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INTEGER POWERS OF SOME UNBOUNDED LINEAR OPERATORS ON P-ADIC HILBERT SPACES

Abstract. We introduce and examine integer powers of the diagonal operators within p-adic framework. The latter enables us to consider integer powers of some particular unbounded linear operators on the so-called p-adic Hilbert space $\mathbb{E}_\omega$. The product and algebraic sum of those linear operators will be discussed.

1. Introduction

We initiate and examine integer powers of the (possibly unbounded) diagonal operators on the so-called p-adic Hilbert space $\mathbb{E}_\omega$ (see [10], [11], and [3]). For that, we first give and recall the required background on author's recent work related to the formalism of unbounded linear operators in the p-adic setting [5]. Next, we shall be dealing with integer powers of the diagonal operators, their product and algebraic sums, and use the definition of integer powers of diagonal operators in order to deal with integer powers to some particular unbounded linear operators on $\mathbb{E}_\omega$. However, let us mention that our objective in the coming years remains to introduce fractional powers of densely defined closed unbounded linear operators on $\mathbb{E}_\omega$ in order to formulate a p-adic analogue of the classical square root problem of Kato (see [6], [7], [8], and [9]). This is actually, the main motivation of this paper.

Let us mention that the p-adic Hilbert space $\mathbb{E}_\omega$ will play a key role throughout the paper. Apart from their intrinsic interests, p-adic Hilbert spaces have found extensive applications in theoretical physics. For more on these and related issues we refer the reader to([10], [11], [3], and [5]) and the references therein.

Let $\mathbb{K}$ be a complete ultrametric valued field. Classical examples of such a field include $\mathbb{Q}_p$ the field of p-adic numbers where $p \geq 2$ is a prime, $\mathbb{C}_p$ the field of p-adic complex numbers, and the field of formal Laurent series([10], [11]).

An ultrametric Banach space $\mathbb{E}$ over $\mathbb{K}$ is said to be a *free Banach space* (see [10], [11], and [3]) if there exists a family $(e_i)_{i\in I}$ (I being an index set) of elements of $\mathbb{E}$ such that each element $x \in \mathbb{E}$ can be written in a unique fashion as

$$x = \sum_{i\in I} x_i e_i, \quad \lim_{i\in I} x_i e_i = 0, \quad \text{and} \quad \|x\| = \sup_{i\in I} |x_i| \|e_i\|.$$

The family $(e_i)_{i\in I}$ is then called an *orthogonal basis* for $\mathbb{E}$, and if $\|e_i\| = 1$, for all $i \in I$, the family $(e_i)_{i\in I}$ is called an *orthonormal basis*. For a detailed description and properties of these spaces, we refer the reader to ([10], [11], [3], and [5]) and the references therein. Up to now, we shall suppose that the index set I is $\mathbb{N}$ the set of all natural numbers.

For a free Banach space $\mathbb{E}$, let $\mathbb{E}^*$ denote its (topological) dual and $B(\mathbb{E})$ the Banach space of all bounded linear operators on $\mathbb{E}$ (see [10], [11], and [3]). Both $\mathbb{E}^*$ and $B(\mathbb{E})$ are equipped with their respective natural norms.

For $(u, v) \in \mathbb{E} \times \mathbb{E}^*$ we define the linear operator $(v \otimes u)$ by setting:

$$\forall x \in \mathbb{E}, \ (v \otimes u)(x) := v(x)\, u = \langle v, x \rangle\, u.$$

It follows that $(v \otimes u) \in B(\mathbb{E})$ and $\|v \otimes u\| = \|v\| \,.\, \|u\|$.

Let $(e_i)_{i \in \mathbb{N}}$ be an orthogonal basis for $\mathbb{E}$. We then define $e_i' \in \mathbb{E}^*$ by

$$x = \sum_{i \in \mathbb{N}} x_i e_i, \quad e_i'(x) = x_i.$$

It turns out that $\|e_i'\| = \dfrac{1}{\|e_i\|}$. Furthermore, each $x' \in \mathbb{E}^*$ can be expressed as a pointwise convergent series: $x' = \sum_{i \in \mathbb{N}} \langle x', e_i \rangle e_i'$. In addition to that, we have that:

$$\|x'\| := \sup_{i \in \mathbb{N}} \frac{|\langle x', e_i \rangle|}{\|e_i\|}.$$

Now let us recall that every bounded linear operator A on $\mathbb{E}$ can be expressed as a pointwise convergent series, that is, there exists an infinite matrix $(a_{ij})_{(i,j) \in \mathbb{N} \times \mathbb{N}}$ with coefficients in $\mathbb{K}$, such that

$$A = \sum_{ij} a_{ij} (e_j' \otimes e_i), \text{ and for any } j \in \mathbb{N}, \ \lim_{i \to \infty} |a_{ij}| \, \|e_i\| = 0. \tag{1}$$

Moreover, for each $j \in \mathbb{N}$, $Ae_j = \sum_{i \in \mathbb{N}} a_{ij}\, e_i$ and its norm is defined by:

$$\|A\| := \sup_{i,j} \frac{|a_{ij}| \, \|e_i\|}{\|e_j\|}.$$

In this paper, we shall make extensive use of the p-adic Hilbert space $\mathbb{E}_\omega$ whose definition is given below. Again, for details, we refer the reader to ([10], [11], and [3]) and the references therein.

Let $\omega = (\omega_i)_{i \in \mathbb{N}}$ be a sequence of non-zero elements in a complete non-Archimedean field $\mathbb{K}$. Define the space $\mathbb{E}_\omega$ by

$$\mathbb{E}_\omega := \left\{ u = (u_i)_{i \in \mathbb{N}} \mid \forall i, \ \ u_i \in \mathbb{K} \text{ and } \lim_{i \to \infty} |u_i| \, |\omega_i|^{1/2} = 0 \right\}.$$

Clearly, $u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega$ if and only if $\lim_{i\to\infty} u_i^2\omega_i = 0$. Actually $\mathbb{E}_\omega$ is an ultrametric Banach space over $\mathbb{K}$ with the norm given by

$$u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega, \quad \|u\| = \sup_{i\in\mathbb{N}} |u_i|\,|\omega_i|^{1/2}.$$

Let us also notice that $\mathbb{E}_\omega$ is a free Banach space (see [10], [11]) and it has a *canonical orthogonal basis*. Namely, $(e_i)_{i\in\mathbb{N}}$, where e_i is the sequence all of whose terms are 0 except the i-th term which is 1, in other words, $e_i = \left(\delta_{ij}\right)_{j\in\mathbb{N}}$, where δ_{ij} is the usual Kronecker symbol. We shall make extensive use of such a canonical orthogonal basis throughout the paper. It should be mentioned that for each i, $\|e_i\| = |\omega_i|^{1/2}$. Now if $|\omega_i| = 1$ we shall refer to $(e_i)_{i\in\mathbb{N}}$ as the *canonical orthonormal basis*.

Let $\langle , \rangle : \mathbb{E}_\omega \times \mathbb{E}_\omega \to \mathbb{K}$ be the $\mathbb{K}$-bilinear form defined by

$$\forall u, v \in \mathbb{E}_\omega,\ u = (u_i)_{i\in\mathbb{N}},\ v = (v_i)_{i\in\mathbb{N}},\ \ \langle u, v\rangle := \sum_{i\in\mathbb{N}} \omega_i\, u_i v_i. \tag{2}$$

Then, $\langle , \rangle$ is a symmetric, non-degenerate form on $\mathbb{E}_\omega \times \mathbb{E}_\omega$ with value in $\mathbb{K}$, and it satisfies the Cauchy-Schwarz inequality:

$$|\langle u, v\rangle| \le \|u\| \,.\, \|v\|, \quad \forall u, v \in \mathbb{E}_\omega.$$

Let us also mention that elements $(e_i)_{i\in\mathbb{N}}$ of the canonical orthogonal basis for $\mathbb{E}_\omega$ satisfy

$$\left\langle e_i, e_j\right\rangle = \omega_i\delta_{ij} = \begin{cases} 0 & \text{if} \quad i \neq j \\ \omega_i & \text{if} \quad i = j. \end{cases}$$

DEFINITION 1. *The space $\mathbb{E}_\omega$ endowed with the bilinear form $\langle , \rangle$ defined in Eq. (2) is called a p-adic Hilbert space.*

It should also be observed that for every bounded linear operator A on $\mathbb{E}_\omega$, the domain $D(A)$ ($D(A) := \{u = (u_i) \in \mathbb{E}_\omega : \lim_{i\to\infty} |u_i|\|Ae_i\| = 0\}$) of A is actually the whole of $\mathbb{E}_\omega$.

It is well-known ([10] and [11]) that one can find bounded linear operators on $\mathbb{E}_\omega$ which do not have adjoint. Similarly, there exist unbounded linear operators on $\mathbb{E}_\omega$ which do not have adjoints (see [5]). Consequently, we denote by $B_0(\mathbb{E}_\omega)$ the space of all bounded linear operators which do have adjoints with respect to the non-degenerate form $\langle , \rangle$ defined in Eq. (2).

Let $\omega = (\omega_i)_{i\in\mathbb{N}}$ be a sequence of nonzero elements in a (complete) non-Archimedean field $\mathbb{K}$ and let $(\mathbb{E}_\omega, \langle , \rangle)$ be the corresponding p-adic Hilbert space. This paper provides a definition of the integer powers of (possibly unbounded) diagonal operators within the p-adic framework, that is, on the p-adic Hilbert spaces $\mathbb{E}_\omega$. As for bounded linear operators on $\mathbb{E}_\omega$, some of the results go along the classical line and others deviate from it. For the most part, the statements of the results are inspired by their classical settings. However their proofs may depend heavily on the ultrametric nature of $\mathbb{E}_\omega$ and the ground field $\mathbb{K}$. We especially emphasis on the integer powers, self-adjointness, product, and algebraic sums of those diagonal operators.

2. Unbounded Linear Operators On $\mathbb{E}_\omega$

Let $\omega = (\omega_i)_{i\in\mathbb{N}}$, $\varpi = (\varpi_i)_{i\in\mathbb{N}}$ be sequences of non-zero elements in a complete non-Archimedean field $\mathbb{K}$, and let $\mathbb{E}_\omega$, $\mathbb{E}_\varpi$ be their corresponding p-adic Hilbert spaces. Suppose that $(e_i)_{i\in\mathbb{N}}$, $(h_j)_{j\in\mathbb{N}}$ are respectively the canonical orthogonal bases associated to the p-adic Hilbert spaces $\mathbb{E}_\omega$ and $\mathbb{E}_\varpi$.

Let $D \subset \mathbb{E}_\omega$ be a subspace and let $A : D \subset \mathbb{E}_\omega \mapsto \mathbb{E}_\varpi$ be a linear transformation. As for bounded linear operators one can decompose A as a pointwise convergent series defined by:

$$A = \sum_{i,j} a_{ij}\, e'_j \otimes h_i \ \text{ and, } \ \forall j \in \mathbb{N}, \ \lim_{i\to\infty} |a_{ij}|\, \|h_i\| = 0.$$

DEFINITION 2. *An unbounded linear operator A from $\mathbb{E}_\omega$ into $\mathbb{E}_\varpi$ is a pair $(D(A), A)$ consisting of a subspace $D(A) \subset \mathbb{E}_\omega$ (called the domain of A) and a (possibly not continuous) linear transformation $A : D(A) \subset \mathbb{E}_\omega \mapsto \mathbb{E}_\varpi$.*

Throughout the paper, we mean by unbounded linear operator on $\mathbb{E}_\omega$, every linear transformation A whose domain $D(A)$ consists of all $u \in \mathbb{E}_\omega$ such that $Au \in \mathbb{E}_\varpi$, that is,

$$(3) \qquad \begin{cases} D(A) := \{u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega : \ \lim_{i\to\infty} |u_i|\, \|Ae_i\| = 0\}, \\[2ex] A = \sum_{i,j\in\mathbb{N}} a_{ij}\, e'_j \otimes h_i, \quad \forall j \in \mathbb{N}, \ \lim_{i\to\infty} |a_{ij}|\, \|h_i\| = 0. \end{cases}$$

We denote the collection of those unbounded linear operators by $U(\mathbb{E}_\omega, \mathbb{E}_\varpi)$. Clearly, $B(\mathbb{E}_\omega, \mathbb{E}_\varpi) \subset U(\mathbb{E}_\omega, \mathbb{E}_\varpi)$.

As mentioned in the introduction, if $A \in B(\mathbb{E}_\omega, \mathbb{E}_\varpi)$ then its domain is the whole of $\mathbb{E}_\omega$. We shall see that one can find elements of $U(\mathbb{E}_\omega, \mathbb{E}_\varpi)$ whose domains differ from $\mathbb{E}_\omega$ (see *Remark 1*). As for bounded linear operators, there exists elements of $U(\mathbb{E}_\omega, \mathbb{E}_\varpi)$ which do not have adjoint (see *Example 1* below). Actually, in the next definition we state conditions which do guarantee the existence of the adjoint. Without lost of generality we shall suppose that $\mathbb{E}_\omega = \mathbb{E}_\varpi$. As usual we denote $U(\mathbb{E}_\omega, \mathbb{E}_\omega)$ by $U(\mathbb{E}_\omega)$.

In what follows, $(\mathbb{K}, |.|)$ denotes a complete non-Archimedean field.

DEFINITION 3. *An operator*

$$\begin{cases} D(A) := \{u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega : \lim\limits_{i\to\infty} |u_i| \, \|Ae_i\| = 0\}, \\ \\ A = \sum\limits_{i,j\in\mathbb{N}} a_{ij} \, e'_j \otimes e_i, \quad \forall j \in \mathbb{N}, \ \lim\limits_{i\to\infty} |a_{ij}| \, \|e_i\| = 0 \end{cases}$$

is said to have an adjoint $A^* \in U(\mathbb{E}_\omega)$ if and only if

$$\lim_{s\to\infty} \left(\frac{|a_{is}|}{|\omega_j|^{1/2}} \right) = 0, \quad \forall i \in \mathbb{N}. \tag{4}$$

In this event, the adjoint A^* of A is uniquely expressed by

$$\begin{cases} D(A^*) := \{v = (v_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega : \lim\limits_{i\to\infty} |v_i| \, \|A^*e_i\| = 0\}, \\ \\ A^* = \sum\limits_{i,j\in\mathbb{N}} a^*_{ij} \, e'_j \otimes e_i, \quad \forall j \in \mathbb{N}, \ \lim\limits_{i\to\infty} |a^*_{ij}| \, |\omega_i|^{\frac{1}{2}} = 0, \end{cases}$$

where $a^*_{ij} = w_i^{-1} w_j a_{j,i}$.

We denote by $U_0(\mathbb{E}_\omega)$, the collection of linear operators in $U(\mathbb{E}_\omega)$ which do have adjoint operators. Clearly, $B_0(\mathbb{E}_\omega) \subset U_0(\mathbb{E}_\omega)$.

EXAMPLE 1. (Unbounded operator with no adjoint). Set $\mathbb{K} = \mathbb{Q}_p$ the field of p-adic numbers endowed with the p-adic norm $|.|$ and let $\omega_i = p^{3i}$ (in the appropriate $\mathbb{Q}_p$) so that $|\omega_i|^{\frac{1}{2}} = p^{-\frac{3}{2}i}$. Define $A = \sum\limits_{i,j} a_{ij} \, (e'_j \otimes e_i)$ by its coefficients:

$$a_{ij} = \begin{cases} p^{-j} & \text{if } i < j \\ 1 & \text{if } i = j \\ p^{-i} & \text{if } i > j. \end{cases} \quad \text{and} \quad |a_{ij}| = \begin{cases} p^{j} & \text{if } i < j \\ 1 & \text{if } i = j \\ p^{i} & \text{if } i > j. \end{cases}$$

We have

PROPOSITION 1. *The linear operator $A = \sum\limits_{i,j} a_{ij} \, (e'_j \otimes e_i)$ defined above is in $U(\mathbb{E}_\omega)$ and does not have an adjoint with respect to the bilinear form defined in Eq. (2).*

Proof. Clearly, $\forall j$, $\lim\limits_{i} |a_{ij}||\omega_i|^{\frac{1}{2}} = \lim\limits_{i>j} p^i \, p^{-\frac{3}{2}i} = 0$, hence A is well-defined. Furthermore, $\forall i, j \in \mathbb{N}$,

$$\lambda_{ij} := \frac{|a_{ij}||\omega_i|^{\frac{1}{2}}}{|\omega_j|^{\frac{1}{2}}} = \begin{cases} p^{\frac{3}{2}(j-i)+j} & \text{if } i < j \\ 1 & \text{if } i = j \\ p^{i+\frac{3}{2}j-\frac{3}{2}i} & \text{if } i > j, \end{cases}$$

and

$$\forall i, j, \quad \lambda_{ij} := \frac{|a_{ij}||\omega_i|^{1/2}}{|\omega_j|^{1/2}} \geq \begin{cases} p^j & \text{if} \quad i < j \\ 1 & \text{if} \quad i = j \\ p^{-\frac{3}{2}i} & \text{if} \quad i > j. \end{cases}$$

Hence $\|A\| := \sup_{i,j} \lambda_{ij} = \infty$, that is, $A \in U(\mathbb{E}_\omega)$. To complete the proof we have to show that $\forall i$, $\lim_j \frac{|a_{ij}|}{|\omega_j|^{\frac{1}{2}}} \neq 0$. Indeed, $\forall i$, $\lim_j \frac{|a_{ij}|}{|\omega_j|^{\frac{1}{2}}} = \lim_{j>i} p^j p^{\frac{3}{2}j} = \infty$, hence the adjoint of A does not exist. □

3. Closed Linear Operators on $\mathbb{E}_\omega$

To deal with the closedness of unbounded operators on $\mathbb{E}_\omega$, one supposes that the characteristic $char(\mathbb{K})$ of the ground field $\mathbb{K}$ is zero (see details in [5]). Note that examples of such fields include $\mathbb{Q}_p$ the field of p-adic numbers.

Let $A \in U(\mathbb{E}_\omega)$. As in the classical setting we define the graph of the linear operator A by

$$\mathcal{G}(A) := \{(x, Ax) \in \mathbb{E}_\omega \times \mathbb{E}_\omega : \ x \in D(A)\}.$$

DEFINITION 4. *An operator $A \in U(\mathbb{E}_\omega)$ is said to be closed if its graph is a closed subspace in $\mathbb{E}_\omega \times \mathbb{E}_\omega$. Similarly, an operator A is said to be closable if it has a closed extension.*

As in the classical setting we characterize the closedness of an operator $A \in U(\mathbb{E}_\omega)$ as follows: $\forall u_n \in D(A)$ such that $\|u - u_n\| \mapsto 0$ and $\|Au_n - v\| \mapsto 0$ ($v \in \mathbb{E}_\omega$) as $n \mapsto +\infty$, then $u \in D(A)$ and $Au = v$.

It is now clear that if $A \in B(\mathbb{E}_\omega)$, it is closed. Indeed since A is bounded, $D(A) = \mathbb{E}_\omega$. Moreover if $x_n \in \mathbb{E}_\omega$ such that $x_n \mapsto x$ on $\mathbb{E}_\omega$, then by the boundedness of A it follows that $Ax_n \mapsto Ax$, that is, $(x_n, Ax_n) \mapsto (x, Ax)$ on $(\mathbb{E}_\omega \times \mathbb{E}_\omega, \|.\|_2)$ (see details on the ultrametric norm $\|.\|_2$ in [5]), hence $\mathcal{G}(A)$ is closed.

We denote the collection of closed linear operators on $\mathbb{E}_\omega$ by $C(\mathbb{E}_\omega)$. In view of the above, $B(\mathbb{E}_\omega) \subset C(\mathbb{E}_\omega)$.

DEFINITION 5. *An operator $A \in U_0(\mathbb{E}_\omega)$ is said to be self-adjoint if $D(A) = D(A^*)$ and $Au = A^*u$ for each $u \in D(A)$.*

The proof of the next theorem can be found in [5].

THEOREM 1. *Let $A \in U_0(\mathbb{E}_\omega)$, then its adjoint A^* is a closed linear operator. In particular if A is self-adjoint, then it is closed.*

Let $A \in U(\mathbb{E}_\omega)$. We define the resolvent set $\rho(A)$ of A as the set of all $\lambda \in \mathbb{K}$

such that the operator $A_\lambda := A - \lambda I$ (I being the identity operator of $\mathbb{E}_\omega$) is one-to-one and that $(A-\lambda I)^{-1} \in B(\mathbb{E}_\omega)$. In that case, the spectrum $\sigma(A)$ of A is defined as the complement of $\rho(A)$ in $\mathbb{K}$.

4. The Diagonal Operator on $\mathbb{E}_\omega$

Let $\omega = (\omega_i)_{i\in\mathbb{N}}$ be a sequence of nonzero terms in $\mathbb{K}$ and let $(\mathbb{E}_\omega, \langle , \rangle)$ be the corresponding p-adic Hilbert space. Define the diagonal operator $A \in U(\mathbb{E}_\omega)$ by:

$$D(A) = \{x = (x_i) \subset \mathbb{K} : \lim_i |\lambda_i||x_i|\|e_i\| = 0\},$$

and

$$Ax = \sum_{i\in\mathbb{N}} \lambda_i x_i e_i, \quad \forall x \in D(A),$$

where $(\lambda_i)_{i\in\mathbb{N}} \subset \mathbb{K}$.

Suppose that $(\lambda_i)_{i\in\mathbb{N}} \subset \mathbb{K}$ is a sequence of nonzero terms satisfying:

$$\lim_{i\to\infty} |\lambda_i| = \infty. \tag{5}$$

PROPOSITION 2. *Suppose that Eq. (5) holds true. Then the operator A is self-adjoint. Furthermore, $\rho(A) = \{\lambda \in \mathbb{K} : \lambda \neq \lambda_i,\ \forall i \in \mathbb{N}\}$, and*

$$\|(A-\lambda)^{-1}\| \leq \frac{1}{\inf_{i\in\mathbb{N}} |\lambda_i - \lambda|}$$

for each $\lambda \in \rho(A)$.

Proof. First of all, let us make sure that the operator A is well-defined. For that, note that $|a_{i,i}| = |\lambda_i|$ and that $|a_{ij}| = 0$ if $i \neq j$, and

$$\lim_i |a_{ij}|\,|\omega_i|^{1/2} = \lim_{i>j} |a_{ij}|\,|\omega_i|^{1/2} = 0,$$

hence A is well-defined. Now, $\dfrac{|a_{ij}|\,|\omega_i|^{1/2}}{|\omega_j|} = |\lambda_i|$ if $i = j$ and 0 if $i \neq j$. It follows that

$$\|A\| := \sup_{i,j} \frac{|a_{ij}|\,|\omega_i|^{1/2}}{|\omega_j|} = \sup_i |\lambda_i| = \infty,$$

hence $A \in U(\mathbb{E}_\omega)$.

Let us show that the adjoint A^* of A does exist. This is actually obvious since $\forall i \in \mathbb{N}$, $\lim_j |a_{ij}|\,|\omega_j|^{-1/2} = \lim_{j>i} |a_{ij}|\,|\omega_j|^{-1/2} = 0$. Now the adjoint A^* is defined by $A^* = \sum_{i,j} b_{ij}\, e'_j \otimes e_i$, where $b_{ij} = \omega_i^{-1}\omega_j a_{j,i} = a_{ij}$ for all $i, j \in \mathbb{N}$. The latter yields $A = A^*$. Notice that A is also closed, by *Theorem 1*.

To complete the proof one needs to compute $\rho(A)$. For that, we have to solve the equation

$$(A - \lambda I)x = y, \tag{6}$$

where $x = \sum_i x_i e_i \in D(A) = D(A - \lambda I)$ and $y = \sum_i y_i e_i \in \mathbb{E}_\omega$.

Considering Eq. (6) on $(e_i)_{i\in\mathbb{N}}$ and using the fact A is self-adjoint it follows that $\forall i \in \mathbb{N},\ (\lambda_i - \lambda).\langle e_i, x\rangle = \langle e_i, y\rangle$. Equivalently, $\forall i \in \mathbb{N}$,

$$(\lambda_i - \lambda).\omega_i x_i = \omega_i y_i. \tag{7}$$

Now if $\forall i \in \mathbb{N},\ \lambda_i \neq \lambda$, Eq. (7) has a unique solution x, moreover,

$$x = (A - \lambda)^{-1} y = \sum_i \frac{y_i}{\lambda_i - \lambda} e_i. \tag{8}$$

Let us show that $x = (A - \lambda)^{-1} y$ given above is well-defined. For that it is sufficient to prove that $\lim_i \frac{|y_i|}{|\lambda_i - \lambda|}\ \|e_i\| = 0$. According to Eq. (5), the sequence $\left(\frac{1}{|\lambda_i - \lambda|}\right)_{i\in\mathbb{N}}$ is bounded, hence $\lim_i \frac{|y_i|}{|\lambda_i - \lambda|}\ \|e_i\| = 0$.

It remains to find conditions on λ so that x defined above belongs to $D(A)$. For that, it is sufficient to find conditions so that:

$$\lim_i \frac{|y_i|}{|\lambda_i - \lambda|} \|Ae_i\| = \lim_i \frac{|\lambda_i|}{|\lambda_i - \lambda|}\ |y_i|\ \|e_i\| = 0.$$

Indeed, since $\lim_i |y_i|\ \|e_i\| = 0$,

$$\begin{aligned} 0 &\leq \lim_i \frac{|y_i|}{|\lambda_i - \lambda|} \|Ae_i\| \\ &\leq \lim_i \frac{|\lambda_i|}{|\,|\lambda_i| - |\lambda|\,|}\ .\ \lim_i |y_i|\ \|e_i\| \\ &= 0. \end{aligned}$$

From Eq. (8) it follows that

$$\begin{aligned} \|(A - \lambda)^{-1} y\| &= \sup_{i\in\mathbb{N}} \frac{|y_i|\|e_i\|}{|\lambda_i - \lambda|} \\ &\leq \|y\|\ .\ \sup_{i\in\mathbb{N}} \frac{1}{|\lambda_i - \lambda|} \\ &\leq \|y\|\ .\ \frac{1}{\inf_{i\in\mathbb{N}} |\lambda_i - \lambda|} \\ &< \infty, \end{aligned}$$

hence $(A-\lambda)^{-1} \in B(\mathbb{E}_\omega)$. And,

$$\|(A-\lambda)^{-1}\| \leq \frac{1}{\inf_{i\in\mathbb{N}} |\lambda_i - \lambda|} < \infty.$$

In summary, $\rho(A) = \mathbb{K} - \{\lambda_i\}_{i\in\mathbb{N}}$. □

REMARK 1. Let us notice that the domain $D(A)$ of the diagonal operator A may not be equal to the whole of $\mathbb{E}_\omega$. To see it, suppose that the ground field $\mathbb{K}$ *contains a square of each of its elements* and choose $\tilde{x} = (\tilde{x}_i)_{i\in\mathbb{N}}$ where $\tilde{x}_i$ is given by: $\tilde{x}_i^2 = \dfrac{1}{\lambda_i^2\, \omega_i}$ for all $i \in \mathbb{N}$. According to the assumption on the field $\mathbb{K}$ it is clear that for all $i \in \mathbb{N}$, $\tilde{x}_i$ lies in $\mathbb{K}$. Now, $\tilde{x} \in \mathbb{E}_\omega$ since

$$\lim_{i\to\infty} |\tilde{x}_i| \|e_i\| = \lim_{i\to\infty} \frac{1}{|\lambda_i|} = 0,$$

by Eq. (5). Meanwhile, one can easily see that $\tilde{x} \notin D(A)$ since

$$\lim_{i\to\infty} |\tilde{x}_i|\, |\lambda_i| \|e_i\| = 1 \neq 0.$$

If $B \in U(\mathbb{E}_\omega)$ is another diagonal operator on $\mathbb{E}_\omega$ defined by,

$$D(B) = \{x = (x_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega : \lim_i |\mu_i||x_i|\|e_i\| = 0\},$$

and

$$Bx = \sum_{i\in\mathbb{N}} \mu_i x_i e_i, \quad \forall x \in D(B),$$

where $(\mu_i)_{i\in\mathbb{N}} \subset \mathbb{K}$, the algebraic sum $A+B$ of A and B is defined by

$$\begin{cases} D(A+B) = D(A) \cap D(B), \\ \\ (A+B)x = Ax + Bx, \end{cases}$$

for all $x \in D(A) \cap D(B)$.

COROLLARY 1. *Under Eq. (5), suppose that $|\mu_i| < |\lambda_i|$ for each $i \in \mathbb{N}$, then $A+B$ is self-adjoint. Furthermore, $\rho(A+B) = \{\lambda \in \mathbb{K} : \lambda \neq \lambda_i + \mu_i,\ \forall i \in \mathbb{N}\}$, and*

$$\|(A+B-\lambda)^{-1}\| \leq \frac{1}{\inf_{i\in\mathbb{N}} |\lambda - (\lambda_i + \mu_i)|}$$

for each $\lambda \in \rho(A+B)$.

Proof. First of all, note that $(A+B)x = \sum_{i\in\mathbb{N}} (\lambda_i + \mu_i) x_i e_i$, for each $x = (x_i)_{i\in\mathbb{N}} \in D(A+B)$, where $D(A+B) = \{x = (x_i)_{i\in\mathbb{N}} : \lim_{i\to\infty} |\lambda_i + \mu_i|\, |x_i|\, \|e_i\| = 0\}$.

Since $|\mu_i| < |\lambda_i|$ for each $i \in \mathbb{N}$ it follows that $|\lambda_i + \mu_i| = |\lambda_i|$ for all $i \in \mathbb{N}$, and so, $D(A+B) = D(A)$. Now, $A+B$ is well-defined since

$$\lim_{i\to\infty} |\lambda_i + \mu_i|\,|x_i|\,\|e_i\| = \lim_{i\to\infty} |\lambda_i|\,|x_i|\,\|e_i\| = 0,$$

by Eq. (5). Now, note that $A+B$ is a diagonal operator with coefficients $\gamma_i = \lambda_i + \mu_i$, where $\lim_{i\to\infty} |\gamma_i| = \lim_{i\to\infty} |\lambda_i| = \infty$, by Eq. (5). So to complete the proof one follows along the same line as in the proof of *Proposition 2*. □

Similarly, the product AB of the diagonal operators A and B is defined by:

$$\begin{cases} D(AB) = \{x \in D(B) : Bx \in D(A)\}, \\ \\ (AB)x = A(Bx), \ \ \forall x \in D(AB). \end{cases}$$

It can be easily checked that, $(AB)x = \sum_{i\in\mathbb{N}} \lambda_i \mu_i x_i e_i$, for each $x = (x_i)_{i\in\mathbb{N}} \in D(AB)$, where $D(AB) = \{x = (x_i)_{i\in\mathbb{N}} : \lim_{i\to\infty} |\lambda_i|\,|\mu_i|\,|x_i|\,\|e_i\| = 0\}$.

We have

COROLLARY 2. *If $\lim_{i\to\infty} |\lambda_i|\,|\mu_i| = \infty$, then the product AB of A and B is self-adjoint. Furthermore, $\rho(AB) = \{\lambda \in \mathbb{K} : \lambda \neq \lambda_i\mu_i, \ \forall i \in \mathbb{N}\}$, and*

$$\|(AB-\lambda)^{-1}\| \leq \frac{1}{\inf_{i\in\mathbb{N}} |\lambda_i\ \mu_i - \lambda|}$$

for each $\lambda \in \rho(AB)$.

5. Integer Powers of Diagonal Operators

Let $(\mathbb{K}, |.|)$ be a complete ultrametric fields and let $\omega = (\omega_i)_{i\in\mathbb{N}} \subset \mathbb{K}$ be a sequence of nonzero elements. Let A be a diagonal linear operator on $\mathbb{E}_\omega$ defined by

$$\begin{cases} D(A) = \{x = (x_i) \subset \mathbb{K} : \displaystyle\lim_{i\to\infty} |\lambda_i||x_i|\|e_i\| = 0\}, \\ \\ Ax = \displaystyle\sum_{i\in\mathbb{N}} \lambda_i x_i e_i, \ \ \forall x = \sum_{i\in\mathbb{N}} x_i e_i \in D(A), \end{cases}$$

where $(\lambda_i)_{i\in\mathbb{N}} \subset \mathbb{K}$ is the so-called corresponding coefficients to A.

For $\mu = (\mu_i)_{i\in\mathbb{N}} \subset \mathbb{K}$, let $\mathbb{J}(\mu)$ denote the collection of $z \in \mathbb{Z}$ such that $\lim_{i\to\infty} |\mu_i{}^z| = \lim_{i\to\infty} |\mu_i|^z = \infty$; ie.

$$\mathbb{J}(\mu) = \{z \in \mathbb{Z} : \lim_{i\to\infty} |\mu_i{}^z| = \lim_{i\to\infty} |\mu_i|^z = \infty\}.$$

REMARK 2. If $\lambda = (\lambda_i)_{i\in\mathbb{N}}$, $\mu = (\mu_i)_{i\in\mathbb{N}}$ are sequences of elements in $\mathbb{K}$, one has the following properties:

(i) if $z, z' \in \mathbb{J}(\lambda)$, then $z + z',\ zz' \in \mathbb{J}(\lambda)$;

(ii) $\mathbb{J}(\lambda) = \mathbb{J}(|\lambda|)$;

(iii) $\mathbb{J}(\lambda + \mu) = \mathbb{J}(\max(|\lambda|, \mu|))$ whenever $|\lambda| \neq |\mu|$;

(iv) $\mathbb{J}(\lambda) \subset \mathbb{J}(\mu)$ whenever $|\mu| \leq |\lambda|$;

(v) $0 \notin \mathbb{J}(\lambda)$ for each λ;

(vi) $\mathbb{J}(0) = \mathbb{Z}^- - \{0\}$, the set of negative integers except zero;

(vii) $\mathbb{J}(\lambda) \cap \mathbb{J}(\lambda^{-1}) = \emptyset$.

DEFINITION 6. *Let $z \in \mathbb{J}(\lambda)$. Define integer powers A^z of the diagonal operator A by:*

$$(9)\qquad \begin{cases} D(A^z) = \{x = (x_i)_{i\in\mathbb{N}} \subset \mathbb{K} : \lim\limits_i |\lambda_i|^z |x_i| \|e_i\| = 0\}, \\ \\ A^z x = \sum\limits_{i\in\mathbb{N}} {\lambda_i}^z x_i e_i, \ \textit{for each}\ x = (x_i)_{i\in\mathbb{N}} \in D(A^z), \end{cases}$$

where $(\lambda_i)_{i\in\mathbb{N}} \subset \mathbb{K}$.

Similarly, we define $A^0 = I$, where I is the identity operator of $\mathbb{E}_\omega$.

EXAMPLE 2. Suppose $\mathbb{K} = \mathbb{Q}_p$ where $p \geq 2$ is a prime number. Consider the diagonal operator A defined by:

$$D(A) = \{x = (x_i)_{i\in\mathbb{N}} \subset \mathbb{Q}_p : \lim_i |\lambda_i| |x_i| \|e_i\| = 0\},$$

and

$$Ax = \sum_{i\in\mathbb{N}} \lambda_i x_i e_i,\ \ \forall x \in D(A),$$

where $\lambda_i = p^{p^i}$ for each $i \in \mathbb{N}$.

It is easy to see that $\mathbb{J}(\lambda) = \mathbb{Z}^- - \{0\}$, that is, the set of all negative integers except zero. In this event, for each $z \in \mathbb{Z} - \{0\}$, one defines A^z by:

$$D(A^z) = \{x = (x_i)_{i\in\mathbb{N}} \subset \mathbb{Q}_p : \lim_i p^{-zp^i} |x_i| \|e_i\| = 0\}$$

and

$$A^z x = \sum_{i\in\mathbb{N}} p^{zp^i} x_i e_i,\ \ \forall x \in D(A).$$

Using previous results, one can easily see that A^z is self-adjoint and that $\rho(A^z) = \{\lambda \in \mathbb{Q}_p : \lambda \neq p^{p^i},\ \ \forall i \in \mathbb{N}\}$.

PROPOSITION 3. *Let $z, z' \in \mathbb{J}(\lambda)$. If A is a diagonal operator with $(\lambda_i)_{i\in\mathbb{N}} \subset \mathbb{K}$ as a corresponding sequence, then*

(i) $A^z.A^{z'} = A^{z+z'}$;

(ii) $(A^z)^{z'} = A^{zz'}$.

Proof. (i) If $x = \sum_{i\in\mathbb{N}} x_i e_i \in D(A^z.A^{z'})$, then $A^z A^{z'} x = \sum_{i\in\mathbb{N}} \lambda_i{}^{z+z'} x_i e_i$. Next, we use the fact that $z, z' \in \mathbb{J}(\lambda)$ yield $z + z' \in \mathbb{J}(\lambda)$ (*Remark* 2(i)).

(ii) Similarly, if $x = \sum_{i\in\mathbb{N}} x_i e_i \in D((A^z)^{z'})$, then $(A^z)^{z'} x = \sum_{i\in\mathbb{N}} \lambda_i{}^{zz'} x_i e_i$. Now since $zz' \in \mathbb{J}(\lambda)$ (*Remark* 2(i)) it is clear that $(A^z)^{z'} = A^{zz'}$.

□

PROPOSITION 4. *If $z \in \mathbb{J}(\lambda)$, then the integer power A^z of the diagonal operator A is self-adjoint. Furthermore, $\rho(A^z) = \{\lambda \in \mathbb{K} : \lambda \neq \lambda_i^z,\ \forall i \in \mathbb{N}\}$, and*

$$\|(A^z - \lambda)^{-1}\| \leq \frac{1}{\inf_{i\in\mathbb{N}} |\lambda_i^z - \lambda|}$$

for each $\lambda \in \rho(A^z)$.

Proof. First of all, note that $A^z x = \sum_{i\in\mathbb{N}} \lambda_i{}^z x_i e_i$, $\forall x = (x_i)_{i\in\mathbb{N}} \in D(A^z)$ with $D(A^z) = \{x = (x_i)_{i\in\mathbb{N}} \subset \mathbb{K} : \lim_{i\to\infty} |\lambda_i|^z\, |x_i|\, \|e_i\| = 0\}$. Since $z \in \mathbb{J}(\lambda)$ it follows that A^z is well-defined. Note that A^z is a diagonal operator corresponding to $\gamma_i = \lambda_i{}^z$ with $\lim_{i\to\infty} |\gamma_i| = \infty$, since $z \in \mathbb{J}(\lambda)$. So to complete the proof one follows along the same line as in the proof of *Proposition 2*. □

PROPOSITION 5. *Let A, B be diagonal operators on $\mathbb{E}_\omega$. If $\lambda = (\lambda_i)_{i\in\mathbb{N}}$, $\mu = (\mu_i)_{i\in\mathbb{N}}$ are respectively the corresponding sequences to the diagonal operators A and B, and if $|\lambda_i| \neq |\mu_i|$ for each $i \in \mathbb{N}$, and if $\mathbb{J}(\lambda) \cap \mathbb{J}(\mu) \cap \mathbb{Z}^+ \neq \emptyset$ ($\mathbb{Z}^+$ being the set of all natural numbers), then*

$$D((A + B)^z) = D(A^z) \cap D(B^z) = D((A + B)^{*z}),$$

for each $z \in \mathbb{J}(\lambda) \cap \mathbb{J}(\mu) \cap \mathbb{Z}^+$.

Proof. Using the argument that $|\lambda_i| \neq |\mu_i|$ for each $i \in \mathbb{N}$ it easily follows that

$$|\lambda_i + \mu_i|^z = \max(|\lambda|^z, |\mu_i|^z) \tag{10}$$

for all $i \in \mathbb{N}$, $z \in \mathbb{Z}^+$.

In particular, Eq. (10) holds for each $z \in \mathbb{J}(\lambda) \cap \mathbb{J}(\mu) \cap \mathbb{Z}^+$. In this event, the operator $(A+B)^z$ is defined by:

$$(A+B)^z x = \sum_{i \in \mathbb{N}} (\lambda_i + \mu_i)^z x_i e_i$$

for each $x = (x_i)_{i \in \mathbb{N}} \in D((A+B)^z)$, where

$$\begin{aligned} D((A+B)^z) &= \{x = (x_i)_{i \in \mathbb{N}} \subset \mathbb{K} : \lim_{i \to \infty} |\lambda_i + \mu_i|^z \, |x_i| \, \|e_i\| = 0\} \\ &= \{x = (x_i)_{i \in \mathbb{N}} \subset \mathbb{K} : \lim_{i \to \infty} \max(|\lambda_i|^z, |\mu_i|^z) \, |x_i| \, \|e_i\| = 0\} \\ &= D(A^z) \cap D(B^z). \end{aligned}$$

It is also clear that $(A+B)^z$ is self-adjoint for each $z \in \mathbb{J}(\lambda) \cap \mathbb{J}(\mu) \cap \mathbb{Z}^+$, and so, $(A+B)^z = (A+B)^{*z}$.

□

6. Integer Powers of Some Particular Unbounded Operators

Let $(\omega_i)_{i \in \mathbb{N}} \subset \mathbb{K}$ be a sequence of nonzero terms and let $(a_{ij})_{i,j \in \mathbb{N}} \subset \mathbb{K}$ be a sequence.

In this section, we examine integer powers of the particular linear operators A defined by

$$(11) \qquad \begin{cases} D(A) = \{x = (x_i)_{i \in \mathbb{N}} \subset \mathbb{K} : \ \lim_{i \to \infty} \left[|x_i| \, |\mu_i| \left(\sup_{j \in \mathbb{N}} |a_{ji}| \|e_j\| \right) \right] = 0\}, \\ \\ Ax := \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \mu_i \, x_i a_{ji} e_j, \ \text{ for each } \ x = (x_i)_{i \in \mathbb{N}} \in D(A), \end{cases}$$

where $(\mu_i)_{i \in \mathbb{N}} \subset \mathbb{K}$ is a given sequence of nonzero terms satisfying

$$(12) \qquad \lim_{i \to \infty} |\mu_i| = \infty.$$

For that, we transform the expression of A so that it can be seen as a diagonal operator in a certain orthogonal base for $\mathbb{E}_\omega$, and next apply the previous results.

Indeed, setting

$$(13) \qquad \forall j \in \mathbb{N}, \quad f_j = \sum_i a_{ij} e_i \ \text{ with } \ \lim_i |a_{ij}| \, \|e_i\| = 0,$$

it is clear that the operator A can be seen as

$$(14) \qquad \begin{cases} D(A) = \{x = (x_i)_{i \in \mathbb{N}} \subset \mathbb{K} : \ \lim_{i \to \infty} |x_i| \, |\mu_i| \, \|f_i\| = 0\}, \\ \\ Ax := \sum_{i \in \mathbb{N}} \mu_i \, x_i f_i, \ \text{ for each } \ x = (x_i)_{i \in \mathbb{N}} \in D(A). \end{cases}$$

Now, to achieve our goal, we have to choose $(f_i)_{i\in\mathbb{N}}$ so that it can be seen as an orthogonal base for $\mathbb{E}_\omega$. In addition to that we shall suppose: there exists a nontrivial isometric linear bijection T such that

$$Te_i = f_i,\ \ \forall i \in \mathbb{N}. \tag{15}$$

In particular $\|e_i\| = \|f_i\|$ for each $i \in \mathbb{N}$.

Throughout the rest of the paper, we suppose that $(f_i)_{i\in\mathbb{N}}$ is an orthogonal base for $\mathbb{E}_\omega$. As a consequence, each $x \in \mathbb{E}_\omega$ can be (uniquely) expressed as

$$x = \sum_{i\in\mathbb{N}} x_i f_i \ \text{ with } \lim_{i\to\infty} |x_i|\|f_i\| = 0,$$

where $\|f_i\| =: |\varpi_i|^{1/2} = |\omega_i|^{1/2},\ \forall i \in \mathbb{N}$, and $\langle f_i, f_j\rangle = \varpi_i\delta_{ij}$ ($(\varpi_i)_{i\in\mathbb{N}} \subset \mathbb{K}$ being a sequence of nonzero terms and δ_{ij} is the classical Kronecker symbol).

Notice that the operator A defined in Eq. (14) is self-adjoint with resolvent $\rho(A) = \{\lambda \in \mathbb{K} : \lambda \neq \mu_i,\ \ \forall i \in \mathbb{N}\}$. Now, let us require that:

$$\lim_{m\mapsto\infty}\left[\frac{\sup_{n\in\mathbb{N}}\left(|a_{nm}|\,|\omega_n|^{1/2}\right)}{|\varpi_m|^{1/2}}\right] = 0. \tag{16}$$

We have

PROPOSITION 6. *Under assumptions Eqs. (12)-(13)-(15), and (16), the operator A is self-adjoint. Furthermore $\rho(A) = \{\lambda \in \mathbb{K} : \lambda \neq \mu_i,\ \ \forall i \in \mathbb{N}\}$, and for each $\mu \in \rho(A)$,*

$$\|(A-\mu)^{-1}\| \leq \frac{\gamma}{\inf_{m\in\mathbb{N}}|\mu_m - \mu|},$$

where $\gamma = \sup_{m\in\mathbb{N}}\left[\frac{\sup_{n\in\mathbb{N}}\left(|a_{nm}|\,|\omega_n|^{1/2}\right)}{|\varpi_m|^{1/2}}\right] < \infty.$

Proof. To prove that A is self-adjoint, one follows along the same line as in the proof of *Proposition 2*. Now let us consider the solvability of the equation

$$Ax - \mu x = y, \tag{17}$$

where $x = \sum_{m\in\mathbb{N}} x_m f_m \in D(A)$ and $y = \sum_{n\in\mathbb{N}} y_n e_n \in \mathbb{E}_\omega$.

From Eq. (17) it follows that $\mu_m \varpi_m x_m - \mu \varpi_m x_m = \sum_{n\in\mathbb{N}} \omega_n \, y_n \, a_{nm}$. And so, for $\mu \neq \mu_m$, $\forall m \in \mathbb{N}$, the coefficients of a solution x to Eq. (17) are given by

$$x_m = \frac{1}{(\mu_m - \mu) \; \varpi_m} \left[\sum_{n\in\mathbb{N}} \omega_n \; y_n \; a_{nm} \right], \quad \forall m \in \mathbb{N}. \tag{18}$$

Since $(f_i)_{i\in\mathbb{N}}$ is an orthogonal basis for $\mathbb{E}_\omega$, it is also clear that $N(A - \mu I) = \{0\}$, where N denotes the kernel. And so, the solution $x = (A - \mu)^{-1} y$ to Eq. (17) is unique. In addition, the coefficients (x_m) given in Eq. (18) are well-defined, by Eq. (16). Now let us show that $x \in D(A - \mu I) = D(A)$. Indeed, from the expression of x_m in Eq. (18), we have:

$$\begin{aligned}
|x_m| \; |\mu_m| \; \|f_m\| &= |x_m| \; |\mu_m| \; |\varpi_m|^{1/2} \\
&= \frac{|\mu_m| \; . \left| \sum_{n\in\mathbb{N}} \omega_n \; y_n \; a_{nm} \right|}{|\varpi_m|^{1/2} \; . \; |\mu_m - \mu|} \\
&\leq \left[\frac{|\mu_m| \; . \; \sup_{n\in\mathbb{N}} \left(|a_{nm}| \; |\omega_n|^{1/2} \right)}{|\mu_m - \mu| \; . \; |\varpi_m|^{1/2}} \right] . \; \|y\| \\
&\leq \left(\frac{|\mu_m|}{|\mu_m - \mu|} \right) \; . \; \left[\frac{\sup_{n\in\mathbb{N}} \left(|a_{nm}| \; |\omega_n|^{1/2} \right)}{|\varpi_m|^{1/2}} \right] . \; \|y\|.
\end{aligned}$$

Passing to the limit ($m \mapsto \infty$) in the previous inequality it follows that

$$\lim_{m\to\infty} |x_m| \; |\mu_m| \; \|f_m\| = 0,$$

by the fact $\lim_{m\to\infty} \left(\frac{|\mu_m|}{|\mu_m - \mu|} \right) = 1$ and Eq. (16), and so $x \in D(A)$.

Similarly, from Eq. (18), one has:

$$
\begin{aligned}
|x_m|\ \|f_m\| &= |x_m|\ |\varpi_m|^{1/2} \\
&= \frac{\left|\sum_{n\in\mathbb{N}} \omega_n\ y_n\ a_{nm}\right|}{|\varpi_m|^{1/2}\ .\ |\mu_m-\mu|} \\
&\leq \left[\frac{\sup_{n\in\mathbb{N}}\left(|a_{nm}|\ |\omega_n|^{1/2}\right)}{|\mu_m-\mu|\ .\ |\varpi_m|^{1/2}}\right]\ .\ \|y\| \\
&= \left(\frac{1}{|\mu_m-\mu|}\right)\ .\ \left[\frac{\sup_{n\in\mathbb{N}}\left(|a_{nm}|\ |\omega_n|^{1/2}\right)}{|\varpi_m|^{1/2}}\right]\ .\ \|y\| \\
&\leq \frac{1}{\inf_{m\in\mathbb{N}}|\mu_m-\mu|}\ .\ \left[\frac{\sup_{n\in\mathbb{N}}\left(|a_{nm}|\ |\omega_n|^{1/2}\right)}{|\varpi_m|^{1/2}}\right]\ .\ \|y\|.
\end{aligned}
$$

Therefore

$$\|(A-\mu)^{-1}\| \leq \frac{\gamma}{\inf\limits_{m\in\mathbb{N}}|\mu_m-\mu|},$$

where $\gamma = \sup\limits_{m\in\mathbb{N}}\left[\frac{\sup_{n\in\mathbb{N}}\left(|a_{nm}|\ |\omega_n|^{1/2}\right)}{|\varpi_m|^{1/2}}\right] < \infty.$

In summary, $\rho(A) = \{\lambda \in \mathbb{K} : \lambda \neq \mu_i,\ \ \forall i \in \mathbb{N}\}$. □

Under previous assumptions, one defines integer powers of A by:

DEFINITION 7. *Let $z \in \mathbb{J}(\mu)$. Define integer powers A^z of the diagonal operator A by:*

$$
(19)\qquad
\begin{cases}
D(A^z) = \{x = (x_i) \subset \mathbb{K} : \lim\limits_i |\mu_i|^z |x_i| \|f_i\| = 0\}, \\[2ex]
A^z x = \sum\limits_{i\in\mathbb{N}} {\mu_i}^z x_i f_i, \quad \text{for each } \ x = (x_i)_{i\in\mathbb{N}} \in D(A^z).
\end{cases}
$$

Similar results as in *Section 5* hold for those type of diagonal operators.

REMARK 3. We complete this paper by mentioning two challenging questions related to powers of linear operators on $\mathbb{E}_\omega$.

(i) In view of our definition (see *Definition 6*) of integer powers of diagonal operators, the point now is how to define integer powers of a general unbounded linear operator A defined by $Ax = \sum_{i,j} a_{ij}(e'_j \otimes e_i)x$ for each $x \in D(A)$?

(ii) In view of integer powers of diagonal operators, it remains to define fractional powers of (diagonal) linear operators on $\mathbb{E}_\omega$. But this seems to depend on the ground field $\mathbb{K}$. Indeed, one should consider ultrametric fields $(\mathbb{K}, |.|)$ having the following property: if $\lambda \in \mathbb{K}$, then $\lambda^q \in \mathbb{K}$ for some $q \in \mathbb{Q}$, and that $|\lambda^q| = |\lambda|^q$. Once such fields are identified, our previous theory should apply in order to deal with fractional powers of diagonal operators on $\mathbb{E}_\omega$. A field satisfying the previous property may be called as a field which has the fractional powers property (FPP).

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