (52). Then we form

$$\begin{aligned} \mathcal{W}_k^{s,\gamma}(\mathbb{X}) := \{&u \in H^s_{\mathrm{loc}}(2X \setminus Z)\,|_{\mathrm{int}\, X} : (\chi_j)_*(\varphi_j u) \in \mathcal{W}^s(\mathbb{R}^{n-2}, E_k) \\ &\text{for all } j, 1 \le j \le L\}, \end{aligned}$$

where we have continuous embeddings $\mathcal{W}^{s,\gamma}_{k+1}(\mathbb{X}) \hookrightarrow \mathcal{W}^{s,\gamma}_{k}(\mathbb{X})$ for all k, and we then set

$$\mathcal{W}^{s,\gamma}_{P}(\mathbb{X}) := \varprojlim_{k\in\mathbb{N}} \mathcal{W}^{s,\gamma}_{k}(\mathbb{X}).$$

In a similar manner we proceed for $\Theta = (-\infty, 0]$ as well in the case of spaces on M. All these spaces are Fréchet in the corresponding projective limit topologies.

REMARK 6. The spaces

$$\mathcal{W}^{\infty,\gamma}_{P}(\mathbb{X}) \quad \text{and} \quad \mathcal{W}^{\infty,\gamma}_{Q}(M) \tag{54}$$

are independent of the choice of the group actions $\{\kappa^{\wedge}_{\lambda}\}_{\lambda\in\mathbb{R}_+}$ and $\{\kappa_{\lambda}\}_{\lambda\in\mathbb{R}_+}$, respectively. This makes the form of singular functions of discrete edge asymptotics particularly simple.

2.5. Green symbols

In this section we introduce so-called *Green symbols* of the (local) pseudo-differential algebra of mixed problems. These will be operator families $g(z,\zeta)$ that are pointwise block matrices of the form

$$g(z,\zeta) = \begin{pmatrix} g_{11} & g_{1+} & g_{1-} & g_{10} \\ g_{+1} & g_{++} & g_{+-} & g_{+0} \\ g_{-1} & g_{-+} & g_{--} & g_{-0} \\ g_{01} & g_{0+} & g_{0-} & g_{00} \end{pmatrix}(z,\zeta),$$

where the block matrix structure corresponds to mappings of the type

$$g(z,\zeta) : \begin{array}{c} \mathcal{K}^{s,\gamma}(I^{\wedge}) \\ \oplus \\ \bigoplus_{k=1}^{M} \mathcal{K}^{s-n^k_+-\frac{1}{2},\gamma-n^k_+-\frac{1}{2}}(\mathbb{R}_+) \\ \oplus \\ \bigoplus_{k=1}^{M} \mathcal{K}^{s-n^k_--\frac{1}{2},\gamma-n^k_--\frac{1}{2}}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{l_-} \end{array} \longrightarrow \begin{array}{c} \mathcal{S}^{\gamma-m}_{P}(I^{\wedge}) \\ \oplus \\ \bigoplus_{l=1}^{N} \mathcal{S}^{\gamma-m^l_+-\frac{1}{2}}_{P_+,l}(\mathbb{R}_+) \\ \oplus \\ \bigoplus_{l=1}^{N} \mathcal{S}^{\gamma-m^l_--\frac{1}{2}}_{P_-,l}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{l_+} \end{array}, \tag{55}$$

for all $s > -\frac{1}{2}$.

Let us set

$$\mathbf{s} := (s; (s-n^k_+-\frac{1}{2})_{k=1,\dots,M}, (s-n^k_--\frac{1}{2})_{k=1,\dots,M}), \tag{56}$$

$$\boldsymbol{\gamma} := (\gamma; (\gamma-n^k_+-\frac{1}{2})_{k=1,\dots,M}, (\gamma-n^k_--\frac{1}{2})_{k=1,\dots,M}), \tag{57}$$

(58) $$\delta := (\gamma - m; (\gamma - m_+^l - \frac{1}{2})_{l=1,\dots,N}, (\gamma - m_-^l - \frac{1}{2})_{l=1,\dots,N})$$

and, for abbreviation

$$\mathcal{K}^{\mathbf{s},\boldsymbol{\gamma}}(I^\wedge, \mathbb{R}_+; M) :=$$
$$\mathcal{K}^{s,\gamma}(I^\wedge) \oplus \bigoplus_{k=1}^{M} \mathcal{K}^{s-n_+^k-\frac{1}{2},\gamma-n_+^k-\frac{1}{2}}(\mathbb{R}_+) \oplus \bigoplus_{k=1}^{M} \mathcal{K}^{s-n_-^k-\frac{1}{2},\gamma-n_-^k-\frac{1}{2}}(\mathbb{R}_+),$$
$$\mathcal{S}_{\mathbf{P}}^{\delta}(I^\wedge, \mathbb{R}_+; N) :=$$
$$\mathcal{S}_P^{\gamma-m}(I^\wedge) \oplus \bigoplus_{l=1}^{N} \mathcal{S}_{P_+,l}^{\gamma-m_+^l-\frac{1}{2}}(\mathbb{R}_+) \oplus \bigoplus_{l=1}^{N} \mathcal{S}_{P_-,l}^{\gamma-m_-^l-\frac{1}{2}}(\mathbb{R}_+)$$

for $\mathbf{P} \in \mathbf{As}([0,\pi], \tilde{\mathbf{g}})$, $\tilde{\mathbf{g}} = (\delta, \Theta)$, $\Theta = (\vartheta, 0]$, $-\infty \le \vartheta < 0$. Here and below we denote by $\mathbf{As}([0,\pi], \tilde{\mathbf{g}})$ for $\tilde{\mathbf{g}} = (\mathbf{e}, \Theta)$, weights $\mathbf{e} = (e_0; (e_{+,l})_{l=1,\dots,L}, (e_{-,l})_{l=1,\dots,L})$ and $\Theta = (\vartheta, 0]$, $-\infty \le \vartheta < 0$, the set of all tuples of asymptotic types $\mathbf{P} = (P; (P_{\pm,l})_{l=1,\dots,L})$ such that $P \in \mathrm{As}([0,\pi], \mathbf{g}_0)$ for $\mathbf{g}_0 = (e_0, \Theta)$ and $P_{\pm,l} \in \mathrm{As}(\mathbf{g}_{\pm,l})$ for $\mathbf{g}_{\pm,l} = (e_{\pm,l}, \Theta)_{l=1,\dots,N}$. Then the operator family (55) takes the form

(59) $$g(z,\zeta) : \begin{array}{c} \mathcal{K}^{\mathbf{s},\boldsymbol{\gamma}}(I^\wedge, \mathbb{R}_+; M) \\ \oplus \\ \mathbb{C}^{l_-} \end{array} \longrightarrow \begin{array}{c} \mathcal{S}_{\mathbf{P}}^{\delta}(I^\wedge, \mathbb{R}_+; N) \\ \oplus \\ \mathbb{C}^{l_+} \end{array}$$

for all $s > -\frac{1}{2}$.

We also need (pointwise) formal adjoints $g^*(z,\zeta)$, defined by

$$(gu, v)_{\mathcal{K}^{0,0}(I^\wedge)\oplus\mathcal{K}^{0,0}(\mathbb{R}_+,\mathbb{C}^{2N})\oplus\mathbb{C}^{l_+}} = (u, g^*v)_{\mathcal{K}^{0,0}(I^\wedge)\oplus\mathcal{K}^{0,0}(\mathbb{R}_+,\mathbb{C}^{2M})\oplus\mathbb{C}^{l_-}}$$

for all $u \in C_0^\infty((\mathrm{int}\, I)^\wedge) \oplus C_0^\infty(\mathbb{R}_+, \mathbb{C}^{2M}) \oplus \mathbb{C}^{l_-}$, $v \in C_0^\infty((\mathrm{int}\, I)^\wedge) \oplus C_0^\infty(\mathbb{R}_+, \mathbb{C}^{2N}) \oplus \mathbb{C}^{l_+}$. Here, $\mathcal{K}^{0,0}(I^\wedge)$ and $\mathcal{K}^{0,0}(\mathbb{R}_+)$ are endowed with the scalar products of $r^{-\frac{1}{2}}L^2(\mathbb{R}_+ \times I)_{dr d\varphi}$ and $L^2(\mathbb{R}_+)$, respectively, and $\mathcal{K}^{0,0}(\mathbb{R}_+, \mathbb{C}^j) := \mathcal{K}^{0,0}(\mathbb{R}_+) \otimes \mathbb{C}^j$.

The non-degenerate pairing

$$\{C_0^\infty(\mathrm{int}\, I)^\wedge) \oplus C_0^\infty(\mathbb{R}_+, \mathbb{C}^L) \oplus \mathbb{C}^l\} \times \{C_0^\infty(\mathrm{int}\, T)^\wedge) \oplus C_0^\infty(\mathbb{R}_+, \mathbb{C}^L) \oplus \mathbb{C}^l\} \to \mathbb{C}$$

defined by $(u,v)_{\mathcal{K}^{0,0}(I^\wedge)\oplus\mathcal{K}^{0,0}(\mathbb{R}_+,\mathbb{C}^L)\oplus\mathbb{C}^l}$ extends to a non-degenerate sesquilinear form

$$\{\mathcal{K}^{0,\gamma}(I^\wedge) \oplus \bigoplus_{k=1}^{L} \mathcal{K}^{0,\gamma_k}(\mathbb{R}_+) \oplus \mathbb{C}^l\} \times \{\mathcal{K}^{0,-\gamma}(I^\wedge) \oplus \bigoplus_{k=1}^{L} \mathcal{K}^{0,-\gamma_k}(\mathbb{R}_+) \oplus \mathbb{C}^l\} \to \mathbb{C}$$

for arbitrary $\gamma, \gamma_1, \dots, \gamma_L \in \mathbb{R}$. For that reason we also introduce dual tuples of weights, namely

$$\begin{aligned} -\boldsymbol{\gamma} &:= (-\gamma; (-\gamma + n_+^k + \frac{1}{2})_{k=1,\dots,M}, (-\gamma + n_-^k + \frac{1}{2})_{k=1,\dots,M}), \\ -\delta &:= (-\gamma + m; (-\gamma + m_+^l + \frac{1}{2})_{l=1,\dots,N}, (-\gamma + m_-^l + \frac{1}{2})_{l=1,\dots,N}). \end{aligned}$$

Concerning the smoothness in the spaces for convenience we set

$$\mathbf{t}^* := (s+m, (s+m_+^l+\frac{1}{2})_{l=1,\dots,N}, (s+m_-^l+\frac{1}{2})_{l=1,\dots,N}). \tag{60}$$

Let us introduce a matrix of orders

$$\mu := (\mu_{lk})_{l=0,\dots,2N+1,k=0,\dots,2M+1}, \qquad \text{where} \qquad \mu_{lk} = \gamma_k - \delta_l$$

with the tuples $((\gamma_k)_{k=0,\dots,2M}, \gamma_{2M+1})$ and $((\delta_l)_{l=0,\dots,2N}, \delta_{2N+1})$, where $(\gamma_k)_{k=0,\dots,2M} := \gamma$ and $(\delta_l)_{l=0,\dots,2N} := \delta$ are the sequences (57) and (58), respectively, while $\gamma_{2M+1} := \gamma - 1$ and $\delta_{2N+1} := \gamma - 1 - m$.

In the following definition we set $\mathbf{g} = (\gamma, \Theta; \delta, \Theta')$, where γ and δ are weight tuples (57) and (58), respectively, and $\Theta = (\vartheta, 0]$ and $\Theta' = (\vartheta', 0]$ fixed weight intervals, $-\infty \le \vartheta, \vartheta' < 0$.

DEFINITION 1. *$\mathcal{R}_G^{\mu,0}(U \times \mathbb{R}^{n-2}; \mathbf{g}; \mathbf{w})$ for $\mathbf{w} := (M, l_-; N, l_+)$, $\mu \in \mathbb{R}, U \subseteq \mathbb{R}^{n-2}$ open, is defined to be the space of all operator families* (59) *such that*

$$g(z,\zeta) \in S_{\text{cl}}^{\mu}\left(U \times \mathbb{R}^{n-2}; \begin{matrix}\mathcal{K}^{\mathbf{s},\gamma}(I^\wedge, \mathbb{R}_+; M) \\ \oplus \\ \mathbb{C}^{l_-}\end{matrix}, \begin{matrix}\mathcal{S}_{\mathbf{P}}^{\delta}(I^\wedge, \mathbb{R}_+; N) \\ \oplus \\ \mathbb{C}^{l_+}\end{matrix}\right) \tag{61}$$

and

$$g^*(z,\zeta) \in S_{\text{cl}}^{\mu^*}\left(U \times \mathbb{R}^{n-2}; \begin{matrix}\mathcal{K}^{\mathbf{t}^*,-\delta}(I^\wedge, \mathbb{R}_+; N) \\ \oplus \\ \mathbb{C}^{l_+}\end{matrix}, \begin{matrix}\mathcal{S}_{\mathbf{Q}}^{-\gamma}(I^\wedge, \mathbb{R}_+; M) \\ \oplus \\ \mathbb{C}^{l_-}\end{matrix}\right) \tag{62}$$

for all $s > -\frac{1}{2}$ and for transposed matrix μ^* of μ with tuples of asymptotic types $\mathbf{P} \in \mathbf{As}([0,\pi], \tilde{\mathbf{g}})$ for $\tilde{\mathbf{g}} = (\delta, \Theta')$ and $\mathbf{Q} \in \mathrm{As}([0,\pi], \tilde{\mathbf{g}})$ for $\tilde{\mathbf{g}} = (-\gamma, \Theta)$ dependent on the symbol g (not on s).

In our application we set $U = \mathbf{\Omega}$ or $U = \mathbf{\Omega} \times \mathbf{\Omega}$ for an open set $\mathbf{\Omega} \subseteq \mathbb{R}^{n-2}$ (in the latter case we also write (z, z') instead of z). As classical symbols Green symbols have a unique sequence of homogeneous components

$$\sigma^{(\mu-j)}(g)(z,\zeta), \;\; j \in \mathbb{N},$$

(here, $\mu - j$ is the matrix with entries $\mu_{lk} - j$, $\mu := (\mu_{lk})_{l=0,\dots,2N+1,k=0,\dots,2M+1}$) that are C^∞ functions in $(z,\zeta) \in U \times (\mathbb{R}^{n-2} \setminus \{0\})$ with values in continuous operators

$$\sigma^{(\mu-j)}(g)(z,\zeta) : \begin{matrix}\mathcal{K}^{\mathbf{s},\gamma}(I^\wedge, \mathbb{R}_+; M) \\ \oplus \\ \mathbb{C}^{l_-}\end{matrix} \longrightarrow \begin{matrix}\mathcal{S}_{\mathbf{P}}^{\delta}(I^\wedge, \mathbb{R}_+; N) \\ \oplus \\ \mathbb{C}^{l_+}\end{matrix}, \tag{63}$$

where the pointwise adjoints act in the same spaces as $g^*(z,\zeta)$ in formula (62).

Setting

$$\kappa_\lambda := \mathrm{diag}\,(\kappa_\lambda^k)_{k=0,\dots,2M+1},\quad \tilde{\kappa}_\lambda := \mathrm{diag}\,(\tilde{\kappa}_\lambda^l)_{l=0,\dots,2N+1},$$

where $\kappa_\lambda^0 = \tilde{\kappa}_\lambda^0 = \kappa_\lambda^\wedge$, $\kappa_\lambda^k = \kappa_\lambda, k = 1,\dots,2M$, $\tilde{\kappa}_\lambda^l = \kappa_\lambda, l = 1,\dots,2N$, $\kappa_\lambda^{2M+1} = \mathrm{id}_{\mathbb{C}^{l_-}}$, $\tilde{\kappa}_\lambda^{2N+1} = \mathrm{id}_{\mathbb{C}^{l_+}}$, and

$$\chi_\lambda := \mathrm{diag}(\lambda^{\alpha_k}\kappa_\lambda^k)_{k=0,\dots,2M+1},\quad \tilde{\chi}_\lambda := \mathrm{diag}(\lambda^{\beta_l}\tilde{\kappa}_\lambda^l)_{l=0,\dots,2N+1},$$

where $\alpha_0 = \beta_0 = 0$, $\alpha_k = n_+^k + \frac{1}{2}, k = 1,\dots,M$, $\alpha_k = n_-^{k-M} + \frac{1}{2}, k = M+1,\dots,2M$, $\beta_l = m_+^l + \frac{1}{2} - m, l = 1,\dots,N$, $\beta_l = m_-^{l-N} + \frac{1}{2} - m, l = N+1,\dots,2N$, $\alpha_{2M+1} = \beta_{2N+1} = 1$, we have

$$\sigma^{(\mu)}(g)(z,\lambda\zeta) = \lambda^m \tilde{\chi}_\lambda \sigma^{(\mu)}(g)(z,\zeta)\chi_\lambda^{-1}$$

for all $(z,\zeta) \in U \times (\mathbb{R}^{n-2} \setminus \{0\})$, $\lambda \in \mathbb{R}_+$. Below we often set

$$\sigma_\wedge^{(\mu)}(g)(z,\zeta) := \sigma^{(\mu)}(g)(z,\zeta)$$

when $\boldsymbol{\Omega} = U$ or, for $U = \boldsymbol{\Omega} \times \boldsymbol{\Omega} \ni (z,z')$

$$\sigma_\wedge^{(\mu)}(g)(z,\zeta) := \sigma^{(\mu)}(g)(z,z',\zeta)\,|_{z'=z}\,.$$

REMARK 7. Let $g(z,\zeta) \in \mathcal{R}_G^{\mu,0}(U\times\mathbb{R}^{n-2};\mathbf{g};\mathbf{w})$ and choose diagonal matrices of scalar symbols

$$\begin{aligned}\mathbf{r}(z,\zeta) &:= \mathrm{diag}\,(r_0(z,\zeta),(r_k(z,\zeta))_{k=1,\dots,2M+1}),\\ \tilde{\mathbf{r}}(z,\zeta) &:= \mathrm{diag}(\tilde{r}_0(z,\zeta),(\tilde{r}_l(z,\zeta))_{l=1,\dots,2N+1}),\end{aligned}$$

where $r_k(z,\zeta) \in S_{\mathrm{cl}}^{\nu_k}(U\times\mathbb{R}^{n-2})$ and $\tilde{r}_l(z,\zeta) \in S_{\mathrm{cl}}^{\tilde{\nu}_l}(U\times\mathbb{R}^{n-2})$ are elliptic scalar symbols of orders $\nu_k, \tilde{\nu}_l \in \mathbb{R}, k = 0,\dots,2M+1, l = 0,\dots,2N+1$, that are all non-vanishing for all $(z,\zeta) \in U\times\mathbb{R}^{n-2}$. Then, setting

$$\tilde{g}(z,\zeta) := \tilde{\mathbf{r}}(z,\zeta)g(z,\zeta)\mathbf{r}(z,\zeta)$$

we get $\tilde{g}(z,\zeta) \in \mathcal{R}_G^{\tilde{\mu},0}(U\times\mathbb{R}^{n-2};\mathbf{g};\mathbf{w})$ for $\tilde{\mu} = (\tilde{\mu}_{lk})_{l=0,\dots,2N+1,k=0,\dots,2M+1}$, where $\tilde{\mu}_{lk} = \mu_{lk} + \nu_k + \tilde{\nu}_l$.

REMARK 8. Let $f(z,\zeta)$ be a C^∞ function in $(z,\zeta) \in U\times(\mathbb{R}^{n-2}\setminus\{0\})$ with values in continuous operators

$$f(z,\zeta) : \begin{array}{c}\mathcal{K}^{\mathbf{s},\gamma}(I^\wedge,\mathbb{R}_+;M)\\ \oplus\\ \mathbb{C}^{l_-}\end{array} \longrightarrow \begin{array}{c}\mathcal{S}_{\mathbf{P}}^{\delta}(I^\wedge,\mathbb{R}_+;N)\\ \oplus\\ \mathbb{C}^{l_+}\end{array},$$

for all $s > -\frac{1}{2}$, such that the pointwise adjoint $f^*(z,\zeta)$ defines a C^∞ family of maps

$$f^*(z,\zeta) : \begin{array}{c}\mathcal{K}^{\mathbf{t}^*,-\delta}(I^\wedge,\mathbb{R}_+;N)\\ \oplus\\ \mathbb{C}^{l_+}\end{array} \longrightarrow \begin{array}{c}\mathcal{S}_{\mathbf{Q}}^{-\gamma}(I^\wedge,\mathbb{R}_+;M)\\ \oplus\\ \mathbb{C}^{l_-}\end{array},$$

for all $s > -\frac{1}{2}$. Further, assume that

(64) $$f(z, \lambda\zeta) = \lambda^m \tilde{\chi}_\lambda f(z,\zeta)\chi_\lambda^{-1}$$

for all $z \in \mathbf{\Omega}$, $\zeta \in \mathbb{R}^{n-2} \setminus \{0\}$ and $\lambda \in \mathbb{R}_+$. Then we have

$$\chi(\zeta) f(z,\zeta) \in \mathcal{R}_G^{\mu,0}(U \times \mathbb{R}^{n-2}; \mathbf{g}; \mathbf{w})$$

for every excision function $\chi(\zeta)$ (i.e., $\chi(\zeta) \in C^\infty(\mathbb{R}^{n-2})$, $\chi(\zeta) = 0$ for $|\zeta| < c_0$, $\chi(\zeta) = 1$ for $|\zeta| > c_1$ for certain $0 < c_0 < c_1$).

REMARK 9. Let $\mathcal{R}_G^{\mu,0}(U \times \mathbb{R}^{n-2}; \mathbf{g}; \mathbf{w})_{\mathbf{P},\mathbf{Q}}$ denote the set of all $g(z,\zeta) \in \mathcal{R}_G^{\mu,0}(U \times \mathbb{R}^{n-2}; \mathbf{g}; \mathbf{w})$, where the tuples $\mathbf{P}$ and $\mathbf{Q}$ of asymptotic types are fixed , cf. the notation in Definition 1. Then $\mathcal{R}_G^{\mu,0}(U \times \mathbb{R}^{n-2}; \mathbf{g}; \mathbf{w})_{\mathbf{P},\mathbf{Q}}$ is a Fréchet space in a canonical way, and

$$D_z^\alpha D_\zeta^\beta : \mathcal{R}_G^{\mu,0}(U \times \mathbb{R}^{n-2}; \mathbf{g}; \mathbf{w})_{\mathbf{P},\mathbf{Q}} \longrightarrow \mathcal{R}_G^{\mu-\beta,0}(U \times \mathbb{R}^{n-2}; \mathbf{g}; \mathbf{w})_{\mathbf{P},\mathbf{Q}}.$$

THEOREM 3. *Let $g_j(z,\zeta) \in \mathcal{R}_G^{\mu-j,0}(U \times \mathbb{R}^{n-2}; \mathbf{g}; \mathbf{w})_{\mathbf{P},\mathbf{Q}}$, $j \in \mathbb{N}$, be an arbitrary sequence , where $\mathbf{P}$ and $\mathbf{Q}$ are independent of j. Then there is a $g(z,\zeta) \in \mathcal{R}_G^{\mu,0}(U \times \mathbb{R}^{n-2}; \mathbf{g}; \mathbf{w})_{\mathbf{P},\mathbf{Q}}$ such that*

$$g(z,\zeta) - \sum_{j=0}^{N} g_j(z,\zeta) \in \mathcal{R}_G^{\mu-(N+1),0}(U \times \mathbb{R}^{n-2}; \mathbf{g}; \mathbf{w})_{\mathbf{P},\mathbf{Q}}$$

for every $N \in \mathbb{N}$, and $g(z,\zeta)$ is unique modulo symbols of order $-\infty$ in ζ.

The proof is analogous to that for the existence of asymptotic sums in the standard sense (i.e., for operator-valued symbols in the set up with group actions, cf. [40]), and we write $g \sim \sum_{j=0}^{\infty} g_j$, called an *asymptotic sum* of the corresponding Green symbols g_j.

DEFINITION 2. *The space $\mathcal{R}_G^{\mu,d}(U \times \mathbb{R}^{n-2}; \mathbf{g}; \mathbf{w})$ of Green symbols of order μ and type $d \in \mathbb{N}$ is defined to be the space of all operator functions*

$$g(z,\zeta) = g_0(z,\zeta) + \sum_{j=1}^{d} g_j(z,\zeta)\,\mathrm{diag}\,(\Phi^j, 0)$$

for arbitrary $g_j(z,\zeta) \in \mathcal{R}_G^{\mu-j,0}(U \times \mathbb{R}^{n-2}; \mathbf{g}; \mathbf{w})$; here, $\mathrm{diag}\,(\Phi^j, 0)$ *is the diagonal matrix, where the upper left corner $\Phi^j := \frac{\partial^j}{\partial\varphi^j}$ is the only non-vanishing entry.*

The above notation and results for $d = 0$ can be generalised to arbitrary $d \in \mathbb{N}$. In particular, we have the spaces

$$\mathcal{R}_G^{\mu,d}(U \times \mathbb{R}^{n-2}; \mathbf{g}; \mathbf{w})_{\mathbf{P},\mathbf{Q}}$$

which means that all g_j in Definition 2 belong to the corresponding symbol classes with subscript **P**, **Q**.

Theorem 3 on asymptotic sums then has an evident version for arbitrary d (it suffices to carry out asymptotic sums of the factors at diag $(\Phi^j, 0)$ separately).

We do not formulate all relations for arbitrary d explicitly but tacitly use them below. Let us only mention that $g(z,\zeta) \in \mathcal{R}_G^{\mu,d}(U \times \mathbb{R}^{n-2}; \mathbf{g}; \mathbf{w})_{\mathbf{P},\mathbf{Q}}$ is an operator-valued symbol in the sense of relations (61) and (62), respectively, for all $s > d - \frac{1}{2}$.

3. Mellin symbols

3.1. Parameter-dependent operators on the interval

If X is any C^∞ manifold with boundary, we have the space $\mathcal{B}^{\mu,d}(X; \mathbf{v})$ of pseudo-differential boundary value problems of order $\mu \in \mathbb{Z}$ and type $d \in \mathbb{N}$, that are continuous operators

$$\mathcal{A}: \begin{array}{c} H^s_{\mathrm{comp}}(X,E) \\ \oplus \\ H^{s-\frac{1}{2}}_{\mathrm{comp}}(\partial X, J_-) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}_{\mathrm{loc}}(X,F) \\ \oplus \\ H^{s-\frac{1}{2}-\mu}_{\mathrm{loc}}(\partial X, J_+) \end{array} \tag{65}$$

for all $s \in \mathbb{R}$, $s > d - \frac{1}{2}$. Here E, F and J_-, J_+ are smooth complex vector bundles on X and ∂X, respectively, and $\mathbf{v}$ is the abbreviation for the tuple of bundles. For simplicity, in our application we assume E and F to be the trivial bundles of fibre dimension 1; then E and F are omitted everywhere. In addition, because of our assumptions in boundary conditions (and then also in potential conditions) we replace in our context the bundles J_-, J_+ by direct sums, and, accordingly, the Sobolev spaces by direct sums of spaces, where the smoothness indices in the components may be different. For the moment we are mainly interested in the case that X is simply the interval $I = [0, \pi]$. Then ∂X consists of two points $\{0\}$ and $\{\pi\}$ and the Sobolev spaces at the boundary as they occur in (65) are to be replaced by finite-dimensional spaces. Because of boundary and potential conditions with respect to $\{0\}$ and $\{\pi\}$ the latter spaces are direct sums

$$\mathbb{C}^{M_+} \bigoplus \mathbb{C}^{M_-} \quad \text{and} \quad \mathbb{C}^{N_+} \bigoplus \mathbb{C}^{N_-},$$

respectively, where M_+, N_+ belong to $\{0\}$ and M_-, N_- to $\{\pi\}$. Because of the nature of our applications we may content ourselves with the case $M := M_+ = M_-$ and $N := N_+ = N_-$, because we start from mixed problems for elliptic differential operators with the same number of boundary conditions on both sides. This equality then remains preserved in all steps of the calculus. In other words, we are interested in operators of the class $\mathcal{B}^{\mu,d}(I; \mathbf{w})$, where $\mathbf{w}$ abbreviates the information on the dimensions M, N; we set in this case $\mathbf{w} = (M, N)$. For purposes below, we also need the parameter-dependent analogue of these spaces, namely

$$\mathcal{B}^{\mu,d}(I; \mathbf{w}; \mathbb{R}^q) \tag{66}$$

for a space of parameters $\mathbb{R}^q \ni \eta$. The space (66) is Fréchet in a natural way and we then also have the spaces

$$C^\infty(U, \mathcal{B}^{\mu,d}(I; \mathbf{w}; \mathbb{R}^q))$$

for any open set $U \subseteq \mathbb{R}^p$. For the applications below we generalise (66) to the case of matrix-valued orders, where we replace μ by $\mu = (\mu_{lk})_{l=0,\dots,2N,k=0,\dots,2M}$. The space $\mathcal{B}^{\mu,d}(I; \mathbf{w}; \mathbb{R}^q)$ is defined to be the set of all block matrix - valued operators $\mathcal{A} = (\mathcal{A}_{lk})_{l=0,\dots,2N,k=0,\dots,2M}$, where ord $\mathcal{A}_{lk} = \mu_{lk}$ in the sense that $\mathcal{A}_{lk} \in \mathcal{B}^{\mu_{lk},d}(\dots)$. Note that integer orders are only assumed in the upper corners.

3.2. Holomorphic Mellin symbols

DEFINITION 3. *$\mathcal{M}_{\mathcal{O}}^{\mu,d}(I; \mathbf{w}; \mathbb{R}^{n-2})$ for $\mathbf{w} = (M, N)$ denotes the subspace of all $h(w, \zeta) \in \mathcal{A}(\mathbb{C}, \mathcal{B}^{\mu,d}(I; \mathbf{w}; \mathbb{R}^{n-2}))$ such that*

$$h(w, \zeta)\,|_{\Gamma_\beta \times \mathbb{R}^{n-2}} \in \mathcal{B}^{\mu,d}(I; \mathbf{w}; \Gamma_\beta \times \mathbb{R}^{n-2})$$

for every $\beta \in \mathbb{R}$, uniformly in $c \leq \beta \leq c'$ for arbitrary $c \leq c'$.

The space $\mathcal{M}_{\mathcal{O}}^{\mu,d}(I; \mathbf{w}; \mathbb{R}^{n-2})$ is Fréchet in a natural way, and we can also talk about the spaces

$$C^\infty(U, \mathcal{M}_{\mathcal{O}}^{\mu,d}(I; \mathbf{w}; \mathbb{R}^{n-2}))$$

for any open set $U \subseteq \mathbb{R}^p$ (or, similarly, with $\overline{\mathbb{R}}_+ \times U$ in place of U).

An basic tool is then the following result.

THEOREM 4. *For every $f(z, w, \zeta) \in C^\infty(U, \mathcal{B}^{\mu,d}(I; \mathbf{w}; \Gamma_\beta \times \mathbb{R}^{n-2}))$ there exists an $h(z, w, \zeta) \in C^\infty(U, \mathcal{M}_{\mathcal{O}}^{\mu,d}(I; \mathbf{w}; \mathbb{R}^{n-2}))$ such that*

$$h(z, w, \zeta)\,|_{\Gamma_\beta} = f(z, w, \zeta) \mod C^\infty(U, \mathcal{B}^{-\infty,d}(I; \mathbf{w}; \Gamma_\beta \times \mathbb{R}^{n-2}))$$

and h is unique mod $C^\infty(U, \mathcal{M}_{\mathcal{O}}^{-\infty,d}(I; \mathbf{w}; \mathbb{R}^{n-2}))$.

Let

$$h(r, z, w, \zeta) := \tilde{h}(r, z, w, r\zeta) \tag{67}$$

for

$$\tilde{h}(r, z, w, \zeta) \in C^\infty(\overline{\mathbb{R}}_+ \times U, \mathcal{M}_{\mathcal{O}}^{\mu,d}(I; \mathbf{w}; \mathbb{R}^{n-2})). \tag{68}$$

For purposes below we set

$$h_0(r, z, w, \zeta) := \tilde{h}(0, z, w, r\zeta). \tag{69}$$

Choose cut-off functions $\omega(r)$ and $\tilde{\omega}(r)$ and form the following family of pseudo-differential operators

$$p(z, \zeta) := \omega(r[\zeta])r^{-\tilde{\mu}}\mathrm{op}_M^{\gamma-\frac{1}{2}}(h)(z, \zeta)\tilde{\omega}(r[\zeta]), \tag{70}$$

where $r^{-\tilde{\mu}} = (r^{-\tilde{\mu}_{lk}})_{l=0,\dots,2N,k=0,\dots,2M}$ with $\tilde{\mu}_{lk} = \mu_{lk}$ for $l = 1,\dots,2N$, $k = 1,\dots,2M$, $\tilde{\mu}_{00} = \mu_{00}$, $\tilde{\mu}_{l0} = \mu_{l0} - \frac{1}{2}$ for $l = 1,\dots,2N$, $\tilde{\mu}_{0k} = \mu_{0k} + \frac{1}{2}$ for $k = 1,\dots,2M$. Here, $\zeta \to [\zeta]$ is a strictly positive function in $C^\infty(\mathbb{R}^{n-2})$ such that $[\zeta] = |\zeta|$ for $|\zeta| \geq c$ for a constant $c > 0$, and $\mathrm{op}_M^\beta(h)$ for any $\beta \in \mathbb{R}$ denotes the weighted Mellin pseudo-differential operator with respect to the weight $\frac{1}{2} - \beta$, cf. [38].

The operators (70) induce continuous mappings

$$p(z,\zeta) : \mathcal{K}^{\mathbf{s},\gamma}(I^\wedge, \mathbb{R}_+; M) \longrightarrow \mathcal{K}^{\tilde{\mathbf{s}},\delta}(I^\wedge, \mathbb{R}_+; N)$$

for all $s > d - \frac{1}{2}$, $\tilde{\mathbf{s}} := (s - m; (s - m_+^l - \frac{1}{2})_{l=1,\dots,N}, (s - m_-^l - \frac{1}{2})_{l=1,\dots,N})$, where $\mathbf{s}, \gamma, \delta$ are as in Section 2.5, and

$$\mathcal{K}^{\tilde{\mathbf{s}},\delta}(I^\wedge, \mathbb{R}_+; N) := \begin{array}{c} \mathcal{K}^{s-m,\gamma-m}(I^\wedge) \\ \oplus \\ \bigoplus_{l=1}^N \mathcal{K}^{s-m_+^l-\frac{1}{2},\gamma-m_+^l-\frac{1}{2}}(\mathbb{R}_+) \\ \oplus \\ \bigoplus_{l=1}^N \mathcal{K}^{s-m_-^l-\frac{1}{2},\gamma-m_-^l-\frac{1}{2}}(\mathbb{R}_+) \end{array} .$$

Assume for a moment that $\tilde{h} = \tilde{h}(z,\omega,\zeta)$ is independent of r. We then have the following homogeneity:

$$p(z,\lambda\zeta) = \lambda^m \tilde{\chi}_\lambda p(z,\zeta) \chi_\lambda^{-1} \tag{71}$$

for all $\lambda \geq 1$ and all $z \in U$, $|\zeta| \geq c$.

PROPOSITION 3. *Given $h(r,z,w,\zeta)$ in the sense of* (67) *we have*

$$p(z,\zeta) \in S^\mu(U \times \mathbb{R}^{n-2}; \mathcal{K}^{\mathbf{s},\gamma}(I^\wedge, \mathbb{R}_+; M), \mathcal{K}^{\tilde{\mathbf{s}},\delta}(I^\wedge, \mathbb{R}_+; N))$$

with respect to the groups $\{\chi_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\{\tilde{\chi}_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively. In particular, if $\tilde{h}$ is independent of r, the symbol $a(z,\zeta)$ is classical.

Proof. By definition $p(z,\zeta)$ is a block matrix of entries

$$p_{lk}(z,\zeta) := \omega(r[\zeta]) r^{-\tilde{\mu}_{lk}} \mathrm{op}_M^{\gamma-\frac{1}{2}}(h_{lk})(z,\zeta)\tilde{\omega}(r[\zeta]),$$

$l = 0,\dots,2N$, $k = 0,\dots,2M$. The assertion consists of

$$\begin{aligned} &p_{lk}(z,\zeta) \in S^{\mu_{lk}}(U \times \mathbb{R}^{n-2}; \mathcal{K}^{s-\alpha_k,\gamma-\alpha_k}(A_1), \mathcal{K}^{s-\beta_l-m,\gamma-\beta_l-m}(A_2)), \\ &k = 0,\dots,2M,\ l = 0,\dots,2N, \end{aligned}$$

where A_1 and A_2 are $I^\wedge$ or $\mathbb{R}_+$, according to the meaning of indices k, l and α_k, β_l, $k = 0,\dots,2M$, $l = 0,\dots,2N$, are as in Section 2.5. However the latter relations are known. □

With $p(z,\zeta)$ (for the case $z \in \Omega$ with $\Omega \subseteq \mathbb{R}^{n-2}$ open) we associate the homogeneous principal edge symbol

$$\sigma_\wedge^\mu(p)(z,\zeta) := \omega(r|\zeta|)r^{-\tilde{\mu}}\mathrm{op}_M^{\gamma-\frac{1}{2}}(h_0)(z,\zeta)\tilde{\omega}(r|\zeta|)$$

for all $(z,\zeta) \in T^*\Omega \setminus 0$, and we get a family of maps

$$\sigma_\wedge^\mu(p)(z,\zeta) : \mathcal{K}^{\mathbf{s},\gamma}(I^\wedge, \mathbb{R}_+; M) \longrightarrow \mathcal{K}^{\tilde{\mathbf{s}},\delta}(I^\wedge, \mathbb{R}_+; N)$$

for all $s > d - \frac{1}{2}$, where

$$\sigma_\wedge^\mu(p)(z,\lambda\zeta) = \lambda^m \tilde{\chi}_\lambda \sigma_\wedge^\mu(p)(z,\zeta)\chi_\lambda^{-1}$$

for all $(z,\zeta) \in T^*\Omega \setminus 0$ and $\lambda \in \mathbb{R}_+$. In addition, for $p(z,\zeta)$ we form a subordinate principal conormal symbol

$$\sigma_M\sigma_\wedge^\mu(p)(z,w) := h_0(0,z,w,0) \tag{72}$$

that we consider for $z \in \Omega$ and $w \in \Gamma_{1-\gamma}$ as a family of operators

$$\sigma_M\sigma_\wedge^\mu(p)(z,w) : \begin{array}{c} H^s(\mathrm{int}\, I) \\ \oplus \\ \mathbb{C}^M \oplus \mathbb{C}^M \end{array} \longrightarrow \begin{array}{c} H^{s-m}(\mathrm{int}\, I) \\ \oplus \\ \mathbb{C}^N \oplus \mathbb{C}^N \end{array} \tag{73}$$

for all $s > d - \frac{1}{2}$.

As an example we want to express the local expressions (5),(7) of mixed problems for differential operators in terms of operator-valued Mellin symbols and operator-valued amplitude functions. Without loss of generality we assume the coefficients $a_{k\beta}(r,z)$ and $b^j_{\pm,k\beta}(r,z)$, $j = 1,\dots,N$, to be independent of r for large r, and we let z vary on an open set $\Omega \subseteq \mathbb{R}^{n-2}$. We then form the column vector

$$f(r,z,w,\zeta) = \begin{pmatrix} \sum_{k+|\beta|\le m} a_{k\beta}(r,z)w^k(r\zeta)^\beta \\ \left(\mathrm{r}^\pm \sum_{k+|\beta|\le m^j_\pm} b^j_{\pm,k\beta}(r,z)w^k(r\zeta)^\beta\right)_{j=1,\dots,N} \end{pmatrix} \tag{74}$$

that equals $h(r,z,w,\zeta)$ in the sense of notation of Theorem 4.

PROPOSITION 4. *Set*

$$a(z,\zeta) := r^{-\mathbf{m}}\mathrm{op}_M^{\gamma-\frac{1}{2}}(f)(z,\zeta),$$

with $f(r,z,w,\zeta)$ *being given by* (74) *and* $r^{-\mathbf{m}} = \mathrm{diag}\,(r^{-m}, (r^{-m^j_\pm})_{j=1,\dots,N})$. *Then*

$$a(z,\zeta) : \mathcal{K}^{s,\gamma}(I^\wedge) \longrightarrow \mathcal{K}^{\tilde{\mathbf{s}},\delta}(I^\wedge, \mathbb{R}_+; N),$$

belongs to $S^\mu(\Omega \times \mathbb{R}^{n-2}; \mathcal{K}^{s,\gamma}(I^\wedge), \mathcal{K}^{\tilde{\mathbf{s}},\delta}(I^\wedge, \mathbb{R}_+; N))$, $\mu = \begin{pmatrix} m \\ (m^j_\pm + \frac{1}{2})_{j=1,\dots,N} \end{pmatrix}$.

Note that when the coefficients $a_{k\beta}$ and $b^j_{\pm,k\beta}$ are independent of r, we have

$$a(z,\zeta) \in S^{\mu}_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n-2}; \mathcal{K}^{s,\gamma}(I^\wedge), \mathcal{K}^{\tilde{\mathbf{s}},\tilde{\gamma}}(I^\wedge, \mathbb{R}_+; N)).$$

We have even in this case

$$a(z,\lambda\zeta) = \lambda^m \tilde{\chi}_\lambda a(z,\zeta)\kappa^{\wedge -1}_\lambda$$

for all $(z,\zeta) \in T^*\Omega \setminus 0$ and $\lambda \in \mathbb{R}_+$.

3.3. Smoothing Mellin operators

Parallel to spaces with discrete asymptotics we now define spaces of Mellin symbols with a meromorphic structure.

DEFINITION 4. *Let $\mathbf{As}^d(I;\mathbf{w})$ for $\mathbf{w} = (M,N)$ denote the set of all sequences*

$$R := \{(r_j, n_j, L_j)\}_{j\in\mathbb{Z}}, \tag{75}$$

where $\pi_{\mathbb{C}} R := \{r_j\}_{j\in\mathbb{Z}}$ has the property $\pi_{\mathbb{C}} R \cap \{w : c \le \mathrm{Re}\, w \le c'\}$ finite for every $c \le c'$, $n_j \in \mathbb{N}$, and $L_j \subset \mathcal{B}^{-\infty,d}(I;\mathbf{w})$ is a finite-dimensional subspace for all $j \in \mathbb{N}$.

DEFINITION 5. *The space $\mathcal{M}^{-\infty,d}_R(I;\mathbf{w})$ for $R \in \mathbf{As}^d(I;\mathbf{w})$ is defined to be the set of all*

$$h(w) \in \mathcal{A}(\mathbb{C} \setminus \pi_{\mathbb{C}} R, \mathcal{B}^{-\infty,d}(I;\mathbf{w}))$$

such that

(i) *for every $\pi_{\mathbb{C}} R$- excision function $\chi(w)$ we have*

$$\chi(w)h(w)\,|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta, \mathcal{B}^{-\infty,d}(I;\mathbf{w}))$$

for every $\beta \in \mathbb{R}$, uniformly in $c \le \beta \le c'$ for arbitrary $c \le c'$,

(ii) *$h(w)$ is meromorphic with poles at r_j of multiplicity $n_j + 1$ and Laurent coefficients at $(z - r_j)^{-(k+1)}$ belonging to L_j for all $0 \le k \le n_j$ and all $j \in \mathbb{Z}$.*

REMARK 10. $\mathcal{M}^{-\infty,d}_R(I;\mathbf{w})$ is a nuclear Fréchet space.

REMARK 11. Let $\omega(r), \tilde{\omega}(r)$ be arbitrary cut-off functions and $h(w) \in \mathcal{M}^{-\infty,d}_R(I;\mathbf{w})$ with $\pi_{\mathbb{C}} R \cap \Gamma_{1-\gamma} = \emptyset$. Then

$$\omega(r)\mathrm{op}_M^{\gamma-\frac{1}{2}}(h)\tilde{\omega}(r) : \mathcal{K}^{\mathbf{s},\gamma}(I^\wedge, \mathbb{R}_+; M) \longrightarrow \mathcal{K}^{\infty,\delta}(I^\wedge, \mathbb{R}_+; N) \tag{76}$$

is continuous for $s > d - \frac{1}{2}$. Moreover, for every $\mathbf{P} \in \mathbf{As}([0,\pi], \tilde{\mathbf{g}})$ for $\tilde{\mathbf{g}} = (\gamma, \Theta)$ there exists a $\mathbf{Q} \in \mathbf{As}([0,\pi], \tilde{\mathbf{g}})$ for $\tilde{\mathbf{g}} = (\delta, \Theta')$ such that (76) induces a continuous operator

$$\omega(r)\mathrm{op}_M^{\gamma-\frac{1}{2}}(h)\tilde{\omega}(r) : \mathcal{K}^{\mathbf{s},\gamma}_{\mathbf{P}}(I^\wedge, \mathbb{R}_+; M) \longrightarrow \mathcal{S}^{\delta}_{\mathbf{Q}}(I^\wedge, \mathbb{R}_+; N)$$

for all $s \in \mathbb{R}$, $s > d - \frac{1}{2}$.

PROPOSITION 5. *Let $j \in \mathbb{N}$, $\alpha \in \mathbb{N}^{n-2}$, $|\alpha| \leq j$, and set*

$$p(z,\zeta) := r^{-\tilde{\mu}+j}\omega(r[\zeta])\mathrm{op}_M^{\gamma-\frac{1}{2}}(h)(z)\zeta^\alpha\tilde{\omega}(r[\zeta]),$$

where $r^{-\tilde{\mu}+j} = (r^{-\tilde{\mu}_{lk}+j})_{l=0,\dots,2N,k=0,\dots,2M}$, and we assume $h(z,w) \in C^\infty(U, \mathcal{M}_R^{-\infty,d}(I;\mathbf{w}))$ for some $R \in \mathbf{As}^d(I;\mathbf{w})$ and $\pi_{\mathbb{C}}R \cap \Gamma_{1-\gamma} = \emptyset$. Then we have

$$p(z,\zeta) \in S_{\mathrm{cl}}^{\mu}(U\times\mathbb{R}^{n-2}; \mathcal{K}_{\mathbf{P}}^{\mathbf{s},\gamma}(I^\wedge,\mathbb{R}_+;M), \mathcal{S}_{\mathbf{Q}}^{\delta}(I^\wedge,\mathbb{R}_+;N)) \tag{77}$$

for every $s > d-\frac{1}{2}$ and for every $\mathbf{P} \in \mathbf{As}([0,\pi],\tilde{\mathbf{g}})$ for $\tilde{\mathbf{g}} = (\gamma,\Theta)$ with some resulting $\mathbf{Q} \in \mathbf{As}([0,\pi],\tilde{\mathbf{g}})$ for $\tilde{\mathbf{g}} = (\delta,\Theta')$. Here $\mathbf{m} = (m_{lk})_{l=0,\dots,2N,k=0,\dots,2M}$ with $m_{lk} = m$ for all $l = 0,\dots,N$, $k = 0,\dots,M$.

Proof. By virtue of (37) we prove that

$$p(z,\zeta) \in S_{\mathrm{cl}}^{\mu-j+|\alpha|}(U\times\mathbb{R}^{n-2}; \mathcal{K}_{\mathbf{P}}^{\mathbf{s},\gamma}(I^\wedge,\mathbb{R}_+;M), \mathcal{S}_{\mathbf{Q}}^{\delta}(I^\wedge,\mathbb{R}_+;N)), \tag{78}$$

where $\mu - j + |\alpha| = (\mu_{lk} - j + |\alpha|)_{l=0,\dots,2N,k=0,\dots,2M}$. Now it is clear that it suffices to prove

$$\begin{aligned}\tilde{p}(z,\zeta) &:= r^{-\tilde{\mu}}\omega(r[\zeta])\mathrm{op}_M^{\gamma-\frac{1}{2}}(h)(z)\tilde{\omega}(r[\zeta])\\ &\in S_{\mathrm{cl}}^{\mu}(U\times\mathbb{R}^{n-2}; \mathcal{K}_{\mathbf{P}}^{\mathbf{s},\gamma}(I^\wedge,\mathbb{R}_+;M), \mathcal{S}_{\mathbf{Q}}^{\delta}(I^\wedge,\mathbb{R}_+;N))\end{aligned} \tag{79}$$

(cf. Remark 7).

Let

$$\mathcal{K}_P^{s,\gamma}(I^\wedge) = \varprojlim_{j\in\mathbb{N}} E^j, \qquad \mathcal{K}_{P_{\pm,k}}^{s-n_\pm^k-\frac{1}{2},\gamma-n_\pm^k-\frac{1}{2}}(\mathbb{R}_+) = \varprojlim_{j\in\mathbb{N}} E_{\pm,k}^j,\ k = 1,\dots,M,$$

$$\mathcal{S}_Q^{\gamma-m}(I^\wedge) = \varprojlim_{j\in\mathbb{N}} F^j, \qquad \mathcal{S}_{Q_{\pm,l}}^{\gamma-m_\pm^l-\frac{1}{2}}(\mathbb{R}_+) = \varprojlim_{j\in\mathbb{N}} F_{\pm,l}^j,\ l = 1,\dots,N,$$

(cf. (51),(52)), for sequences of Hilbert spaces E^j, F^j and $E_{\pm,k}^j$, $F_{\pm,l}^j$ with strongly continuous groups of isomorphisms $\kappa_\lambda^\wedge$ and κ_λ, respectively, for all $j \in \mathbb{N}$. Then we have

$$\tilde{p}(z,\zeta) \in C^\infty(U\times\mathbb{R}^{n-2}, \mathcal{L}(\tilde{E}^{m_j}, \tilde{F}^{n_j}))$$

for every $n_j \in \mathbb{N}$ and for some resulting $m_j \in \mathbb{N}$ (here, $\tilde{E}^{m_j}$ is E^j or $E_{\pm,k}^j$ and $\tilde{F}^{n_j}$ is F^j or $F_{\pm,l}^j$). Further

$$\tilde{p}(z,\lambda\zeta) = \tilde{\chi}_\lambda\tilde{p}(z,\zeta)\chi_\lambda^{-1}$$

for all $z \in U$, $|\zeta| \geq c$, $\lambda \geq 1$ with an $c > 0$. This completes the proof. □

4. Construction of elliptic edge conditions

4.1. Elliptic mixed problems

We now return to mixed problems $\mathcal{A}$ for differential operators A with differential boundary conditions $T_\pm$ in the notation of Section 1.1. By virtue of Proposition 4 our operator $\mathcal{A}$ is continuous in the sense of a map (4). This follows from the nature of weighted edge Sobolev spaces, cf. formula (47) and Remark 3, and general continuity results of pseudo-differential operators with operator-valued symbols, cf. Theorem 2.

With $\mathcal{A}$ we have associated the tuple of principal symbols

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_{\partial,\pm}(\mathcal{A}), \sigma_\wedge(\mathcal{A})), \tag{80}$$

where $\sigma_\psi(\mathcal{A}) := \sigma_\psi^m(A)$, cf. formulas (8),(10),(23). In addition, we have the subordinate conormal symbol (30).

In the literature on concrete mixed problems, e.g., for the Laplace operator with the Zaremba problem or the Lamé system with other types of boundary conditions, e.g., jumping oblique derivative conditions, it is customary to call $\mathcal{A}$ *elliptic* if A is $(\sigma_\psi, \sigma_{\partial,\pm})$- elliptic, i.e., $\sigma_\psi^m(A)(x,\xi) \neq 0$ (or $\det \sigma_\psi^m(A)(x,\xi) \neq 0$ in the case of systems) for $(x,\xi) \in T^*X \setminus 0$, and the boundary conditions $T_\pm$ satisfy the Shapiro-Lopatinskij condition on the $\pm$ parts $\operatorname{int} Y_\pm$ which means that (11) are isomorphisms for all sufficiently large s, for $(y,\eta) \in T^*(\operatorname{int} Y_\pm) \setminus 0$.

However, this does not imply, in general, the Fredholm property of the operator (4). To associate with (4) a Fredholm operator we have to pase extra elliptic conditions on the inner boundary $Z = Y_+ \cap Y_-$. This requires an additional assumption on the weight γ, namely, that (30) is a family of isomorphisms for all $w \in \Gamma_{1-\gamma}$ and all $z \in Z$, cf. Proposition 1. Here, $\Gamma_\beta = \{w \in \mathbb{C} : \operatorname{Re} w = \beta\}, \beta \in \mathbb{R}$. By Theorem 1 we then know that the principal edge symbol is a family of Fredholm operators for all $(z,\zeta) \in T^*Z \setminus 0$.

In the following section we construct additional entries of a block matrix

$$\mathcal{M} := \begin{pmatrix} \mathcal{A} & \mathcal{K} \\ \mathcal{T} & \mathcal{Q} \end{pmatrix} : \begin{array}{c} \mathcal{W}^{s,\gamma}(\mathbb{X}) \\ \oplus \\ H^{s-1}(Z, J_-) \end{array} \longrightarrow \begin{array}{c} \mathcal{V}^{s-m,\gamma-m}(\mathbb{X}) \\ \oplus \\ H^{s-1-m}(Z, J_+) \end{array}, \tag{81}$$

where $J_\pm$ are smooth complex vector bundles on Z and $H^r(Z, J_\pm)$ corresponding Sobolev spaces of distributional sections in the respective bundles of smoothness $r \in \mathbb{R}$, while

$$\mathcal{V}^{s-m,\gamma-m}(\mathbb{X}) := \begin{array}{c} \mathcal{W}^{s-m,\gamma-m}(\mathbb{X}) \\ \oplus \\ \oplus_{j=1}^N \mathcal{W}^{s-m_+^j-\frac{1}{2},\gamma-m_+^j-\frac{1}{2}}(\operatorname{int} Y_+) \\ \oplus \\ \oplus_{j=1}^N \mathcal{W}^{s-m_-^j-\frac{1}{2},\gamma-m_-^j-\frac{1}{2}}(\operatorname{int} Y_-) \end{array}.$$

4.2. Additional edge conditions and the Fredholm property

To summarize the information so far, we assume the mixed problem $\mathcal{A}$ to be $(\sigma_\psi, \sigma_{\partial,\pm})$-elliptic, and we choose a weight $\gamma \in \mathbb{R}$ such that (30) is a family of isomorphisms for all $z \in Z$ and $w \in \Gamma_{1-\gamma}$ (that is also assumed to be possible and guaranteed in many concrete examples). We then have our family of Fredholm operators

$$\sigma_\wedge(\mathcal{A})(z,\zeta) : \mathcal{K}^{s,\gamma}(I^\wedge) \longrightarrow \mathcal{L}^{s-m,\gamma-m}(I^\wedge) \tag{82}$$

for $(z,\zeta) \in T^*Z \setminus 0$, $s > m_\pm^j + \frac{1}{2}$ for all $j = 1, \dots, N$, where

$$\mathcal{L}^{s-m,\gamma-m}(I^\wedge) := \begin{array}{c} \mathcal{K}^{s-m,\gamma-m}(I^\wedge) \\ \oplus \\ \bigoplus_{j=1}^N \mathcal{K}^{s-m_+^j-\frac{1}{2},\gamma-m_+^j-\frac{1}{2}}(\mathbb{R}_+) \\ \oplus \\ \bigoplus_{j=1}^N \mathcal{K}^{s-m_-^j-\frac{1}{2},\gamma-m_-^j-\frac{1}{2}}(\mathbb{R}_+) \end{array}.$$

By virtue of the homogeneity (25) it suffices to assume $|\zeta| = 1$, i.e., $(z,\zeta) \in S^*Z$, where S^*Z is the unit cosphere bundle induced by T^*Z. Since Z is compact, also S^*Z is a compact topological space.

As is well-known the dimensions of $\ker \sigma_\wedge(\mathcal{A})(z,\zeta)$ and $\operatorname{coker} \sigma_\wedge(\mathcal{A})(z,\zeta)$ are not necessarily constant with respect to $(z,\zeta) \in S^*Z$. However, using the theory of elliptic boundary value problems on the infinite cone, cf. Kapanadze and Schulze [18, Chapter 3], we have the following result:

PROPOSITION 6. *There is an $l_- \in \mathbb{N}$ and a map*

$$k : \mathbb{C}^{l_-} \longrightarrow \begin{array}{c} C_0^\infty(\mathbb{R}_+ \times I) \\ \oplus \\ \bigoplus_{j=1}^N C_0^\infty(\mathbb{R}_+) \\ \oplus \\ \bigoplus_{j=1}^N C_0^\infty(\mathbb{R}_+) \end{array}$$

such that

$$a(z,\zeta) := (\sigma_\wedge(\mathcal{A})(z,\zeta) \quad k) : \begin{array}{c} \mathcal{K}^{s,\gamma}(I^\wedge) \\ \oplus \\ \mathbb{C}^{l_-} \end{array} \longrightarrow \mathcal{L}^{s-m,\gamma-m}(I^\wedge) \tag{83}$$

is a family of surjective operators for all $(z,\zeta) \in S^*Z$ and all $s > m_\pm^j + \frac{1}{2}$, $j = 1, \dots, N$.

Proof. First we know that kernels and cokernels of (82) are independent of the specific s, cf. [18, Section 1.2.7]. This allows us to fix any sufficiently large $s \in \mathbb{R}$. In this proof let us simply set $K := \mathcal{K}^{s,\gamma}(I^\wedge)$, $L := \mathcal{L}^{s-m,\gamma-m}(I^\wedge)$. There is then a finite-dimensional subspace $\tilde{M} \subset L$ (of dimension l_-) and an isomorphism $\tilde{k} : \mathbb{C}^{l_-} \to \tilde{M}$

such that

$$(\sigma_\wedge(\mathcal{A})(z,\zeta) \quad \tilde{k}) : \begin{array}{c} K \\ \oplus \\ \mathbb{C}^{l_-} \end{array} \longrightarrow L$$

is surjective for all $(z,\zeta) \in S^*Z$. By virtue of the fact that the space $M := C_0^\infty(\mathbb{R}_+ \times I) \oplus \bigoplus_{j=1}^{2N} C_0^\infty(\mathbb{R}_+)$ is dense in L we can approximate $\tilde{k}$ by an isomorphism $k : \mathbb{C}^{l_-} \to M$ such that

$$a(z,\zeta) : \begin{array}{c} K \\ \oplus \\ \mathbb{C}^{l_-} \end{array} \longrightarrow L \tag{84}$$

is also surjective for all $(z,\zeta) \in S^*Z$ (here, we use that the space of surjective operators between Hilbert spaces is open in the operator norm toplogy). Then, since the surjectivity is independent of s, we also get the surjectivity of (83) for all $s > m_\pm^j + \frac{1}{2}$, $j = 1, \ldots, N$. □

The operators of the family (84) are Fredholm and surjective for all $(z,\zeta) \in S^*Z$. Assume, for simplicity, that S^*Z is connected (otherwise, we may argue for the connected components separately). Then $\dim \ker a(z,\zeta) =: l_+$ is a constant, though the directions of $\ker a(z,\zeta)$ smoothly vary in the space $K \oplus \mathbb{C}^{l_-}$. As is well-known, $\ker a := \bigcup \{\ker a(z,\zeta);\ (z,\zeta) \in S^*Z\}$ form a vector bundle L_+ of fibre dimension l_+ on the space S^*Z.

A basic (topological) assumption on our problem is now that (when we choose the above dimension l_- sufficiently large) the bundle L_+ is the pull-back of a bundle J_+ on Z with respect to the canonical projection $\pi : S^*Z \to Z$, $\pi : (z,\zeta) \to z$. In other words, we require $L_+ = \pi^* J_+$. If $[L_+, \mathbb{C}^{l_-}]$ denotes the element in the K-group of S^*Z, represented by the pair of bundles L_+ and $\mathbb{C}^{l_-}$ $(:= S^*Z \times \mathbb{C}^{l_-})$, then the so-called index element

$$\mathrm{ind}_{S^*Z}\sigma_\wedge(\mathcal{A}) := [L_+, \mathbb{C}^{l_-}] \in K(S^*Z)$$

(that is independent of the choice of the above-mentioned map k) is required to be in the image under the pull-back $\pi^* : K(Z) \to K(S^*Z)$. This is an analogue of the well-known topological obstruction for the existence of Shapiro - Lopatinskij elliptic boundary conditions in the standard theory of boundary value problems, cf. Atiyah and Bott [2], Boutet de Monvel [4].

We now construct a homomorphism

$$b : \begin{array}{c} K \\ \oplus \\ \mathbb{C}^{l_-} \end{array} \longrightarrow J_+,$$

i.e., a smooth (z,ζ) - dependent family of linear maps $b(z,\zeta) : \begin{array}{c} K \\ \oplus \\ \mathbb{C}^{l_-} \end{array} \to J_{+,z}$ for every $(z,\zeta) \in S^*Z$ (with $J_{+,z}$ being the fibre of J_+ over the point z) of the form

$$b := b_0 P,$$

where P is the family of orthogonal projections $P(z,\zeta) : \begin{matrix} K \\ \oplus \\ \mathbb{C}^{l-} \end{matrix} \to \ker a(z,\zeta)$ (with respect to any fixed choice of a scalar product in $\begin{matrix} K \\ \oplus \\ \mathbb{C}^{l-} \end{matrix}$) and $b_0 : \ker a \to J_+$ an arbitrary isomorphism. From the results of [38] or [18] we know that the elements of $\ker a(z,\zeta)$ are vectors of the form $(u(r,\varphi), c)$, where $u(r,\varphi) \in \mathcal{S}^{\gamma}_{P}(I^{\wedge})$ for some discrete asymptotic type $P \in \mathrm{As}([0,\pi],(\gamma,(-\infty,0]))$, (dependent on z), and $c \in \mathbb{C}^{l-}$. We may forget about the specific P when we are only interested in the nature of additional conditions to be constructed here, but identify $u(r,\varphi)$ with an element in $\mathcal{S}^{\gamma}_{P_0}(I^{\wedge})$, where $P_0 \in \mathrm{As}([0,\pi],(\gamma,(-\varepsilon,0]))$ encodes flatness of order $\varepsilon > 0$ with respect to the weight γ, independent of $z \in Z$. Let $\Omega \subseteq \mathbb{R}^{n-2}$ be an open set such that $\ker a(z,\zeta)$ is trivial over Ω (recall that this is always the case when Ω is a ball or any other contractible open set). Choosing a base

$$\begin{pmatrix} u_j(r,\varphi;z,\zeta) \\ c_j(z,\zeta) \end{pmatrix}_{j=1,\dots,l_+},$$

of sections in $\ker a(z,\zeta)\,|_{\Omega}$ our map $b(z,\zeta)$ can be written in the following form:

$$b(z,\zeta)\begin{pmatrix} u \\ d \end{pmatrix} = $$
$$b_0(z,\zeta)\sum_{j=1}^{l_+}\left(\begin{pmatrix} u(r,\varphi) \\ d \end{pmatrix}, \begin{pmatrix} u_j(r,\varphi;z,\zeta) \\ c_j(z,\zeta) \end{pmatrix}\right)_{K\oplus\mathbb{C}^{l-}} \begin{pmatrix} u_j(r,\varphi;z,\zeta) \\ c_j \end{pmatrix}$$

for $u \in K,\ d \in \mathbb{C}^{l-}$.

We have constructed in this way a family of isomorphisms

$$\mathbf{a}_1(z,\zeta) = \begin{pmatrix} \sigma_{\wedge}(\mathcal{A})(z,\zeta) & k \\ b_1(z,\zeta) & b_2(z,\zeta) \end{pmatrix} : \begin{matrix} K \\ \oplus \\ \mathbb{C}^{l-} \end{matrix} \longrightarrow \begin{matrix} L \\ \oplus \\ J_{+,z} \end{matrix} \tag{85}$$

smoothly dependent on $(z,\zeta) \in S^*Z$. Here, $b(z,\zeta) = (b_1(z,\zeta), b_2(z,\zeta))$.

The next step is to extend (85) from S^*Z to $T^*Z\setminus 0$ to a family of isomorphisms

$$\mathbf{a}(z,\zeta) : \begin{matrix} K \\ \oplus \\ \mathbb{C}^{l-} \end{matrix} \longrightarrow \begin{matrix} L \\ \oplus \\ J_{+,z} \end{matrix} \tag{86}$$

such that

$$\mathbf{a}(z,\lambda\zeta) = \lambda^m \tilde{\chi}_\lambda \mathbf{a}(z,\zeta)\hat{\kappa}_\lambda^{-1}, \tag{87}$$

for all $\lambda \in \mathbb{R}_+$, $\zeta \neq 0$, where $\hat{\kappa}_\lambda = \operatorname{diag}(\kappa_\lambda^\wedge, \mathrm{id}_{\mathbb{C}^{l_-}})$. In other words, we set

$$\mathbf{a}(z,\zeta) := |\zeta|^m \tilde{\chi}_{|\zeta|} \mathbf{a}_1(z, \frac{\zeta}{|\zeta|}) \hat{\kappa}_{|\zeta|}^{-1}. \tag{88}$$

Let us write

$$\mathbf{a}(z,\zeta) = \begin{pmatrix} \sigma_\wedge(\mathcal{A})(z,\zeta) & k_{(m)}(\zeta) \\ t_{(m)}(z,\zeta) & q_{(m)}(z,\zeta) \end{pmatrix}. \tag{89}$$

We then set

$$t(z,\zeta) := \chi(\zeta) t_{(m)}(z,\zeta)\omega, \;\; k(\zeta) := \omega\chi(\zeta) k_{(m)}(\zeta), \;\; q(z,\zeta) := \chi(\zeta) q_{(m)}(z,\zeta).$$

Here $\chi(\zeta)$ is an excision function and $\omega(r)$ a cut-off function. Then

$$g(z,\zeta) := \begin{pmatrix} 0 & k(\zeta) \\ t(z,\zeta) & q(z,\zeta) \end{pmatrix} \tag{90}$$

is an operator-valued symbol in the sense

$$g(z,\zeta) \in S^{\mu}_{\mathrm{cl}} \left(\Omega \times \mathbb{R}^{n-2}; \begin{array}{c} \mathcal{K}^{s,\gamma}(I^\wedge) \\ \oplus \\ \mathbb{C}^{l_-} \end{array}, \begin{array}{c} \mathcal{L}^{s-m,\gamma-m}(I^\wedge) \\ \oplus \\ \mathbb{C}^{l_+} \end{array} \right),$$

where $J_+ \;|_\Omega = \Omega \times \mathbb{C}^{l_+}$ is the chosen trivialisation of J_+ over Ω and $\mu = (\mu_{lk})_{l=0,\dots,2N+1,k=0,1}$, where $\mu_{lk} = \gamma_k - \delta_l$ with $\gamma_0 = \gamma$, $\gamma_1 = \gamma - 1$ and the tuple $((\delta_l)_{l=0,\dots,2N}, \delta_{2N+1})$ as in Section 2.5.

Writing $g(z,\zeta) = (g_{ij}(z,\zeta))$ we then form the block matrix

$$\mathcal{G} := \mathrm{Op}_z(g) = (\mathrm{Op}_z(g_{ij}))$$

of pseudo-differential operators with our operator-valued symbol (90). This refers first to a fixed $\Omega \subseteq \mathbb{R}^{n-2}$ corresponding to any chart $\chi_j : V_j \to \Omega$ on Z, where $V_j := U_j \cap Z$, $j = 1, \dots, L$, cf. the beginning of Section 2.3. Let $\{\psi_1, \dots, \psi_L\}$ denote a partition of unity on Z subordinate to $\{V_1, \dots, V_L\}$. Furthermore, let $\{\tilde{\psi}_1, \dots, \tilde{\psi}_L\}$ be functions $\tilde{\psi}_j \in C_0^\infty(V_j)$ such that $\psi_j \tilde{\psi}_j = \psi_j$ for all $j = 1, \dots, L$.

Let $k_j(\zeta)$ denote the above-mentioned symbol $k(\zeta)$ on Ω that belongs to the chart $\chi_j : V_j \to \Omega$. We then have the pull-back of the pseudo-differential operator $\mathrm{Op}_z(k_j)$ to V_j, namely $(\chi_j^{-1})_* \mathrm{Op}_z(k_j)$. We then set

$$\mathcal{K} := \sum_{j=1}^{L} \psi_j \{(\chi_j^{-1})_* \mathrm{Op}_z(k_j)\} \tilde{\psi}_j$$

and obtain our potential entry

$$\mathcal{K} : H^{s-1}(Z, J_-) \longrightarrow \mathcal{V}^{s-m,\gamma-m}(\mathbb{X}),$$

where $J_- = Z \times \mathbb{C}^{l_-}$. Similarly, let $t_j(z,\zeta)$ and $q_j(z,\zeta)$ denote the symbols belonging to the chart $\chi_j : V_j \to \Omega$. We then set

$$(\mathcal{T} \quad \mathcal{Q}) = \left(\sum_{j=1}^{L} \psi_j \{(\chi_j^{-1})_* \mathrm{Op}_z(t_j)\} \tilde{\psi}_j \quad \sum_{j=1}^{L} \psi_j \{(\chi_j^{-1})_* \mathrm{Op}_z(q_j)\} \tilde{\psi}_j \right)$$

and obtain in this way the second row of our block matrix (81), namely

$$(\mathcal{T} \quad \mathcal{Q}) : \begin{array}{c} \mathcal{W}^{s,\gamma}(\mathbb{X}) \\ \oplus \\ H^{s-1}(Z, J_-) \end{array} \longrightarrow H^{s-1-m}(Z, J_+).$$

Clearly, in the pull-backs of operators under χ_j we have tacitly integrated the cocycle of the bundle J_+. An important property of our construction is that the operators

$$\begin{pmatrix} 0 & \mathcal{K} \\ \mathcal{T} & \mathcal{Q} \end{pmatrix} : \begin{array}{c} \mathcal{W}^{s,\gamma}(\mathbb{X}) \\ \oplus \\ H^{s-1}(Z, J_-) \end{array} \longrightarrow \begin{array}{c} \mathcal{V}^{s-m,\gamma-m}(\mathbb{X}) \\ \oplus \\ H^{s-1-m}(Z, J_+) \end{array}$$

only change by compact operators when we change the excision function χ, the cut-off function ω or the charts and the functions ψ_j, $\tilde{\psi}_j$.

THEOREM 5. *If the additional conditions $\mathcal{K}, \mathcal{T}$ and $\mathcal{Q}$ to the operator $\mathcal{A}$ are chosen in the above-mentioned way, the operator $\mathcal{M}$ is elliptic in the sense of edge-boundary value problems and hence* (81) *is Fredholm for all $s \in \mathbb{R}$, $s - m_\pm^j - \frac{1}{2} > 0$, $j = 1, \ldots, N$.*

Proof. Our construction of elliptic edge conditions $\mathcal{K}, \mathcal{T}$ and $\mathcal{Q}$ to $\mathcal{A}$ has reached a variant of the calculus of boundary value problems on a manifold with edges. In fact, an inspection of the proof of [18, Theorem 4.5.11] shows that we can generalise the arguments to the present case of Douglis-Nirenberg orders. □

REMARK 12. The difference of the situation considered in [18, Chapters 4,5] and here lies in the fact that we consider the "realistic" orders from the problem, while those in [18] are thought to be obtained by an order reduction. The construction of order reducing objects within our operator spaces is a rather voluminous program that will be carried out in a future paper.

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