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E. Gover

MAXIMAL POINCARÉ SERIES AND BOUNDS FOR BETTI NUMBERS

Dedicated to Paolo Valabrega on the occasion of his 60 th birthday

Abstract. We give a new proof of an existing theorem concerning maximal growth rates of Betti numbers. Other results that use Poincaré series formulas to give upper bounds for the growth of Betti numbers of finitely generated modules over local rings are then surveyed.

1. Introduction

Consider Notherian local rings $(A, \mathfrak{m}, k)$ and $(B, \mathfrak{n}, k)$, a finitely generated B-module M, and a local morphism $f : A \to B$ inducing an isomorphism of residue class fields $A/\mathfrak{m} \cong B/\mathfrak{n} \cong k$ and through which B and M become finite A-modules. Denote the n^{th} Betti numbers of B and M over A by $b_n^A(B)$ and $b_n^A(M)$ respectively, and the n^{th} Betti number of M over B by $b_n^B(M)$. The corresponding Poincaré series are then the formal power series $P^A(B)$, $P^A(M)$, and $P^B(M)$ whose coefficients are those Betti numbers.

With the additional technical assumption that $\operatorname{Tor}^A(k, B)$ is a k-vector space when considered as a B-module (which is automatically satisfied when f is surjective), an inequality was established by the author and P. Salmon in [9] that relates the coefficients of the Poincaré series:

$$(1_M) \qquad P^B(M) \preceq \frac{P^A(M)}{1 - t(P^A(B) - 1)}.$$

(The symbol $\preceq$ signifies that the inequality is among corresponding coefficients of the power series in the variable t on either side.) In terms of Betti numbers, (1_M) can be expressed as

$$b_n^B(M) + b_{n-1}^B(M) \leq \sum_{j=1}^{n} b_{n-j}^B(M) b_{j-1}^A(B) + b_n^A(M)$$

for $n \geq 1$, and $b_0^B(M) \leq b_0^A(M)$. The relationship is proved by considering the change of rings spectral sequence with

$$E_{p,q}^2 = \operatorname{Tor}_p^B\left(\operatorname{Tor}_q^A(k, B), M\right) \underset{p}{\Rightarrow} \operatorname{Tor}_{p+q}^A(k, M)$$

and then counting dimensions of appropriate k-vector subspaces. The inequality is the natural extension to finitely generated modules of one originally stated by Serre [19] in the special case where $M = k$. In its general form, (1_M) gives implicit bounds for the growth rates of the Betti numbers of M.

Questions concerning the growth rates of Betti numbers for classes of modules over local rings have been considered by several investigators, especially Avramov. In [2], he proved that the Betti numbers of arbitrary finitely generated modules over local rings have at most strong exponential growth. The purpose of the present article is to reprove that and a related result directly from inequality (1_M). Then, with an upper bound for the growth of Betti numbers in place, a review will be given of growth rates at the extremes, namely, for modules over complete intersections and Golod rings.

2. Exponential bounds for the growth rates of Betti numbers

We may assume, using the Cohen structure theorem, that all local rings considered are homomorphic images of regular local rings. That is, we may take the $\mathfrak{n}$-adic completion $\widehat{B}$ of local ring B to be isomorphic to $A/\mathfrak{b}$ where $(A, \mathfrak{m}, k)$ is regular and $\mathfrak{b} \subseteq \mathfrak{m}^2$. As $\widehat{B}$ is flat over B, the homological data including the Betti numbers and Poincaré series will be the same for B and $\widehat{B}$, and the former may be replaced by the latter. The condition $\mathfrak{b} \subseteq \mathfrak{m}^2$, which can always be achieved, assures that A and B have the same embedding dimension: $\operatorname{e.dim} A = \dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim_k(\mathfrak{n}/\mathfrak{n}^2) = \operatorname{e.dim} B$. Bounds for the growth rates of Betti numbers now follow using (1_M).

THEOREM 1. *The Betti numbers of a finitely generated module M over any local ring B have at most termwise exponential growth; that is, for a given module, there exists a constant $\alpha > 1$ (which depends on M and B) and a sequence of positive integers $\{c_n\}$ such that $b_n^B(M) \leq c_n$ for each n, and $c_n \geq \alpha\, c_{n-1}$ for all $n \geq \dim_k(\mathfrak{m}/\mathfrak{m}^2)$. The bounding sequence $\{c_n\}$ also exhibits strong exponential growth; namely, there exist constants $1 < \beta \leq \gamma$ such that $\beta^n < c_n < \gamma^n$ for all $n \geq 2\dim_k(\mathfrak{m}/\mathfrak{m}^2)$, and therefore $b_n^B(M) < \gamma^n$ for those n.*

Proof. Both assertions follow from inequality (1_M) applied to the surjective natural map $f\colon A \to A/\mathfrak{b} = B$ where A is regular and $\mathfrak{b} \subseteq \mathfrak{m}^2$. Define $\{c_n\}$ by setting

$$\sum_{n=0}^{\infty} c_n t^n = \frac{P^A(M)}{1 - t(P^A(B) - 1)}. \tag{2}$$

Then, by (1_M), $b_n^B(M) \leq c_n$ for each n. The next part of the argument parallels Peeva [18, Proposition 5] but in a slightly different context. As A is regular, both the A-free minimal resolutions of M and B, $X^A(M)$ and $X^A(B)$, are finite, so the Euler characteristics of these resolutions vanish implying $t = -1$ is a root of both $P^A(M) = 0$ and $P^A(B) = 0$. Thus, we may write

$$P^A(M) = (1 + t)p_M(t) \quad \text{and} \quad P^A(B) = (1 + t)p_B(t)$$

where $p_M(t)$ and $p_B(t)$ are polynomials in t of degree $< \varepsilon = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$.

When B is not a hypersurface ring (so that $b_1^A(B) \geq 2$ and $a_2 = b_1^A(B) - b_0^A(B) \neq 0$), denominator $1 - t(P^A(B) - 1) = 1 + t - tP^A(B) = (1+t)(1 - tp_B(t))$ has the form

$$1 - t(P^A(B) - 1) = (1+t)(1 - t - a_2t^2 - \cdots - a_rt^r)$$

where $a_n = \sum_{i=1}^{n}(-1)^{i+1}b_{n-i}^A(B)$ for $2 \leq n \leq r \leq \varepsilon$ and $a_2, a_r \neq 0$. It follows that equation (2) can be rewritten as

$$\sum_{n=0}^{\infty} c_n t^n = \frac{p_M(t)}{1 - t - a_2t^2 - \cdots - a_rt^r}$$

from which for each $n \geq \varepsilon$,

(3) $$c_n = c_{n-1} + a_2c_{n-2} + \cdots + a_rc_{n-r}.$$

Localizing at (0) and considering the Euler characteristic of the appropriate exact sequence, $\dim_k\left(\mathrm{Syz}_n^A(B)_{(0)}\right) - a_n = 0$ (where $\mathrm{Syz}_n^A(B)$ is the n^{th} syzygy module of $X^A(B)$), thereby showing for each $n \leq r$ that a_n is a positive integer. Meanwhile, starting from the bottom using (2), $c_n = b_n^A(M) \geq 1$ for $n = 0, 1$ (assuming M is not A-free) and after simplification,

$$c_n = \sum_{j=2}^{n} c_{n-j}b_{j-1}^A(B) + b_n^A(M)$$

for $n \geq 2$, which shows that $c_n \geq 1$ for all $n \leq \varepsilon$. When combined with (3), this shows that $c_n > c_{n-1}$ for all $n \geq \varepsilon$, provided B is not a hypersurface ring.

Once the c_n strictly increase for $n \geq \varepsilon - 1$, set $\alpha = \min\left\{\frac{c_\varepsilon}{c_{\varepsilon-1}}, \frac{c_{\varepsilon+1}}{c_\varepsilon}, \ldots, \frac{c_{2\varepsilon-1}}{c_{2\varepsilon-2}}\right\}$. Then $\alpha > 1$, and for all n with $\varepsilon \leq n \leq 2\varepsilon - 1$, it follows that $c_n \geq \alpha c_{n-1}$. Suppose that for some $n \geq 2\varepsilon$ it has already been shown that $c_j \geq \alpha c_{j-1}$ for all $\varepsilon \leq j \leq n-1$. Then,

$$c_n = c_{n-1} + a_2c_{n-2} + \cdots + a_rc_{n-r} \geq \alpha c_{n-2} + a_2\alpha c_{n-3} + \cdots + a_r\alpha c_{n-r-1} = \alpha c_{n-1}.$$

The inequality $c_n \geq \alpha c_{n-1}$ now holds for all $n \geq \varepsilon$ by induction, which establishes the first assertion of the theorem for all except hypersurface rings. For those rings, $\sum_{n=0}^{\infty} c_nt^n = p_M(t)/(1-t)$, in which case $c_n = c_{n-1}$ for $n \geq \varepsilon$. Thus, the Betti numbers of finitely generated modules over hypersurface rings are bounded and hence eventually constant by a result of Eisenbud [4].

Strong exponential bounds for the Betti numbers over non-hypersurface rings can now be found using (3). To find an upper bound $\gamma > 1$ such that $c_n < \gamma^n$ for $n \gg 0$, start by considering (3) with $n = 2\varepsilon$:

$$c_{2\varepsilon} = c_{2\varepsilon-1} + a_2c_{2\varepsilon-2} + \cdots + a_rc_{2\varepsilon-r}.$$

Set $\lambda = 1 + \sum_{j=2}^{r} a_j$. Then, because $c_{2\varepsilon-1} > \cdots > c_{2\varepsilon-r}$, it follows that $c_{2\varepsilon} < \lambda c_{2\varepsilon-1}$. Repeating the calculation using $n = 2\varepsilon + 1$ gives $c_{2\varepsilon+1} < \lambda c_{2\varepsilon} < \lambda^2 c_{2\varepsilon-1}$.

By induction, $c_{2\varepsilon+s} < \lambda^{s+1}c_{2\varepsilon-1}$ for any $s \geq 0$. Moreover, $c_{2\varepsilon-1} > 1$ so it may be rewritten as $c_{2\varepsilon-1} = \mu^{2\varepsilon-1}$ with $\mu > 1$. Thus, $c_{2\varepsilon+s} < \lambda^{s+1}\mu^{2\varepsilon-1}$ for any $s \geq 0$. If $\gamma = \max\{\lambda, \mu\}$, then $c_{2\varepsilon+s} < \gamma^{s+1}\gamma^{2\varepsilon-1} = \gamma^{2\varepsilon+s}$ with $\gamma > 1$. In other words, $c_n < \gamma^n$ for all $n \geq 2\varepsilon$.

Finding a lower bound β for which $\beta^n < c_n$ is even easier and so that argument will be skipped. Note that the lower bound holds for $n \geq \varepsilon + 1$. Taken together, when $n \geq 2\varepsilon$, both bounds apply and $\beta^n < c_n < \gamma^n$. □

REMARK 1. In [2], Avramov proved the second part of the theorem in a different way by applying a theorem of Fatou, which says that when $\sum_{n\geq 0} c_n t^n$ represents a rational function and the coefficients c_n are eventually non-negative and non-decreasing, then the c_n exhibit strong exponential growth if and only if the radius of convergence of the power series is less than 1. For the rational function considered in equation (2) of Theorem 1, the denominator

$$1 - t(P^A(B) - 1) = 1 + t - tP^A(B) = 1 - b_1^A(B)t^2 - \cdots - b_r^A(B)t^{r+1},$$

is a polynomial with a single positive root $\alpha < 1$ while the numerator has no positive root. Hence, the radius of converge is < 1 and so the growth of the c_n is strong exponential by Fatou's result.

3. Cases where Poincaré series inequality (1_M) becomes an equality

We recall some relevant terminology:

DEFINITION 1. *A local morphism* $f\colon (A, \mathfrak{m}, k) \to (B, \mathfrak{n}, k)$ *for which inequality* (1_M) *is an equality when* $M = k$ *and for which* $\mathfrak{n}\widetilde{H}(X^A(k) \otimes_A B) = 0$ *holds, is called a* Golod homomorphism. *In the second condition,* $X^A(k)$ *denotes a minimal A-free resolution of k over A and* $\widetilde{H}$ *signifies reduced homology. This condition holds automatically when* f *is surjective. When A is regular and the natural map* $f\colon A \to \widehat{B} = A/\mathfrak{b}$ *is a Golod homomorphism with* $\mathfrak{b} \subseteq \mathfrak{m}^2$*, the ring B is called a* Golod ring. *If M is a finitely generated B-module for which* (1_M) *is an equality, M is called an* f-Golod module.

There are various characterizations of Golod rings, Golod homomorphisms, and f-Golod modules. For example, results of Levin from [14], [15], [16] give conditions equivalent to the defining ones and are summarized in the next two theorems.

THEOREM 2. *The following are equivalent:*

(1) f *is a Golod homomorphism; that is,* $P^B(k) = \dfrac{P^A(k)}{1 - t(P^A(B) - 1)}$ *and* $\mathfrak{n}\widetilde{H}(X^A(k) \otimes_A B) = 0$.

(2) *The induced maps* $\operatorname{Tor}^A(k, k) \to \operatorname{Tor}^B(k, k)$ *and* $\operatorname{Tor}^A(k, \mathfrak{n}) \to \operatorname{Tor}^B(k, \mathfrak{n})$ *are injective.*

(3) *The induced map* $\mathrm{Tor}^A(k,k) \to \mathrm{Tor}^B(k,k)$ *is injective and* $\mathrm{Tor}^A(k,B)$ *has trivial Massey products.*

(4) *There exists a minimal set of generators* V *for* $\widetilde{H}(X^A(k) \otimes_A B)$ *and a trivial Massey operation* γ *on* V *with* $\mathrm{Im}(\gamma) \subset \mathfrak{n}(X^A(k) \otimes_A B)$.

THEOREM 3. *The following are equivalent:*

(1) *The* B*-module* M *is* f*-Golod; that is,* $P^B(M) = \dfrac{P^A(M)}{1 - t(P^A(B) - 1)}$.

(2) f *is a Golod homomorphism and the induced map* $\mathrm{Tor}^A(k,M) \to \mathrm{Tor}^B(k,M)$ *is injective.*

(3) $\mathrm{Tor}^A(k,M) \to \mathrm{Tor}^B(k,M)$ *and* $\mathrm{Tor}^A(k,\mathrm{Syz}_1^B(M)) \to \mathrm{Tor}^B(k,\mathrm{Syz}_1^B(M))$ *are both injective.*

(4) *For every* $i > 0$, $\mathrm{Syz}_i^B(M)$ *is* f*-Golod.*

Some comments clarify these equivalences. Regarding the first three conditions in each theorem, recall that inequality (1_M) is obtained from the change of rings spectral sequence with

$$E^2_{p,q}(M,f) = \mathrm{Tor}^B_p(\mathrm{Tor}^A_q(k,B),M) \underset{p}{\Rightarrow} \mathrm{Tor}^A_{p+q}(k,M).$$

Meanwhile, the induced maps mentioned in the second and third conditions of each theorem are specific cases of

$$T_M(f)\colon \mathrm{Tor}^A(k,M) \cong H(X \otimes_A M) \cong H((X \otimes_A B) \otimes_B M) \xrightarrow{H(\bar{f} \otimes 1_M)} \mathrm{Tor}^B(k,M)$$

where $X = X^A(k)$ and $Y = Y^B(k)$ are minimal resolutions of k over A and B respectively, and $\bar{f}\colon X \otimes_A B \to Y$ is a lifting of 1_k. It turns out that the edge maps of the spectral sequence together with the filtration that results from its convergence lead to a factorization of $T_M(f)$ that forces injectivity of $T_M(f)$ when inequality (1_M) is an equality. Conversely, if $T_k(f)$ and $T_M(f)$ are injective, inequality (1_M) becomes an equality. This suggests how the equivalence (1) $\Leftrightarrow$ (2) in each theorem can be obtained.

For properties of Massey products see, for example, [11]. Massey products are used in essentially two ways in Theorem 2. First, Golod's original idea [6], updated by Gulliksen [10] can be expressed in the following way:

LEMMA 1. *If a connected* DG A*-algebra* Λ *has trivial Massey products and is free of finite type as an* A*-module, and if* $T(L)$ *denotes the tensor algebra of the graded* A*-module* L *where* $L_0 = 0$ *and* L_n *is a free* A*-module of rank* $= \dim_k(\widetilde{H}_{n-1}(\Lambda) \otimes_A k)$, *then the differential on* Λ *can be extended to a differential on* $Y = \Lambda \otimes_A T(L)$ *so that* Y *becomes an* A*-free resolution of* k. *If, moreover,* $\partial\Lambda \subset \mathfrak{m}\Lambda$ *and a trivial Massey operation* γ *can be chosen for a minimal set of generators of* $\widetilde{H}(\Lambda)$ *so that* $\mathrm{Im}(\gamma) \subset \mathfrak{m}\Lambda$, *then* Y *is a minimal resolution of* k.

Basically, trivial Massey products guarantee existence of a trivial Massey operator γ, which is used to extend the differential to $Y = \Lambda \otimes_A T(L)$ in such a way that Y is acyclic and so becomes an A-free algebra resolution of k. The additional conditions $\partial\Lambda \subset \mathfrak{m}\Lambda$ and $\mathrm{Im}(\gamma) \subset \mathfrak{m}\Lambda$ ensure that this resolution is minimal. When the lemma is applied in the context of the local morphism f described in Theorem 2, the result is:

COROLLARY 1. *Set* $X = X^A(k)$. *If there exists a trivial Massey operation* γ *defined on a minimal set of generators for* $\widetilde{H}(X \otimes_A B)$ *with* $\mathrm{Im}(\gamma) \subset \mathfrak{n}(X \otimes_A B)$, *then* $\mathrm{Tor}^A(k, k) \to \mathrm{Tor}^B(k, k)$ *is injective and inequality* (1_M) *with* $M = k$ *is an equality.*

Proof. Let L be the free, graded B-module with $L_0 = 0$ and rank $|L_n| = |\widetilde{H}_{n-1}(X \otimes_A B)|$ for $n \geq 1$. By the lemma, $(X \otimes_A B) \otimes_B T(L)$ is a minimal B-free resolution of k and therefore $\mathrm{Tor}^B(k, k) \cong (X \otimes_A k) \otimes_B T(L)$. At the same time, $\mathrm{Tor}^A(k, k) \cong X \otimes_A k$ and $T_0(L) = B$, so the natural map $X \otimes_A k \to (X \otimes_A k) \otimes_B T(L)$ is injective, thus demonstrating the first assertion.

The second assertion also follows from $\mathrm{Tor}^B(k, k) \cong (X \otimes_A k) \otimes_B T(L)$. In terms of Hilbert series, this identification becomes

$$P^B(k) = \mathcal{H}(X \otimes_A k)\mathcal{H}(T(L)) = P^A(k)/(1 - \mathcal{H}(L)),$$

while $\mathcal{H}(L) = t(P^A(B) - 1)$ holds by virtue of the construction of L. $\square$

The corollary establishes (4) ⇒ (1) and parts of (4) ⇒ (2) and (3) of Theorem 2. The remaining implications (2) ⇒ (3) ⇒ (4) of that theorem follow by interpreting the kernels of the maps $T_M(f)$ in terms matric Massey products, which are generalizations of Massey products that are due to May [17]. Details of how such products are used in Theorems 2 and 3 can be found in [15]. Details of the remaining implications of Theorem 3 can be found in [16].

Many Golod homomorphisms are surjective with $f: A \to A/\mathfrak{b} = B$ where $\mathfrak{b} \subseteq \mathfrak{m}^2$. The ideal $\mathfrak{b}$ is called a *Golod ideal*. Examples of Golod ideals include the following:

- $\mathfrak{b} = 0$. (In other words, the identity $1_A : A \to A$ is a Golod homomorphism.)
- $\mathfrak{b} = xI$ where $x \in \mathfrak{m}$ is regular and either ideal I is proper or $x \in \mathfrak{m}^2$. [15]
- $\mathfrak{b} = \sum_{i=1}^{n} y_i I_i$ where $I_1 \subseteq \cdots \subseteq I_n \subseteq \mathfrak{m}$ and $y_i \in \mathfrak{m}$ are such that y_1 is regular, and for $j > 1$, y_j is regular on $A/\sum_{i<j} y_i I_i$. (These ideals include $\mathfrak{a}_1^{s_1} \cdots \mathfrak{a}_n^{s_n}$ where the $\mathfrak{a}_i$ are generated by disjoint parts of a regular sequence.) [9]
- $\mathfrak{b} = \mathfrak{m}^t$ for all $t \gg 0$. [14]
- $\mathfrak{b} = I(r, s) =$ ideal of $s \times s$ minors of an $r \times s$ matrix ($r \geq s \geq 2$) with entries in $\mathfrak{m}$ and $\mathrm{depth}_A I(r, s) = r - s + 1$. [1]
- $\mathfrak{b} = (0 : \mathfrak{m})$ where A is a 0-dimensional Gorenstein ring of e.dim > 1. [15]

Another example of a Golod homomorphism [10] is the natural homomorphism $A \to A(M)$ where $A(M)$ is the trivial extension of A by an A-module M.

In each of the first five examples, if the ring A is regular, $A/\mathfrak{b}$ is a Golod ring. In the fourth example, $A/\mathfrak{m}^n$ is a Golod ring for any $n \geq 2$ when A is regular as a consequence of either the third or the fifth example, or of [7].

By Theorem 2, Golod rings are also rings where $\mathrm{Tor}^A(k, B)$ has trivial Massey products. $\mathrm{Tor}^A(k, B)$ can be computed as $H(X^A(k) \otimes_A B)$ or as $H(k \otimes_A X^A(B))$. Resolving the first argument with A regular, $X^A(k)$ is just the Koszul complex K^A defined by a minimal set of generators for maximal ideal $\mathfrak{m}$ of A. The condition $\mathfrak{b} \subseteq \mathfrak{m}^2$ ensures that $X^A(k) \otimes_A B = K^A \otimes_A B = K^B$, which is the corresponding Koszul complex over B defined by a minimal set of generators for maximal ideal $\mathfrak{n}$. A Golod ring is therefore one where K^B has trivial Massey products. Note that if B is a complete intersection, $H(K^B)$ is an exterior algebra, so K^B has trivial Massey products only if $H_2(K^B) = 0$ and K^B has just one generator. Thus, hypersurface rings are both complete intersections and Golod rings. In all other cases, complete intersections behave quite differently from Golod rings. That difference will be discussed in the next section.

When $\mathrm{Tor}^A(k, B)$ is computed using a resolution of the second argument, B is a Golod ring provided that $k \otimes_A X^A(B)$ has trivial Massey products where $X^A(B)$ is a minimal resolution. In fact, it suffices that $k \otimes_A W^A(B)$ has trivial Massey products where $W^A(B)$ is any free resolution of B over A. Rings of the form $B = A/I(r, s)$ where A is regular and $\mathrm{depth}_A I(r, s) = r - s + 1$ were first proved to be Golod by this approach using a generalized Koszul complex to resolve B [7]. Later, when an algebra structure was given for the Eagon-Northcott complex [21], which serves as the minimal resolution for such rings, triviality of Massey products was noted there as well.

Examples of f-Golod modules include:

- $M = 0$ over any A for any f.
- $M =$ any finitely generated module over any A and $f = 1_A\colon A \to A$.
- $M =$ any finitely generated module over any A and $f\colon A \to A/(x)$, where x is a non-zero-divisor and $x \in \mathfrak{m}(\mathrm{ann}\, M)$. [20]
- $M = k$ and $M = \mathrm{Syz}_i^B(k)$ for all i when f is a Golod homomorphism, by Theorem 3. [16]
- $M =$ any finitely generated module over $A/\mathfrak{m}^n$ for $n \gg 0$ such that $\mathfrak{m}^{n-1}M = 0$ and $f\colon A \to A/\mathfrak{m}^n$. [15]
- $M =$ any finitely generated B-module such that $(0 : \mathfrak{n})M = 0$ in the case where $f\colon A \to B$ is a *strong Golod homomorphism*, that is, where there is a chain map $H(X^A(k) \otimes_A B) \to X^A(k) \otimes_A B$ inducing an isomorphism on homology where $H(X^A(k) \otimes_A B)$ is regarded as a complex with trivial differential. (See [15].) Such modules include, in particular, the proper ideals of B. It is shown in [15] that strong Golod implies Golod. The maps $f\colon A \to A/\mathfrak{m}^n$ for $n \gg 0$ of the

preceding example are strong Golod homomorphisms (and the f-Golod modules of that example are precisely the ones with $(0 : \mathfrak{n})M = 0$.) Other examples of strong Golod homomorphisms are trivial extensions $A \to A(M)$ where M is a k-vector space, and $A \to A/\mathfrak{n}^r$ where $t \in \mathfrak{n} - \mathfrak{n}^2$ is such that $t^2 = 0$ and $\mathfrak{n}^r = t\mathfrak{n}^{r-1}$.

- $N =$ any finitely generated A-module regarded as an $A(M)$-module and $f : A \to A(M)$ is the natural map to the trivial extension of A by the finitely generated A-module M, [16]. In this case, N becomes an $A(M)$-module via the projection map $A(M) \to A$.

For a Golod ring, the proof of Theorem 1 applies directly to the sequence of Betti numbers of the residue field and shows that this sequence exhibits strong exponential growth. Put another way, over a Golod ring B the residue field k is f-Golod, where $f : A \to A/\mathfrak{b} = \widehat{B}$ is the map defining B as a Golod ring. By Theorem 3, all $\mathrm{Syz}_n^B(k)$ are f-Golod as well. The sequences of Betti numbers of these syzygy modules therefore also exhibit strong exponential growth, again using the proof of Theorem 1.

Even if A is not regular, $f_n : A \to A/\mathfrak{m}^n$ is a strong Golod homomorphism for sufficiently large n (determined using the Artin-Rees lemma). If M is a finitely generated $(A/\mathfrak{m}^n)$-module, then $\mathfrak{m}^{n-1}\mathrm{Syz}_1^{A/\mathfrak{m}^n}(M) = 0$, so this syzygy module and therefore also all higher syzygies are f_n-Golod. This was used in [8] to show for rings of Krull dimension $d \geq 2$ that the Betti numbers $b_i^{A/\mathfrak{m}^n}(M)$ of non-free, finitely generated $(A/\mathfrak{m}^n)$-modules strictly increase for all $i \geq 2$. The result was later improved by Lescot [12] to show for $d \geq 2$, any $n \geq 2$, and $i \geq 1$ that the sequence $\{b_i^{A/\mathfrak{m}^n}(M)\}$ strictly increases.

4. Rings and modules whose Betti numbers grow at the extremes of the permissible range

Two questions: When are the bounds given in Theorem 1 for the growth rates of Betti numbers achieved? What are the possible growth rates for sequences of Betti numbers of finitely generated modules over a particular ring?

For Golod rings, the story is found in [2] and [18], which use results from [5] and [13] to obtain:

THEOREM 4. *For a finitely generated module over a Golod ring A, precisely one of the following situations must occur:*
(1) $\mathrm{pd}_A M < \infty$.
(2) *A is a hypersurface ring,* $\mathrm{pd}_A M = \infty$*, and the Betti numbers of M are eventually constant and nonzero after* $\operatorname{depth} A - \operatorname{depth} M + 1$*. Moreover,* $\mathrm{Syz}_n^A(M)$ *(and therefore the minimal resolution) is periodic of period* 2 *after that point.*
(3) *A is not a hypersurface ring,* $\mathrm{pd}_A M = \infty$*, and the Betti sequence of M has strong exponential growth. In more detail, the Betti sequences of such modules strictly increase after degree* $2\varepsilon - 1$ *where* ε *is the embedding dimension of A, exhibit termwise*

exponential growth after degree 2ε*, and have strong exponential growth after degree* 3ε*.*

The situation for complete intersections is in marked contrast, except for the shared case of a hypersurface ring. A *complete intersection* is a local ring B whose completion is of the form $\widehat{B} \cong A/\mathfrak{a}$ with A regular and $\mathfrak{a}$ generated by an A-regular sequence $x_1, \dots, x_r \subset \mathfrak{m}^2$. If the embedding dimension of B is $\mathrm{e.dim}\, B = \varepsilon$ and the codimension is $r = \varepsilon - \mathrm{depth}\, B$, then B is characterized [22] by the form of the Poincaré series associated with its residue field, k, which is:

$$P^B(k) = \frac{(1+t)^{\varepsilon}}{(1-t^2)^r}.$$

A hypersurface ring is a complete intersection B with $r = 1$. In this case, $X^A(B)$ is just $0 \to A \xrightarrow{x_1} A \to A/(x_1) \to 0$, making $1 - t(P^A(B) - 1) = 1 - t^2$. The denominator thus has the right form for a Golod ring. The numerator also has the right form: $(1+t)^{\varepsilon} = P^A(k)$ because with A regular, the Koszul complex K^A defined from $A \to A/\mathfrak{m}$ is a minimal A-free resolution of k and also an exterior algebra on ε generators. Therefore, in this special case, B is both a complete intersection and a Golod ring. In all other cases, complete intersections and Golod rings behave quite differently from each other.

The form of $P^B(k)$ for complete intersections implies that the Betti numbers $b_n^B(k)$ for all $n \geq 0$ are given by a polynomial in n of degree $r - 1$:

$$b_n^B(k) = \sum_{i=0}^{\varepsilon - r} \binom{\varepsilon - r}{i} \binom{n + r - 1 - i}{r - 1}.$$

Hence, the growth of these Betti numbers is polynomial of degree $r - 1$. It turns out that for finitely generated modules over complete intersections, all growth of Betti numbers is polynomial of specific degrees. The description, due to Avramov, Gasharov and Peeva [3], utilizes Avramov's notion of complexity.

DEFINITION 2. *The* complexity of M over A*, denoted* $\mathrm{cx}_A M$*, is* d *if* $d - 1$ *is the smallest degree of a polynomial in* n *that bounds* $b_n^A(M)$ *from above. The zero polynomial is assigned degree* -1*, and* $\mathrm{cx}_A M = 0$ *means the zero polynomial eventually bounds the Betti numbers; in other words,* $\mathrm{pd}_A M < \infty$*.*

Complexity $\mathrm{cx}_A\, M = 1$ means that a constant bounds the Betti numbers. Complexity $\mathrm{cx}_A\, M = \infty$ signifies that no bounding polynomial exists. Polynomial growth that is of degree $r - 1$, for example, the growth of the Betti numbers $b_n^B(k)$, is expressed by saying that $\mathrm{cx}_B(k) = r = \mathrm{e.dim}\, B - \mathrm{depth}\, B$. For all other finitely generated M, formula (8.5) of [3] gives

$$P^B(M) = \frac{p_M(t)}{(1-t)^d (1+t)^e} \qquad (4_M)$$

with $p_M(t) \in \mathbb{Z}[t]$ such that $p_M(\pm 1) \neq 0$. (Note that integer e should not be confused with $\text{e.dim}\, B = \varepsilon$.) In this setting, Theorems (8.1) and (8.6) of [3] become the following:

THEOREM 5. *If B is a complete intersection and M is a finitely generated B-module whose Poincaré series is given by* (4_M)*, then* $\text{cx}_B\, M = d \leq \text{e.dim}\, B - \text{depth}\, B$, $p_M(1) > 0$*, and one of the following cases holds:*

(0) $d = 0 : e < 0$ *or* $e = 0$ *with* $p_M(-1) > 0$*; also* $\deg p_M(t) = \text{depth}\, B - \text{depth}_B M + e$*,*

(1) $d = 1 : e \leq 0$ *and* $\deg p_M(t) = \text{depth}\, B - \text{depth}_B M + e$*.*

(2) $d \geq 2 : e < d - 1$ *or* $e = d - 1$ *with* $p_M(1) > |p_M(-1)|$*.*

For case (2) *with* $n \gg 0$*, the Betti numbers* $b_n^B(M)$ *are given by polynomials* $b_+(n)$ *when n is even and* $b_-(n)$ *when n is odd, with* $b_+(t), b_-(t) \in \mathbb{Q}[t]$ *and*

$$b_\pm(t) = \frac{b}{2^e(d-1)!}\, t^{d-1} + \frac{c_\pm}{2^d(d-2)!}\, t^{d-2} + \textit{lower order terms}$$

with integers b*,* $c_\pm$*, and* e *such that either* $0 \leq e \leq d-2$*,* $c_+ = c_-$*, and* $b > 0$*, or else* $e = d - 1$ *and* $b > |c_+ - c_-|$*. In particular, both difference polynomials* $b_\pm(t+1) - b_\mp(t)$ *have degree* $d - 2$ *and positive leading coefficients.*

REMARKS 1. In case (0) of the theorem where $d = 0$, $\text{pd}_B M < \infty$ and $P^B(M)$ is a polynomial. In case (1) of the theorem, $d = 1$ means the Betti numbers are bounded. Eisenbud [4] showed that bounded sequences of Betti numbers for finitely generated modules over complete intersections are eventually constant and that they become periodic of period 2 after at most $(\dim B) + 1$ steps.

In case (2), $d \geq 2$ and $\lim_{n\to\infty} b_n^B(M)/n^{d-1} =$ the common leading coefficient of the polynomials $b_\pm(t)$. When it comes to $\lim_{n\to\infty}(b_n^B(M) - b_{n-1}^B(M))/n^{d-2}$, however, the situation is different—the limit exists when $c_+ = c_-$ but does not exist when they are unequal.

To illustrate this, consider $A = k[\![X, Y]\!]$ and $B = A/(X^3, Y^3)$. Thus, B is a complete intersection with $\varepsilon = 2$ and codimension $r = 2$, so by the Tate formula shown above, $P^B(k) = (1+t)^2/(1-t^2)^2 = 1/(1-t)^2$, which implies $b_n^B(B/\mathfrak{n}) = b_n^B(k) = n + 1$ for each $n \geq 0$. On the other hand, $b_n^B(B/\mathfrak{n}^2) = \frac{3}{2}n + 1$ for even $n \geq 0$, and $b_n^B(B/\mathfrak{n}^2) = \frac{3}{2}n + \frac{3}{2}$ for odd $n \geq 1$. (See [2].) These Betti numbers give different values, namely 1 and 2, for (even) $-$ (odd) as opposed to (odd) $-$ (even), so the limit of the differences does not exist.

Thus, there is a complete description of the asymptotic behavior of Betti numbers of finitely generate modules over complete intersections and a different description over Golod rings. For complete intersections, growth is polynomially bounded with detail added using the notion of complexity; for Golod rings that are not hypersurface rings, all growth is exponential. There are other rings over which all growth is either polynomial or exponential and where some of each occurs.

THEOREM 6. [2], [3]. *Let B a local ring that satisfies one of the conditions:*
(1) *B is one link from a complete intersection;*
(2) *B is two links from a complete intersection and B is Gorenstein;*
(3) $\mathrm{e.dim}\, B - \mathrm{depth}\, B \leq 3$*;*
(4) $\mathrm{e.dim}\, B - \mathrm{depth}\, B = 4$ *and B is Gorenstein.*

If M is a finite B-module whose Poincaré series has radius of convergence $\rho \geq 1$, then $\mathrm{cx}_B\, M = d \leq \mathrm{e.dim}\, B - \mathrm{depth}\, B$ and there exist polynomials Δ_1 and Δ_2, each of degree $d-2$ with positive leading coefficients, such that for $n \gg 0$,

$$\Delta_1(n) \leq b_n^B(M) - b_{n-1}^B(M) \leq \Delta_2(n).$$

In particular, $\{b_n^B(M)\}$ is eventually either constant or eventually strictly increasing and bounded by polynomial growth.

If M is a finite B-module whose Poincaré series has radius of convergence $\rho < 1$, then, $\{b_n^B(M)\}$ eventually strictly increases with strong exponential growth.

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Eugene GOVER, Department of Mathematics, Northeastern University, Boston, MA 02115, U.S.A.
e-mail: `e.gover@neu.edu`

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R. Hartshorne

LIAISON WITH COHEN–MACAULAY MODULES

Dedicated to Paolo Valabrega on the occasion of his 60 th birthday

Abstract. We describe some recent work concerning Gorenstein liaison of codimension two subschemes of a projective variety. Applications make use of the algebraic theory of maximal Cohen–Macaulay modules, which we review in an Appendix.

1. Introduction

The purpose of this paper is to report on some recent work in the area of Gorenstein liaison. For me this is a pleasant topic, because it illustrates the field of algebraic geometry at its best. After all, algebraic geometry could be described as the use of algebraic techniques in geometry and the use of geometric methods to understand algebra. In the work I describe here, we found an unexpected connection between the theory of maximal Cohen–Macaulay modules about which there is considerable algebraic literature, and the notion of Gorenstein liaison, which has emerged recently as geometers attempted to generalize results about curves in $\mathbb{P}^3$ to varieties of higher codimension.

In Section 2, we review the "classical" case of curves in $\mathbb{P}^3$. In §3 we describe generalizations of the notion of liaison to schemes of higher dimension and higher codimension. Sections 4 and 5 develop the main new idea, which is instead of working directly with schemes of codimension ≥ 3 in $\mathbb{P}^n$, to consider subschemes of codimension 2 of an arithmetically Gorenstein scheme X in $\mathbb{P}^n$. Any liaison in X is also a liaison in $\mathbb{P}^n$, so this method is useful to establish existence of liaisons in $\mathbb{P}^n$, but it cannot give negative results. We hope that the study of liaison on X may be interesting in its own right, and give more insight into the nature of liaison in general.

Section 6 gives some applications, and Section 7 describes an interesting open problem. The algebraic theory of maximal Cohen–Macaulay modules is reviewed in an Appendix.

The principal new results described here are joint work with Marta Casanellas and Elena Drozd, given in detail in the papers [3] and [4]. For background on liaison, I recommend the book of Migliore [16], and for information on Cohen–Macaulay modules, the book of Yoshino [18].

It was a pleasure to attend the conference Syzygy 2005 in Torino in honor of Paolo Valabrega's sixtieth birthday, and I dedicate this paper respectively to him.

2. Curves in $\mathbb{P}^3$

We review the case of curves in $\mathbb{P}^3$, which has been known for some time, as a model for the more general situations that we will consider below.

We work over an algebraically closed field k. A *curve* is a purely one-dimensional scheme without embedded points. If C_1 and C_2 are curves in $\mathbb{P}^3$, we say they are *linked* by a complete intersection curve Y, if $C_1 \cup C_2 = Y$ and $\mathcal{I}_{C_i,Y} \cong \mathcal{H}om(\mathcal{O}_{C_j}, \mathcal{O}_Y)$ for $i, j = 1, 2, i \neq j$. The equivalence relation generated by chains of linkages is called *liaison*. If a liaison is accomplished by an even number of linkages, it is called *even liaison*. To any curve C in $\mathbb{P}^3$ we associate its *Rao module* $M_C = H^1_*(\mathcal{I}_C) = \bigoplus_{n\in\mathbb{Z}} H^1(\mathbb{P}^3, \mathcal{I}_{C,\mathbb{P}^3}(n))$. The basic results about curves in $\mathbb{P}^3$ are the following

THEOREM 1 (Rao [17]). a) *Two curves C_1 and C_2 are in the same even liaison equivalence class if and only if their Rao modules M_1, M_2 are isomorphic, up to a shift in degrees.*

b) *For any finite-length graded $R = k[x_0, x_1, x_2, x_3]$-module M, there exists a nonsingular irreducible curve C in $\mathbb{P}^3$, whose Rao module is isomorphic to a shift of M.*

Thus the even liaison equivalence classes of curves in $\mathbb{P}^3$ are in one-to-one correspondence with finite length graded R-modules, up to shift.

For the next statement we need the notion of biliaison. If a curve C_1 lies on a surface S, and if $C_2 \sim C_1 + mH$, meaning linear equivalence in the sense of generalized divisors [7] on S, where H is the hyperplane section of S, and m is an integer, then we say that C_2 is obtained by an *elementary biliaison* of height m from C_1. If $m \geq 0$ it is an *ascending* elementary biliaison. It is easy to see that an elementary biliaison gives an even liaison between C_1 and C_2. It is also easy to calculate numerical invariants of C_2, such as degree, genus, and postulation, from those of C_1 in terms of m and the degree of S.

THEOREM 2 (Lazarsfeld–Rao property [1], [15]). a) *In any even liaison equivalence class of curves in $\mathbb{P}^3$, the* minimal curves *(meaning those of minimal degree) form an irreducible family.*

b) *Any curve that is not minimal in its even liaison equivalence class can be obtained by a sequence of ascending elementary biliaisons from some minimal curve.*

REMARK 1. These results generalize well to subschemes V of codimension two in $\mathbb{P}^n$. The Rao module has to be replaced by a series of higher deficiency modules $H^i_*(\mathcal{I}_V)$ for $0 < i \leq \dim V$ and certain extensions between them: the best way to express this is by an element of the derived category. Or one can use the so-called $\mathcal{E}$-type resolution, in which case the set of even liaison equivalence classes of schemes V of codimension two is in one-to-one correspondence with coherent sheaves $\mathcal{E}$ (satisfying some additional conditions), up to stable equivalence and shift, and this in turn is in one-to-one correspondence with the quasi-isomorphism classes of certain complexes in the derived category replacing the Rao module.

The Lazarsfeld–Rao property also generalizes to codimension two subschemes of quite general schemes. See for example [9] for precise statements and further references.

3. Generalizations

When we consider curves in $\mathbb{P}^4$, or more generally, subschemes of codimension ≥ 3 in any $\mathbb{P}^n$, the direct analogue of Rao's theorem fails. There are infinitely many distinct even liaison equivalence classes of curves all having the same Rao module, which can be distinguished by other cohomological invariants [11]. It seems that liaison using complete intersections, as we have defined it, is much too rigid to give an analogous theory in higher codimension.

The notion of Gorenstein liaison seems to be a better candidate for generalizing the theory.

DEFINITION 1. *Two subschemes V_1, V_2 of $\mathbb{P}^n$, equidimensional and without embedded components, are G-*linked *by an arithmetically Gorenstein scheme Y (meaning the homogeneous coordinate ring of Y is a Gorenstein ring) if $V_1 \cup V_2 = Y$ and $\mathcal{I}_{V_i,Y} \cong \mathcal{H}om(\mathcal{O}_{V_j}, \mathcal{O}_Y)$ for $i, j = 1, 2$, $i \neq j$. The equivalence relation generated by chains of G-links is called* Gorenstein liaison *(or G-liaison for short), and if a G-liaison can be accomplished by an even number of G-links, it is called* even G-liaison.

It is easy to see for curves in $\mathbb{P}^n$ that even G-liaison preserves the Rao module (up to shift), as in the case of $\mathbb{P}^3$, and this naturally leads to the converse problem:

PROBLEM 3. If two curves in $\mathbb{P}^n$ have isomorphic Rao modules (up to shift), are they in the same even G-liaison class?

This problem is open at present. The special case when the Rao module is zero is the case of *arithmetically Cohen–Macaulay* (ACM) curves, meaning that the homogeneous coordinate ring is a Cohen–Macaulay ring. This includes in particular the complete intersection curves. So the problem, which now can be stated for schemes of any dimension is

PROBLEM 4. If V is an ACM scheme in $\mathbb{P}^n$, is V in the Gorenstein liaison class of a complete intersection (glicci for short)?

This problem is also open at present, though many special cases are known (see for example [11]). There are also candidates for counterexamples (as yet unproven), such as 20 general points in $\mathbb{P}^3$, or a general curve of degree 20 and genus 26 in $\mathbb{P}^4$ [8].

Our approach in this paper, instead of studying the problem directly in $\mathbb{P}^n$, will be to study codimension two subscheme of an arithmetically Gorenstein variety X in $\mathbb{P}^n$. Liaisons in X can also be considered to be liaisons in $\mathbb{P}^n$, and thus we study the problem of higher codimension subschemes in $\mathbb{P}^n$ indirectly. While most of our results are valid for X of any dimension, for simplicity in this paper we will stick to dimension 3.

So here is the set-up. Let X be a fixed normal arithmetically Gorenstein subvariety of dimension 3 in $\mathbb{P}^n$. We also keep fixed the embedding and hence the sheaf $\mathcal{O}_X(1)$ on X that defines the class of a hyperplane section H of X.

If C_1 and C_2 are curves in X, we say that C_1 and C_2 are *linked* by a curve Y in X if $C_1 \cup C_2 = Y$ and $\mathcal{I}_{C_i,Y} \cong \mathcal{H}om(\mathcal{O}_{C_j}, \mathcal{O}_Y)$ for $i, j = 1, 2, i \neq j$. If Y is a *complete intersection* in X, meaning that Y is the intersection of surfaces defined by sections of $\mathcal{O}_X(a), \mathcal{O}_X(b)$ in X, then we say it is a *CI-linkage*. If Y is arithmetically Gorenstein (in the ambient $\mathbb{P}^n$), it is a *G-linkage*. These linkages give rise to the equivalence relations of *CI-liaison* and *even CI-liaison* and *G-liaison* and *even G-liaison* as before.

Note that a CI-liaison in X is not necessarily a CI-liaison in $\mathbb{P}^n$, unless X itself is a complete intersection. However, a G-liaison in X is also a G-liaison in $\mathbb{P}^n$.

If S is a surface in X containing a curve C, and if C' is another curve on S, with $C' \sim C + mH$, meaning linear equivalence of generalized divisors on S, where H is the hyperplane section, we say C' is obtained from C by an *elementary biliaison* from C. If S is a complete intersection in X (corresponding to $\mathcal{O}_X(a)$ for some a) it is a *CI-biliaison*. If S is an ACM scheme (in $\mathbb{P}^n$) it is a *G-biliaison*. It is easy to see that a CI-biliaison is an even CI-liaison. In fact, the equivalence relation generated by CI-biliaisons is the same as even CI-liaison (proof similar to [7, 4.4]). One can show also that a G-biliaison is an even G-liaison [11], [10, 3.6], however in general the equivalence relation generated by G-biliaisons is not the same as even G-liaison, as we can see from the following example.

EXAMPLE 1. Let X be a nonsingular quadric hypersurface in $\mathbb{P}^4$. Every surface on X is a complete intersection, and in particular has even degree. Thus G-biliaisons preserve the parity of the degree of a curve. On the other hand, the union of a rational quartic curve with a line meeting it at two points is an arithmetically Gorenstein elliptic quintic, so the two curves are G-linked. One line can also be linked to another line by a conic, so we see that even G-liaison does not preserve parity of degree.

In studying G-liaison and G-biliaison on X, an important role is played by the category of ACM sheaves on X. An ACM *sheaf* is a coherent sheaf $\mathcal{E}$ on X that is locally Cohen–Macaulay and has vanishing intermediate cohomology: $H^i_*(\mathcal{E}) = 0$ for $i = 1, 2$. If X is $\mathbb{P}^3$, the only ACM sheaves are the *dissocié* sheaves, i.e., direct sums of line bundles $\mathcal{O}_X(a_i)$, by a theorem of Horrocks. However, if X is not $\mathbb{P}^3$, there are others, and the category of these sheaves reflects interesting properties of X.

To see why these sheaves are important for G-liaison and G-biliaison, we first mention the following result relating them to ACM surfaces in X and arithmetically Gorenstein (AG) curves in X.

PROPOSITION 1. a) *If S is an* ACM *surface in X, then its ideal sheaf $\mathcal{I}_{S,X}$ is a rank* 1 ACM *sheaf on X. Conversely if $\mathcal{L}$ is a rank* 1 ACM *sheaf on X, then for any $a \gg 0$, the sheaf $\mathcal{L}(-a)$ is isomorphic to the ideal sheaf $\mathcal{I}_{S,X}$ of an* ACM *surface in X.*

b) *If Y is an* AG *curve in X, then there is an exact sequence*

$$0 \to \mathcal{O}_X(-a) \to \mathcal{N} \to \mathcal{I}_{Y,X} \to 0$$

for some $a \in \mathbb{Z}$, where $\mathcal{N}$ is a rank 2 ACM *sheaf on X with $c_1(\mathcal{N}) = -a$. Conversely if $\mathcal{N}$ is any* orientable *(meaning $c_1(\mathcal{N}) = \mathcal{O}_X(a)$ for some $a \in \mathbb{Z}$) rank* 2 ACM *sheaf* *on X, and if s is a sufficiently general section of $\mathcal{N}(a)$ for $a \gg 0$, then s induces an exact sequence*

$$0 \to \mathcal{O}_X \xrightarrow{s} \mathcal{N}(a) \to \mathcal{I}_{Y,X}(b) \to 0$$

for some AG *curve Y in X and some $b \in \mathbb{Z}$.*

Proof. Part a) is elementary, while part b) is the usual Serre correspondence [4, 2.9]. □

Thus we see that the ACM surfaces and AG curves, which are used to define G-biliaison and G-liaison, respectively, correspond in a natural way to rank 1 and rank 2 ACM sheaves on X. In the following two sections, we will study G-biliaison and G-liaison separately.

4. Gorenstein biliaison

As in the previous section, we consider a normal arithmetically Gorenstein 3-fold X, and we will consider Gorenstein biliaison of curves on X.

First of all, let's see what happens with a single elementary Gorenstein biliaison. Let S be an ACM surface in X, let C be a curve in S, and let $C' \sim C + mH$ on S. Then by construction, $\mathcal{I}_{C',S} \cong \mathcal{I}_{C,S}(-m)$. Thus we can write exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{I}_S & \to & \mathcal{I}_{C'} & \to & \mathcal{I}_{C',S} & \to & 0 \\ & & & & & & \| & & \\ 0 & \to & \mathcal{I}_S(-m) & \to & \mathcal{I}_C(-m) & \to & \mathcal{I}_{C,S}(-m) & \to & 0. \end{array}$$

If we let $\mathcal{F}$ be the fibered sum of $\mathcal{I}_{C'}$ and $\mathcal{I}_C(-m)$ over $\mathcal{I}_{C',S} = \mathcal{I}_{C,S}(-m)$, we obtain sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{I}_S & \to & \mathcal{F} & \to & \mathcal{I}_C(-m) & \to & 0 \\ 0 & \to & \mathcal{I}_S(-m) & \to & \mathcal{F} & \to & \mathcal{I}_{C'} & \to & 0. \end{array}$$

Note here that the same coherent sheaf $\mathcal{F}$ appears in the middle of each sequence, and that the sheaves on the left are rank 1 ACM sheaves on X that are isomorphic, up to twist.

Conversely, given exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{L} & \to & \mathcal{F} & \to & \mathcal{I}_C(a) & \to & 0 \\ 0 & \to & \mathcal{L}' & \to & \mathcal{F} & \to & \mathcal{I}_{C'}(a') & \to & 0 \end{array}$$

with the same coherent sheaf $\mathcal{F}$ in the middle, where C, C' are curves in X, a, a' integers, and $\mathcal{L}, \mathcal{L}'$ rank 1 ACM sheaves that are isomorphic up to twist, it follows that C' is obtained by a single elementary G-biliaison from C. The idea of proof is to consider the composed map $\mathcal{L}' \to \mathcal{F} \to \mathcal{I}_C(a)$. If this map is 0, then $C' = C$, which is a trivial G-biliaison. If it is not zero, composing with the inclusion $\mathcal{I}_C(a) \subseteq \mathcal{O}_X(a)$ identifies $\mathcal{L}'(-a)$ with the ideal sheaf $\mathcal{I}_S$ of an ACM surface on X and then one sees easily that $C' \sim C + (a' - a)H$ on S [3, 3.1, 3.3].

With a little more work, one can arrive at an analogous criterion for two curves to be related by a finite succession of elementary G-biliaisons.

THEOREM 3 ([3, 3.1]). *Two curves C, C' on the normal arithmetically Gorenstein 3-fold X are in the same Gorenstein biliaison equivalence class if and only if there exist exact sequences*

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{E} & \to & \mathcal{N} & \to & \mathcal{I}_C(a) & \to & 0 \\ 0 & \to & \mathcal{E}' & \to & \mathcal{N} & \to & \mathcal{I}_{C'}(a') & \to & 0 \end{array}$$

with the same coherent sheaf $\mathcal{N}$ in the middle, where a, a' are integers, and where $\mathcal{E}$ and $\mathcal{E}'$ are ACM *sheaves each having a filtration whose quotients are rank* 1 ACM *sheaves (we call them* layered ACM *sheaves), and such that the rank* 1 *quotients of these filtrations of $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic up to order and twists.*

The point is that each G-biliaison contributes a rank 1 ACM factor, but that these make up the two sheaves $\mathcal{E}$ and $\mathcal{E}'$ in a different order, and with different twists associated to each.

If $\mathcal{E}$ is a layered ACM sheaf as above, the filtration with rank 1 ACM quotients may not be unique. Taking advantage of this are two "exchange lemmas" [3, 3.4, 4.6] that allow one to replace one $\mathcal{E}$ by another $\mathcal{E}'$ having the same factors, in sequences as in Theorem 3, after passing to another curve in the same G-biliaison class. These form a sort of converse to Theorem 3, and allow us to formulate a necessary and sufficient condition for the property analogous to Problem 4 on X, namely that every ACM curve on X should be in the G-biliaison class of a complete intersection on X. This condition is a bit complicated to state (see [3, 4.2, 4.3]), so instead here we will explain the result only in one interesting special case.

THEOREM 4 ([3, 6.2]). *Let X be the cone over a nonsingular quadric surface in $\mathbb{P}^3$. (Thus X is a normal quadric hypersurface in $\mathbb{P}^4$ having one double point.) Then two curves C and C' on X are in the same Gorenstein biliaison equivalence class if and only if their Rao modules are isomorphic, up to shift. In particular, all* ACM *curves are equivalent for G-biliaison.*

Idea of Proof. It is obvious that Gorenstein biliaison preserves the Rao module, up to shift, so one direction is clear.

For the other direction, let C be any curve in X, with Rao module M. Our strategy is to construct another curve C' that depends only on M, and then show that C and C' are in the same G-biliaison equivalence class, which will prove the theorem.

Given M, let M^* be the dual module, and take a resolution

$$0 \to G \to F_2 \to F_1 \to F_0 \to M^* \to 0$$

over R, the homogeneous coordinate ring of X, where the F_i are free graded R-modules, and G is the kernel. Let $\mathcal{N}'$ be the sheaf associated to $G^\vee$, and let $\mathcal{L}'$ be a dissocié sheaf of rank one less mapping to $\mathcal{N}'$ so as to define a curve C' by its cokernel:

$$0 \to \mathcal{L}' \to \mathcal{N}' \to \mathcal{I}_{C'}(a') \to 0.$$

On the other hand, let

$$0 \to \mathcal{L} \to \mathcal{N} \to \mathcal{I}_C \to 0$$

be an $\mathcal{N}$*-type resolution* of C, i.e., with $\mathcal{L}$ dissocié and $\mathcal{N}$ coherent, locally Cohen–Macaulay, and $H^1_*(\mathcal{N}) \cong M$ and $H^2_*(\mathcal{N}) = 0$. Let $N = H^0_*(\mathcal{N})$ and take a resolution

$$0 \to P \to L_1 \to L_0 \to N \to 0$$

over R with L_i free and P the kernel. Dualizing gives an exact sequence

$$0 \to N^\vee \to L_0^\vee \to L_1^\vee \to P^\vee \to M^* \to 0.$$

Now there is a natural map of the earlier free resolution of M^* into this one, and this gives us a map of G to $N^\vee$, from which we obtain a natural map $\mathcal{N} \to \mathcal{N}'$. By adding extra free factors if necessary, we may assume it is surjective, and then let $\mathcal{E}$ be the kernel:

$$0 \to \mathcal{E} \to \mathcal{N} \to \mathcal{N}' \to 0.$$

Since $\mathcal{N}$ and $\mathcal{N}'$ both have $H^1_* = M$ and $H^2_* = 0$, we see that $\mathcal{E}$ is an ACM sheaf on X. Furthermore, taking the composed map from $\mathcal{N}$ to $\mathcal{I}_{C'}(a')$ we obtain an exact sequence

$$0 \to \mathcal{E} \oplus \mathcal{L} \to \mathcal{N} \to \mathcal{I}_{C'}(a') \to 0.$$

In order to apply the criterion of Theorem 3 we now need to use the special property of the quadric 3-fold X (see Appendix), which tells us first that every ACM sheaf on X is layered, secondly that the only rank 1 ACM sheaves on X (up to twist) are $\mathcal{O}_X$, $\mathcal{I}_D$, and $\mathcal{I}_E$, where D, E represent the two types of planes in X, and thirdly that there is an exact sequence

$$0 \to \mathcal{I}_D \to \mathcal{O}_X^2 \to \mathcal{I}_E(1) \to 0.$$

In the ACM sheaf $\mathcal{E}$, copies of $\mathcal{I}_D$ and $\mathcal{I}_E$ (and their twists) must occur in equal numbers, because $\mathcal{E}$ is orientable. Then the exchange lemmas referred to above allow us to replace an $\mathcal{I}_D$ plus an $\mathcal{I}_E$ by an $\mathcal{O}_X^2$. Thus $\mathcal{E} \oplus \mathcal{L}$ is replaced by a dissocié sheaf, and then Theorem 3 tells us that C and C' are in the same G-biliaison class. (For more details see [3, 4.7, 6.2].)

5. Gorenstein liaison

Let us consider a normal AG 3-fold X, as before, and study Gorenstein liaison equivalence of curves in X. Since the AG curves in X are associated to rank 2 ACM sheaves on X, as we saw above, we expect to see them play a role.

First of all, let us see what happens with a single Gorenstein liaison. We track this behavior using the $\mathcal{N}$-type resolution of a curve C.

PROPOSITION 2. *Let C be a curve in X with $\mathcal{N}$-type resolution $0 \to \mathcal{L} \to \mathcal{N} \to \mathcal{I}_C \to 0$, and suppose that C is linked to a curve C' by the* AG *curve Y. Then C' has an $\mathcal{N}$-type resolution of the form*

$$0 \to \mathcal{L}' \to \mathcal{N}' \to \mathcal{I}_{C'}(a') \to 0$$

with $\mathcal{L}'$ dissocié, and where $\mathcal{N}'$ is an extension

$$0 \to \mathcal{L}^\vee \oplus \mathcal{E}^\vee \to \mathcal{N}' \to \mathcal{N}^{\sigma\vee} \to 0,$$

where $\mathcal{E}$ is the rank 2 ACM *sheaf associated to Y, and $\mathcal{N}^{\sigma\vee}$ denotes the dual of the first syzygy sheaf of $\mathcal{N}$.*

To prove this (see [4, 3.2]) one first uses the usual cone construction of the map $\mathcal{I}_Y \subseteq \mathcal{I}_C$, and this gives the sequence

$$0 \to \mathcal{N}^\vee \to \mathcal{L}^\vee \oplus \mathcal{E}^\vee \to \mathcal{I}_{C'}(a) \to 0.$$

This is not an $\mathcal{N}$-type resolution, but by using the syzygy sheaf $\mathcal{N}^\sigma$ of $\mathcal{N}$, one can transform it into the desired $\mathcal{N}$-type resolution.

Note what happens to the Rao module M. From the definition of the syzygy sheaf

$$0 \to \mathcal{N}^\sigma \to \mathcal{F} \to \mathcal{N} \to 0$$

with $\mathcal{F}$ dissocié, we see that $M \cong H^1_*(\mathcal{N}) \cong H^2_*(\mathcal{N}^\sigma)$. By Serre duality then $H^1_*(\mathcal{N}^{\sigma\vee}) \cong M^*$, the dual of M, and this shows that the Rao module of C' is M^* shifted, as we would expect from a single liaison.

This proposition shows us that a single G-liaison complexifies the $\mathcal{N}$-type resolution by throwing in a dual of a syzygy, and adding an extension by a rank 2 ACM sheaf. There is a sort of converse to this, showing how to simplify an $\mathcal{N}$-type resolution by removing a rank 2 ACM sheaf. In general, this cannot be accomplished by a single G-liaison, but requires a more complicated procedure.

PROPOSITION 3. *Let C be a curve with an $\mathcal{N}$-type resolution*

$$0 \to \mathcal{L} \to \mathcal{N} \to \mathcal{I}_C \to 0,$$

and suppose given an exact sequence

$$0 \to \mathcal{E} \to \mathcal{N} \to \mathcal{N}' \to 0$$

with $\mathcal{E}$ a rank 2 ACM *sheaf and $\mathcal{N}'$ a locally* CM *sheaf of rank ≥ 2. Then there is a curve C' in the same even G-liaison equivalence class as C having an $\mathcal{N}$-type resolution*

$$0 \to \mathcal{L}' \to \mathcal{N}' \to \mathcal{I}_{C'}(a') \to 0.$$

Proof. See [4, 3.4]. □

Using these two propositions, it is possible to give a criterion, in terms of the $\mathcal{N}$-type resolutions, for when two curves are in the same G-liaison class [4, 5.1]. The exact statement, which involves successive extensions by rank 2 ACM sheaves and their syzygy duals (which may no longer be of rank 2), is rather complicated, so we omit it here. Using this theorem, one can also give a criterion for every ACM curve to be in the Gorenstein liaison class of a complete intersection [4, 5.4]. Here we will just give one special case, albeit an interesting one.

THEOREM 5. *Let X be a nonsingular quadric hypersurface in $\mathbb{P}^4$. Then two curves are in the same even G-liaison class if and only if their Rao modules are isomorphic, up to shift.*

Sketch of Proof (cf. [4, 6.2]). Let C be any curve, with Rao module M, and let C' be another curve with the same Rao module M, constructed as in the proof of Theorem 4 above. Following the plan of that proof we have $\mathcal{N}$-type resolutions

$$0 \to \mathcal{L} \to \mathcal{N} \to \mathcal{I}_C \to 0$$

$$0 \to \mathcal{L}' \to \mathcal{N}' \to \mathcal{I}_{C'}(a') \to 0$$

and an exact sequence

$$0 \to \mathcal{E} \to \mathcal{N} \to \mathcal{N}' \to 0$$

where $\mathcal{E}$ is an ACM sheaf on X.

Now we invoke the special property of the nonsingular quadric 3-fold X, which is that every ACM sheaf is a direct sum of a dissocié sheaf and copies of twists of a single rank 2 ACM sheaf $\mathcal{E}_0$, associated to a line in X (see Appendix). We apply Proposition 3 repeatedly to remove copies of $\mathcal{E}_0$ and its twists from $\mathcal{N}$, thus eventually obtaining a curve C'', in the same even G-liaison class as C, and having an $\mathcal{N}$-type resolution whose middle sheaf $\mathcal{N}''$ differs from $\mathcal{N}'$ only by a dissocié sheaf. Then $\mathcal{N}'$ and $\mathcal{N}''$ are stably equivalent, and so C'' and C' are in the same even CI-liaison class, by Rao's theorem, and a fortiori in the same even G-liaison class.

Note that in the case of the nonsingular quadric 3-fold, G-biliaison is just the same as CI-biliaison, hence is much too restrictive to provide a result like this theorem.

6. Applications

In [11, 8.10] the authors showed, by an exhaustive listing of all possible ACM curves on these surfaces, that any ACM curve lying on a general smooth rational ACM surface in $\mathbb{P}^4$ is glicci. The rational ACM surfaces in $\mathbb{P}^4$ (not counting those in $\mathbb{P}^3$, for which the theorem is known) are the cubic scroll, the Del Pezzo surface of degree 4, the Castelnuovo surface of degree 5, and the Bordiga surface of degree 6.

For the cubic scroll, the Del Pezzo, and the Castelnuovo surface, this result is an immediate consequence of our Theorem 4, because each of these surfaces is contained in a quadric 3-fold with one double point. Our method does not apply to the Bordiga surface, which is not contained in any quadric hypersurface.

In his paper [14], Lesperance studied curves in $\mathbb{P}^4$ of the following form. Let C be the disjoint union $C_1 \cup C_2$ of two plane curves C_1, C_2, lying in two planes that meet at a single point P. Let them have degrees d_1, d_2, and assume either a) $2 \le d_1 \le d_2$ or b) $2 \le d_1$ and C_2 contains the point P. The Rao module is then $M \cong R/(I_P + R_{\ge d_1})$, which depends only on the point P and the integer d_1. Lesperance shows that all the curves of type a) and some of those of type b) are in the same Gorenstein liaison class, by using explicitly constructed G-liaisons.

Since a union of two planes meeting at a point is contained in a quadric hypersurface with one double point, it follows from our Theorem 4 that all the above curves with the same Rao module are equivalent for G-liaison [3, 6.4].

A third application is the following

THEOREM 6. *Any arithmetically Gorenstein scheme V in $\mathbb{P}^n$ is in the Gorenstein liaison class of a complete intersection (glicci).*

For the proof [4, 7.1] we use the higher-dimensional analogues of the results described in this paper for an AG 3-fold X. By a Bertini-type theorem of Altman and Kleiman, one can find a complete intersection scheme X in $\mathbb{P}^n$, containing V, of dimension two greater than V, and smooth outside of V. Then X is normal and AG, and V is a codimension two AG scheme in X, so there is an exact sequence

$$0 \to \mathcal{O}_X(-a) \to \mathcal{E} \to \mathcal{I}_{V,X} \to 0$$

where $\mathcal{E}$ is a rank 2 ACM sheaf on X. Let $\mathcal{M}$ be a rank 2 dissocié sheaf on X and consider the new $\mathcal{N}$-type resolution of V,

$$0 \to \mathcal{O}_X(-a) \oplus \mathcal{M} \to \mathcal{E} \oplus \mathcal{M} \to \mathcal{I}_{V,X} \to 0.$$

Then we apply the analogue of Proposition 3, which is [4, 3.4], to remove $\mathcal{E}$ and obtain another subscheme $V' \subseteq X$, in the same even G-liaison class as V, with an $\mathcal{N}$-type resolution

$$0 \to \mathcal{L}' \to \mathcal{M} \to \mathcal{I}_{V',X}(a') \to 0.$$

Since $\mathcal{M}$ is rank 2 dissocié, it follows that V' is a complete intersection in X, and since X is itself a complete intersection in $\mathbb{P}^n$, V' is also a complete intersection in $\mathbb{P}^n$. Since G-liaisons in X are also G-liaisons in $\mathbb{P}^n$, we find that V is glicci, as required.

7. An open problem

If there is a moral to all the investigations of Gorenstein liaison so far, it seems to me that good results are obtained for schemes with some special structure, such as determinantal schemes [11], or schemes of codimension 2 in low-degree hypersurfaces, such as the ones considered in Sections 4,5 above.

To describe a situation on the border between what is known and what is not known, I would like to consider the case of zero-dimensional subschemes of a nonsingular cubic surface in $\mathbb{P}^3$. Though of one dimension lower than the discussions earlier in this paper, I think it is a good arena to test the essential difficulties of the subject.

So, let X be a nonsingular cubic surface in $\mathbb{P}^3$. We consider zero-schemes $Z \subseteq X$. Any zero-scheme is ACM, so there are two problems to consider.

PROBLEM 5. Is every zero-scheme $Z \subseteq X$ in the G-biliaison equivalence class of a point?

PROBLEM 6. Is every zero-scheme $Z \subseteq X$ in the G-liaison equivalence class of a point?

Both problems are open at present. I will discuss what is known about them so far.

Using explicit G-liaisons and G-biliaisons on ACM curves on X, one can show that any set Z of n points in general position on X is G-liaison equivalent to a point [8, 2.4]. The proof of this result is curious, in that one uses sequences of liaisons where the number of points may have to increase before it decreases. For example, starting with 18 general points, one makes links to the following numbers of points (always in general position): $18 \to 20 \to 28 \to 22 \to 16 \to 13 \to 7 \to 5 \to 3 \to 1$. For points in special position, it seems hopeless to generalize this method.

Another approach, more in the spirit of this paper, is to study the category of ACM sheaves on X. Faenzi [6] has classified the rank 2 ACM sheaves on X. Up to twist, there is a finite number of possible Chern classes, and for fixed Chern classes, the possible sheaves form algebraic families of dimensions ≤ 5. Already the presence of families of dimension > 1 shows that we are in a situation of "wild CM-type" (see Appendix). Looking at Faenzi's results, again it seems hopeless to achieve a complete classification of ACM sheaves of all ranks on X. One can show, however, that there are families of arbitrarily high dimension of indecomposable ACM sheaves of higher rank.

However, to answer the two problems above, one would not need a complete classification of ACM sheaves on X. For an affirmative answer to Problem 5, it would be sufficient to show [3, 4.3].

($*$) Every orientable ACM sheaf $\mathcal{E}$ on X has a resolution

$$0 \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{E} \to 0$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are layered ACM sheaves (i.e., successive extensions of rank 1 ACM sheaves).

In regard to this property, there are examples of rank 2 ACM sheaves $\mathcal{E}$ on X, that are not layered themselves, but do have a resolution of this form. So there seems to be some hope that this may hold.

For an affirmative answer to Problem 6, it would be sufficient to show [4, 5.4].

($**$) Every orientable ACM sheaf on $\mathcal{E}$ is stably equivalent to a *double-layered* sheaf on X (which is a successive extension of rank 2 ACM sheaves and their syzygies).

Appendix. MCM modules and ACM sheaves

In this appendix we give a brief outline of some algebraic results that are needed to justify the results on ACM sheaves on quadric hypersurfaces used in Sections 4,5 above.

Let $R, \mathfrak{m}$ be a Cohen–Macaulay local ring. A *maximal Cohen–Macaulay* (MCM) *module* is a finitely generated R-module M, with depth $M = \dim R$ and Supp $M = \operatorname{Spec} R$.

For example, if R is a regular local ring, every MCM module has homological dimension zero, and so is free. Conversely, if $R, \mathfrak{m}$ is a Cohen–Macaulay local ring over which every MCM is free, then R is regular. Indeed, for any R-module N, consider a free resolution of length $n = \dim R$, and let M be the kernel at the last step. Then M is an MCM module, hence free, and so N has a finite free resolution. Thus the ring R has finite global homological dimension, and by a theorem of Serre, this implies that R is regular.

Thus the presence of non-trivial MCM modules characterizes non-regular local rings, and the category of MCM modules is an interesting measure of the complexity of the singularity of the local ring.

In certain circumstances, Y. Drozd [5] has shown that local rings can be divided into three classes, depending on the behavior of the MCM modules. It is true in any case that an MCM module can be written uniquely as a direct sum of *indecomposable* MCM modules, namely those that allow no further direct sum decomposition. We say that R is of *finite* CM-*type* if there is only a finite number of indecomposable MCM modules. We say R is of *tame* CM-*type* if the indecomposable MCM modules form a countable number of families of dimension at most one. We say R is of *wild* CM-*type* if there are families of arbitrarily large dimension of indecomposable MCM modules. The tame-wild dichotomy theorem says (in certain cases) that only these cases can occur. While to my knowledge this has not been proved in general, we can keep it in mind as a principle of what to expect when studying MCM modules.

The same definitions apply to the case of graded rings and graded modules, and thus admit a translation into sheaves on projective schemes. If X is a ACM scheme in $\mathbb{P}^n$, we have defined an ACM *sheaf* on X to be a locally Cohen–Macaulay coherent sheaf $\mathcal{E}$ on X with no intermediate cohomology: $H^i_*(\mathcal{E}) = 0$ for $0 < i < \dim X$. If R is the homogeneous coordinate ring of X, then we obtain a correspondence between ACM sheaves on X and graded MCM modules on R by sending a sheaf $\mathcal{E}$ to the module $E = H^0_*(\mathcal{E})$, and sending the module E to the associated sheaf $\mathcal{E} = \tilde{E}$. In carrying over results and definitions from the local case, we should consider graded modules up to shift, and ACM sheaves up to twist. So we can say X is of finite CM-type if there is only a finite number (up to twist) of isomorphism classes of indecomposable ACM sheaves.

To illustrate the different CM-types in the projective case, note that if X is a nonsingular curve in $\mathbb{P}^n$, then an ACM sheaf on X is just a locally free sheaf $\mathcal{E}$, also called a vector bundle. If X is rational, of degree d, the only indecomposable vector bundles (up to twist by $\mathcal{O}_X(1)$) are line bundles of degrees $0 \le e < d$, so X is of finite CM-type. If X is an elliptic curve, then by the classification theorem of Atiyah, for each rank and degree (mod $d = \deg X$) there is a one-parameter family of isomorphism classes of indecomposable vector bundles of rank r and degree e. Thus X is of tame CM-type. And if the genus of X is $g \geq 2$, then as the rank grows, so does the dimension of the moduli space of stable vector bundles, so X is of wild CM-type.

In the complex-analytic and complete local ring case, those local rings of isolated hypersurface singularities of finite CM type have been classified [13], [2]. They are the local rings of simple singularities in the sense of Arnol'd; in each dimension they are associated with Dynkin diagrams A_n, D_n, E_6, E_7, E_8, and their equations can be written explicitly.

Carrying these results over to the graded case, one obtains a list of all projective schemes of finite CM-type [18], namely, projective spaces, nonsingular quadric hypersurfaces in any dimension, the rational cubic scroll in $\mathbb{P}^4$, and the Veronese surface in $\mathbb{P}^5$. Furthermore, the indecomposable ACM sheaves on these varieties can be described explicitly, and this is where we find that there is just one non-trivial indecomposable ACM sheaf on the nonsingular quadric 3-fold, mentioned in the proof of Theorem 5.

The main tools for studying MCM modules on hypersurface singularities, or ACM sheaves on hypersurfaces in $\mathbb{P}^n$, are the matrix factorization, and the double branched covers and periodicity theorems of Knörrer [13]. We explain these in the projective case.

Let X be a hypersurface in $\mathbb{P}^n$, and let $\mathcal{E}$ be an ACM sheaf on X. Since the associated graded module $E = H^0_*(\mathcal{E})$ has depth n over the coordinate ring $P = k[x_0, \dots, x_n]$ of $\mathbb{P}^n$, there is a resolution

$$0 \to \mathcal{L}_1 \xrightarrow{\varphi} \mathcal{L}_0 \to \mathcal{E} \to 0$$

by dissocié sheaves $\mathcal{L}_i$ on $\mathbb{P}^n$ of the same rank m. This gives a square matrix φ of homogeneous forms in P. Then one shows that there is another matrix ψ of the same rank, with the property that $\psi \cdot \varphi = \varphi \cdot \psi = f \cdot id$, where f is the equation of the hypersurface. This is called a *matrix factorization* of f. One sees also that $\det \varphi = f^r$, where $r = \operatorname{rank} \mathcal{E}$. These constraints allow one to gain information about the possible ACM sheaves $\mathcal{E}$ when the numbers are small enough.

The other technique is Knörrer's double branched cover, and periodicity theorems, which allow one to pass from a hypersurface X in $\mathbb{P}^n$ defined by a polynomial $f \in P$ to the hypersurface X' in $\mathbb{P}^{n+1}$ defined by $f + x^2$, or the hypersurface X'' in $\mathbb{P}^{n+2}$ defined by $f + x^2 + y^2$, where x and y are new variables.

In the paper [3], we use these techniques to show that the singular quadric 3-fold X in $\mathbb{P}^4$ with one double point is of *countable* CM-*type*, namely it has only countably many indecomposable ACM sheaves (up to twist), and these are $\mathcal{O}_X, \mathcal{I}_D, \mathcal{I}_E$, where D, E are the two types of planes in X, and two infinite sequences $\mathcal{E}_\ell$ and $\mathcal{E}'_\ell$, for $\ell = 1, 2, \dots$, of rank 2 ACM sheaves that are each extensions of suitable twists of $\mathcal{I}_D$ and $\mathcal{I}_E$ [3, 6.2], hence layered. This is the result needed for the proof of Theorem 4 above.

A good reference for the material described in this appendix, besides the original papers, is the survey article [12] and the book of Yoshino [18].

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Robin HARTSHORNE, Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA
e-mail: `robin@math.berkeley.edu`

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S. Nollet

DEFORMATIONS OF SPACE CURVES: CONNECTEDNESS OF HILBERT SCHEMES

Dedicated to Paolo Valabrega on the occasion of his 60 th birthday

Abstract. We survey the Hilbert schemes $H_{d,g}$ of Cohen-Macaulay space curves having degree d and genus g, giving their geography and the current state of the connectedness problem. Focusing on a specific example, we then describe the irreducible families of curves in $H_{4,-99}$ and explain the connectedness, paying special attention to certain deformations on the double quadric surface. We close with some new results, determining which families of degree four curves are subcanonical and showing how some examples of Chiantini and Valabrega fit into this classification.

1. Introduction

Early in the development of scheme theory in algebraic geometry, Grothendieck constructed the fine moduli space for flat families of subschemes in $\mathbb{P}^n$, known as the Hilbert scheme [15]. Since the Hilbert polynomial is constant for flat families over a connected base, the Hilbert scheme $\mathrm{Hilb^n}$ can be written as a disjoint union of pieces $\mathrm{Hilb^n_{p(z)}}$ indexed by the corresponding Hilbert polynomials. As a fine moduli space, these schemes come equipped with universal flat family

$$\begin{array}{ccc} X & \subset & \mathrm{Hilb^n_{p(z)}} \times \mathbb{P}^n \\ & & \downarrow \\ & & \mathrm{Hilb^n_{p(z)}} \end{array} \tag{1}$$

having fibres with Hilbert polynomial $p(z)$ such that for any flat family

$$\begin{array}{ccc} Y & \subset & T \times \mathbb{P}^n \\ & & \downarrow \\ & & T \end{array} \tag{2}$$

with fibres of Hilbert polynomial $p(z)$, there is a unique map $T \to \mathrm{Hilb^n_{p(z)}}$ such that diagram (2) is obtained from diagram (1) by pull-back. Thus one studies the Hilbert scheme by producing flat families. As Grothendieck showed that $\mathrm{Hilb^n_{p(z)}}$ is projective over $\mathrm{Spec}\,\mathbb{Z}$, the set of all projective subschemes is encoded by equations with integer coefficients.

Since flat families over a connected base have constant Hilbert polynomial, it's natural to ask whether the converse is true: given two subschemes in $\mathbb{P}^n$ with the same Hilbert polynomial, is there a connected flat family of which both are a member? Equivalently, is the Hilbert scheme connected? This was answered by Hartshorne in his PhD thesis [19].

THEOREM 1 (Hartshorne, 1962). *For any $p(z) \in \mathbb{Q}[z]$ and any field k, the Hilbert scheme $\mathrm{Hilb}^{\mathrm{n}}_{\mathrm{p(z)}}$ for closed subschemes $X \subset \mathbb{P}^n_k$ with Hilbert polynomial $p(z)$ is connected whenever it is non-empty.*

The *geography* is an important aspect of any moduli problem: for which natural invariants of the problem is the moduli space non-empty? There are at least three characterizations of the polynomials $p(z) \in \mathbb{Q}[z]$ for which there is a subscheme $V \subset \mathbb{P}^n$ having Hilbert polynomial $p(z)$. One follows from Macaulay's theorem on the growth of the Hilbert function of a standard k-algebra [17], another is a consequence of Hartshorne's thesis [19] and a third occurs naturally from Green's interpretation of Macaulay's bound in terms of restricted linear series [14]: a summary and comparison is given in [4].

We now specialize to space curves: take $n = 3$ and let $\mathrm{Hilb}_{d,g}$ denote the Hilbert scheme of subschemes in $\mathbb{P}^3$ with Hilbert polynomial $p(z) = dz + 1 - g$, the curves of degree d and arithmetic genus g. Classically one is interested in the open subscheme

$$H^0_{d,g} \subset \mathrm{Hilb}_{d,g}$$

corresponding to smooth connected curves. The geography for this problem (the pairs (d, g) for which $H^0_{d,g}$ is non-empty) was known to Halphen and completely proved by Gruson and Peskine a hundred years later [16]. As to connectedness, we have the following results of Harris [18] and Ein [10].

THEOREM 2 (Harris, 1982). *$H^0_{d,g}$ is irreducible if $d \geq \frac{5}{4}g + 1$.*

THEOREM 3 (Ein, 1986). *$H^0_{d,g}$ is irreducible if $d \geq g + 3$.*

EXAMPLE 1. The Hilbert schemes $H^0_{d,g}$ are not connected in general: the smallest example is $H^0_{9,10}$ [20, IV, Ex. 6.4.3], which has two connected components, the curves of type $(3, 6)$ on a smooth quadric and complete intersections of two cubics. More generally, $H^0_{d,g}$ is not connected for $d \geq 9$ and $g = 2d - 8$. Indeed, the curves C of type $(3, d-3)$ on a smooth quadric satisfy $h^0\mathcal{O}_C(2) = 9$ and $h^0\mathcal{I}_C(2) = 1$ while curves D not lying on a quadric satisfy $h^0\mathcal{O}_D(2) \geq 10$ and $h^0\mathcal{I}_D(2) = 0$. By semi-continuity, it follows that the curves of type $(3, d-3)$ form a connected component of $H^0_{d,2d-8}$. Note that there exist other components, as such curves exist on a cubic or quartic surface. Guffroy conjectures that $H^0_{d,g}$ is irreducible for $g < 2d - 8$ (i.e. $d > \frac{1}{2}g + 4$) and proves it for $d \leq 11$ [17]. If true, the conjecture would strongly improve the results above.

The subject of this survey is yet a third moduli space, namely the Hilbert scheme of locally Cohen-Macaulay curves without isolated points, the pure one-dimensional subschemes of $\mathbb{P}^3$ of degree d and genus g. Following Martin-Deschamps and Perrin [27, 29], we denote these Hilbert schemes by $H_{d,g}$, which sit between the two extremes considered above:

$$H^0_{d,g} \subset H_{d,g} \subset \mathrm{Hilb}_{d,g}.$$

The Hilbert schemes $H_{d,g}$ are natural from the perspective of liaison theory, which has seen a great deal of activity over the last 25 years: Migliore's book [31] provides an excellent survey of this work. The point is that liaison preserves the property of being locally Cohen-Macaulay [31, Cor. 5.2.12] but does not preserve geometric properties such as smoothness, irreducibility, or reducedness. On the other hand, even the most general locally Cohen-Macaulay curves can be brought to the classical curves through a sequence of liaisons, as proved by Rao [38, Thm. 2.6].

THEOREM 4 (Rao, 1979). *Every liaison class contains a smooth connected curve.*

Thus the schemes $H_{d,g}$ are the result of starting with the smooth connected curves and closing off under the equivalence relation of liaison. In view of the connectivity results above, the following question is natural:

PROBLEM 7. For which pairs (d, g) is $H_{d,g}$ connected?

REMARK 1. This does not follow in any easy way from the proof of Theorem 1, as Hartshorne constructs deformations which typically pass through (non-reduced) subschemes having embedded points. The real question here is whether curves with embedded points can be avoided.

In addressing the status of Problem 7, we begin with the geography of locally Cohen-Macaulay space curves in §2. This includes (a) the determination of the pairs (d, g) for which $H_{d,g}$ is non-empty and (b) the cohomological bounds leading to the special families of extremal and subextremal curves. The extremal curves become prominent in §3 when we give connectedness results for the Hilbert schemes. We follow this up with an example in §4, describing all the irreducible components of the Hilbert scheme $H_{4,-99}$ and explaining why this scheme is connected. In §5 we discuss deformations of curves on a double surface and show how a disjoint union of two double lines can be deformed to a multiplicity four line without adding embedded points, a crucial part of the proof that $H_{4,-99}$ is connected. Finally, in §6 we determine which families of degree four curves are sub-canonical. In particular, we show how examples of Chiantini and Valabrega [6, Ex. 3.1 and 3.2] fit into our classification.

The author thanks E. Cabral Balreira for his help with making the figures and Mario Valenzano for corrections on the first draft.

2. The geography of Cohen-Macaulay curves

In this section we describe the pairs (d, g) for which our Hilbert schemes $H_{d,g}$ are non-empty. As a byproduct of the proof, we will encounter the extremal curves, which play an important role in the following section. The starting point is the following theorem [28, Thm. 2.5 and Cor. 2.6].

THEOREM 5 (Martin-Deschamps and Perrin, 1993). *Assume char $k = 0$. If*

$C \in H_{d,g}$ is non-planar, then the Rao function $h^1\mathcal{I}_C(n)$ is bounded by the function depicted in Figure 1. In particular, $g \le \binom{d-2}{2}$.

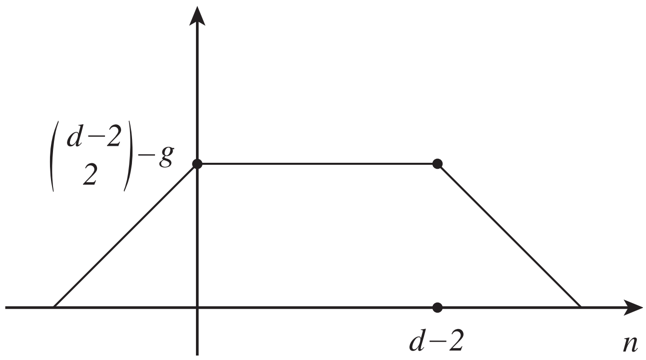

Figure 1: Bound of Theorem 5 on $h^1\mathcal{I}_C(n)$ for non-planar curves

This generalizes to curves in higher dimensional projective space [8], though the bounding function is more complicated. The characteristic zero hypotheses is used to prove that if C is a curve of degree $d \ge 3$ not contained in a plane, then the general hyperplane section $H \cap C$ is not contained in a line. While this fails in characteristic $p > 0$ [21, Ex. 2.3], the bound on cohomology still holds [32, Prop. 2.1], as does the bound on the genus [21, Cor. 3.6]:

THEOREM 6 (Hartshorne 1994). *The Hilbert scheme $H_{d,g}$ is non-empty if and only if either*

(*a*) *$d \ge 1$ and $g = \binom{d-1}{2}$, or*

(*b*) *$d \ge 2$ and $g \le \binom{d-2}{2}$.*

One way to prove that $H_{d,g}$ is non-empty for $g \le \binom{d-2}{2}$ is to observe that there are curves which achieve equality in Theorem 5 [29, Prop. 0.5]:

THEOREM 7 (Martin-Deschamps and Perrin, 1996). *For all $d \ge 2$ and $g \le \binom{d-2}{2}$ there are curves $C \in H_{d,g}$ giving equality in Theorem 5 for all n.*

The curves of Theorem 7 are called *extremal curves* and have some interesting properties. For example, the subset of extremal curves forms an irreducible component $E \subset H_{d,g}$ [29, Thm. 3.7], which is non-reduced except when $d = 2$ (double lines), $g = \binom{d-2}{2}$ (ACM extremal curves) or $d = 3$ and $g = -1$ [29, Thm. 5.3].

REMARK 2. The following are equivalent:

1. C is an extremal curve.

2. C is a minimal curve for a complete intersection module annihilated by two linear forms (this allows one to write the total ideal and minimal resolutions for extremal curves [29, Prop. 0.5, 0.6 and Thm. 1.1]).

3. C is non-planar of degree d and contains a planar subcurve of degree $d-1$ ([11, §2, Thm. 8] or [32, Prop. 2.2]).

Assuming char $k = 0$, Ellia observed [11, §2, Prop. 9] that a curve which is neither planar nor extremal satisfies even stronger bounds on the Rao function. Using Schlesinger's *spectrum* of a curve [40], this bound was refined while removing the characteristic zero hypothesis [32, Thm. 2.11]:

THEOREM 8 (Ellia and Nollet, 1997). *If $C \in H_{d,g}$ is a non-planar and non-extremal, then the Rao function $h^1\mathcal{I}_C(n)$ is bounded by the function depicted in Figure 2. In particular, $g \leq \binom{d-3}{2} + 1$.*

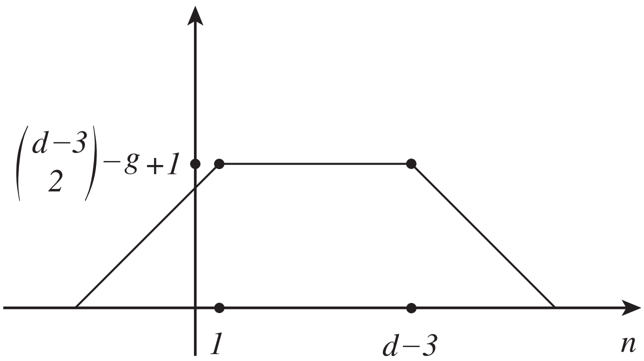

Figure 2: The bound of Theorem 8 on $h^1\mathcal{I}_C(n)$ for non-extremal curves

A curve $C \in H_{d,g}$ is *subextremal* if it achieves the bound of Theorem 8 for all n. A curve $C \in H_{d.g}$ is subextremal if and only if it is a height one elementary biliaison of an extremal curve $C' \in H_{d-2,g+3-d}$ on a quadric surface [32, Thm. 2.14] and hence exist for all $d \geq 4$ and $g \leq \binom{d-3}{2} + 1$: letting $S \subset H_{d,g}$ denote the family of subextremal curves, the universal biliaison scheme of Martin-Deschamps and Perrin shows that S is irreducible. Indeed, if $E \subset H_{d-2,g+3-d}$ is the extremal component, we can consider the set B of triples (C, C', Q) for which C is a height one biliaison of C' on the quadric surface Q. The natural projections

$$\begin{array}{ccc} B & \xrightarrow{p_1} & S \\ {\scriptstyle p_2}\downarrow & & \\ E & & \end{array}$$

are smooth and irreducible [27, VII, §4], hence irreducibility of E implies irreducibility of S.

REMARK 3. Given Theorem 5 and Theorem 8, one might expect that curves which are neither planar nor extremal nor subextremal should satisfy even stronger bounds. This fails, however: there are curves which give equality in Theorem 8 for some values of n, but not others [32, Ex. 2.15 and 2.17].

REMARK 4. As the extremal curves form an irreducible component, one might expect that the closure of the subextremal curves $S \subset H_{d,g}$ to form an irreducible component as well (though S itself is not closed: its closure contains extremal curves [34]). Uwe Nagel has informed me that this is indeed true and is current joint work between he, Nadia Chiarli and Silvio Greco.

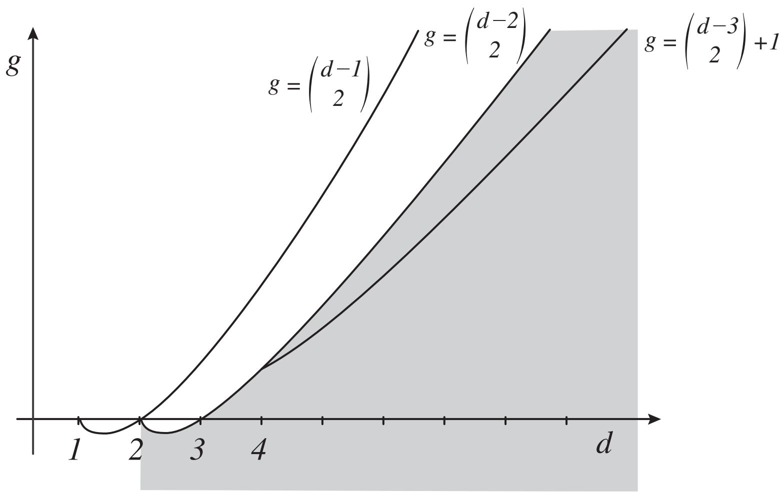

Figure 3: The geography for locally Cohen-Macaulay curves

3. Connectedness results

In this section we summarize the current state of Problem 7. We will begin with some general results about families of curves that can be deformed to extremal curves and then proceed to particular ranges. In terms of the geography of Cohen-Macaulay curves (Figure 3), we will see in Theorems 10 and 11 that $H_{d,g}$ is connected for pairs (d, g) near the boundaries at the top and to the left.

Many families of curves can be deformed to extremal curves (without passing through curves with embedded points).

THEOREM 9. *The following families of curves can be deformed in $H_{d,g}$ to extremal curves.*

(1) *Disjoint unions of lines.*

(2) *Smooth rational curves.*

(3) *Smooth connected curves with $d \geq g + 3$.*

(4) *ACM curves.*

(5) *The disjoint union of an extremal curve and a line.*

(6) *The union of an extremal curve and a line meeting at a point.*

(7) *Any curve in the liaison class of an extremal curve.*

Proof. (1)-(6) are results of Hartshorne [22] and (7) is due to Perrin [18]. □

When the arithmetic genus g is large relative to the degree d, the Hilbert scheme $H_{d,g}$ has few irreducible components, making it relatively easy to check connectedness. The following result is the work of several authors.

THEOREM 10. *If $g \geq \binom{d-3}{2} - 1$, then $H_{d,g}$ is connected.*

Proof. According to Theorem 6, either $g = \binom{d-1}{2}$ (in which case $H_{d,g}$ is the irreducible family of plane curves) or $g \leq \binom{d-2}{2}$. In the range $\binom{d-3}{2} + 1 < g \leq \binom{d-2}{2}$, Theorem 8 shows that $H_{d,g} = E$ is the family of extremal curves, which is irreducible by the work of Martin-Deschamps and Perrin [29].

There are three more arithmetic genera to check, but things become more delicate, as $H_{d,g}$ is not irreducible.

If $g = \binom{d-3}{2} + 1$, then Theorem 8 shows that each curve $C \in H_{d,g}$ is extremal or ACM, since the bound on $h^1\mathcal{I}_C(n)$ is zero. Conversely each ACM curve in $H_{d,g}$ is subextremal by definition, hence $H_{d,g} = E \cup S$ consists only of extremal and subextremal curves. Finally $E \cap \overline{S} \neq \emptyset$ by [34] and $H_{d,g}$ is connected.

If $g = \binom{d-3}{2}$, then the non-extremal curves C satisfy $h^1\mathcal{I}_C(n) \leq 1$. Samir Aït-Amrane showed [1] that $H_{d,g}$ has three irreducible components for large d: (a) extremal curves, (b) subextremal curves and (c) bilinks of height one from a double line of genus -1 on a surface of degree $d - 2$. Both families (b) and (c) specialize to family (a) by Theorem 9(7), but Samir's method was to use the triads developed by Hartshorne, Martin-Deschamps and Perrin [23].

If $g = \binom{d-3}{2} - 1$, then the non-extremal curves C satisfy $h^1\mathcal{I}_C(n) \leq 2$. Irene Sabadini showed [39] that $H_{d,g}$ has 4 irreducible components for $d \geq 9$: (a) extremal curves, (b) subextremal curves, (c) bilinks of height one from a double line of genus -2 on a surface of degree $d - 2$ and (d) disjoint unions of an ACM extremal curve of degree $d - 1$ and a line. Families (b) and (c) specialize to (a) by Theorem 9(7) and family (d) specializes to (a) by Theorem 9(5). □

THEOREM 11. *For $d \leq 4$, $H_{d,g}$ is connected whenever it is non-empty.*

Proof. Since $H_{d,g}$ is irreducible for $g = \binom{d-1}{2}$, we may assume that $d \geq 2$ and $g \leq \binom{d-2}{2}$ by Theorem 6. There are just three cases to consider.

If $d = 2$, then $H_{2,g}$ consists only of double lines, which were classified by Migliore [30]. These form an irreducible family.

If $d = 3$, then $H_{3,g}$ has exactly $\lceil \frac{4-g}{3} \rceil$ irreducible components, most consisting only of triple lines. In this case there are curves which lie in the intersection of all the irreducible components [33, Prop. 3.6 and Remark 3.9], hence $H_{3,g}$ is connected.

Finally if $d = 4$, then $H_{4,g}$ has roughly $\frac{g^2}{24}$ irreducible components, most of the families consisting of 4-lines (there are roughly $\frac{-3g}{2}$ families whose general member is not supported on a line). In work of the author and Enrico Schlesinger [36], these components were classified and connectedness was established through a variety of methods (see next two sections). One new feature to this example is the existence of an irreducible component which does *not* intersect the extremal component: the general curve is a multiplicity four structure on a line which has generic embedding dimension three. □

Looking at the number of irreducible components of the Hilbert schemes, one might guess that $H_{d,g}$ has on the order of g^{d-2} irreducible components, at least for $g << 0$. For degrees $d = 2$ and $d = 3$, the reason for the large number of components is the number of different families of multiplicity structures on a line. Will this behavior persist for larger d? At the other edge, there are few components for $g \sim \binom{d-3}{2}$. Can one find an upper bound on the number?

PROBLEM 8. How many irreducible components does $H_{d,g}$ have?

(*a*) For $g << 0$? Is it of order g^{d-2}? Can one show this is a lower asymptotic bound?

(*b*) For g near $\binom{d-3}{2}$? Can one find an upper bound?

4. The Hilbert scheme $H_{4,-99}$

In this section we fully describe an example, the Hilbert scheme $H_{4,-99}$. We list the irreducible components and their dimensions, as well as describing the general curve in the corresponding family. Complete proofs for general arithmetic genus g can be found in [36].

REMARK 5. The following refer to Table 1.

(a) Notation: L always denotes a line, D a smooth conic, Z a curve of degree two with given genus, and W a triple line.

(b) In family $G_{10,m}$, we set $\epsilon(m) = 0$ for $m > 1$, $\epsilon(1) = 1$ and $\epsilon(0) = 3$.

(c) Most of the families consist of multiplicity structures on a line.

(1) The *thick* 4-lines that occur in family G_4 are curves C with linear support L such that $\mathcal{I}_C \subset \mathcal{I}_L^2$ (they contain $L^{(1)}$).

(2) A multiplicity structure C of degree k on a line L which is not thick is called *quasi-primitive* [2] and has a Cohen-Macaulay filtration

$$L \subset Z_2 \subset Z_3 \subset \cdots \subset Z_k = C$$

with quotients $\mathcal{I}_L/\mathcal{I}_{Z_2} \cong \mathcal{O}_L(a)$, $\mathcal{I}_{Z_2}/\mathcal{I}_{Z_3} \cong \mathcal{O}_L(2a+b)$ and (if necessary) $\mathcal{I}_{Z_4}/\mathcal{I}_{Z_3} \cong \mathcal{O}_L(3a+c)$ with $b \leq c$. The numbers a, b and c give the *type* of the multiple line: thus a double line has type a, a triple line has type (a, b) and a quadruple line has type (a, b, c). We do not give the type for double lines, because the type is determined by the genus.

(d) The last five families listed come with parameters, meaning that there are several irreducible components. For example, there are actually 32 irreducible families of curves of $G_{9,a}$ (each consists of a disjoint union of a triple line and a reduced line), one for each $1 \leq a \leq 32$. Similarly there are 33 families of type $G_{7,a}$, 32 of type $G_{8,a}$, 50 of type $G_{10,m}$ and 376 of type $G_{11,a,b}$, for a total of 529 irreducible components.

(e) We prove connectedness by the following plan:

$$\begin{array}{ccccccccc}
G_{9,a} & & G_{10,0}/G_{10,1} & & G_6 & & G_3 & & \\
 & \searrow & \downarrow & \swarrow & & & \downarrow & & \\
G_{8,a} & \rightarrow & E = G_1 & \leftarrow & G_2 & \rightarrow & G_4 & \leftarrow & G_{11,a,b} \\
 & \nearrow & \uparrow & & & & \uparrow & & \\
G_{7,a} & & G_5 & & & & G_{11,0,m-1} & \subset & G_{10,m>1}
\end{array}$$

Each arrow represents a specialization of curves. The extremal component G_1 draws several arrows. The arrows $G_6 \rightarrow G_1, G_{8,a} \rightarrow G_1, G_{9,a} \rightarrow G_1$ and $G_{10,0}/G_{10,1} \rightarrow G_1$ follow from Theorem 9, parts (5) and (6) and results in [33]. The arrows $G_2 \rightarrow G_1$ and $G_5 \rightarrow G_1$ can be found in [25], as the relevant curves lie on a double plane. The arrow $G_{7,a} \rightarrow G_1$ is obtained by actually writing down equations of the deformation. The arrows $G_2 \rightarrow G_4$ and $G_3 \rightarrow G_4$ arise by varying a resolution for the Rao module [36, Prop. 4.2 and 4.3], while the arrow $G_{11,a,b} \rightarrow G_4$ arises by a tricky deformation of a resolution for the ideals, using the Buchsbaum-Eisenbud criterion [5] to check exactness [36, Prop. 2.4].

Finally, the curves in $G_{10,m}$ with $m > 1$ consist of disjoint unions of double lines of genus < -1. As the support of these curves lies on a smooth quadric, the curves themselves lie on a double quadric. On this surface we were able to deform these curves to a quasi-primitive 4-line in $G_{11,0,m-1}$ on a fixed double quadric: we explain this in the next section. The quasi-primitive 4-lines deform to G_4 as in arrow $G_{11,a,b} \rightarrow G_4$.

Table 1: The 529 Irreducible Components of $H_{4,-99}$

Label	General Curve	Dimension
G_1 Extremal curves	$D \cup Z$ D smooth conic $p_a(Z) = -102$, length$(D \cap Z) = 4$	213
G_2 Subextremal curves	$L_1 \cup_{2P} Z \cup_{2Q} L_2$ $L_1 \cap L_2 = \emptyset$ $p_a(Z) = -101$	211
G_3	$D \cup_{2P} Z$ D smooth conic $p_a(Z) = -100$	211
G_4	thick 4-line	306
G_5	double conic	211
G_6	$Z \cup_{2P} L_1 \dot{\cup} L_2$ $p_a(Z) = -99$	209
$G_{7,a}$ $1 \leq a \leq 33$	$W \cup_{3P} L$ W quasiprimitive 3-line of type $(a, 99 - 3a)$	$209 - a$
$G_{8,a}$ $1 \leq a \leq 32$	$W \cup_{2P} L$ W quasiprimitive 3-line of type $(a, 98 - 3a)$	$208 - a$
$G_{9,a}$ $1 \leq a \leq 32$	$W \dot{\cup} L$ W quasiprimitive 3-line of type $(a, 96 - 3a)$	$206 - a$
$G_{10,m}$ $0 \leq m \leq 49$	$Z_1 \dot{\cup} Z_2$ $p_a(Z_1) = -m$ $p_a(Z_2) = m - 98$	$206 + \epsilon(m)$
$G_{11,a,b}$ $1 \leq a \leq 16$ $0 \leq b \leq 48 - 3a$	Quasiprimitive 4-line of type $(a, b, c = 96 - 6a - b)$	$205 - 3a$

5. Curves on the double quadric

Hartshorne and Schlesinger gave a satisfying classification of curves lying on the double plane [25], describing all the irreducible components and showing connectedness. Their primary tool was a certain triple associated to such a curve (Definition 1 below). In this section we describe joint work of Enrico Schlesinger and the author [35], which uses these triples on a double surface to give a criterion for when the underlying triple of a curve can be spread out in a flat family. As an application we obtain in Example 3

(a) the inclusion

$$G_{11,0,m-1} \subset G_{10,m} \tag{3}$$

needed to show connectedness of $H_{4,-99}$ (see Remark 5 (e)).

To set the scene, let F be a smooth surface on a smooth threefold T with doubling $X = 2F$. More generally one can take X to be a *ribbon* over F in the sense of Eisenbud and Bayer [3].

DEFINITION 1. *For each curve $C \subset X$, the triple $T(C) = \{Z, R, P\}$ is defined as follows:*

1. *P is the support of C, the one dimensional part of $C \cap F$.*
2. *R is the curve part of C residual to P.*
3. *Z is the zero-dimensional part of $C \cap F$, so $\mathcal{I}_{C\cap F,F} \cong \mathcal{I}_{Z,F}(-P)$.*

REMARK 6. If $T(C) = \{Z, R, P\}$, then $Z \subset R$ is zero-dimensional and Gorenstein [35, Prop. 2.1] and $R \subset P$ are divisors on F. The arithmetic genus is given by

$$p_a(C) = p_a(P) + p_a(R) + \deg_R \mathcal{O}_R(F) - \deg Z - 1. \tag{4}$$

EXAMPLE 2. We show below that both families of curves involved in inclusion (3) lie on a double quadric in $\mathbb{P}^3$ and compute their triples.

(a) A curve C in the family $G_{10,m}$ is a disjoint union $C = D_1 \cup D_2$ of double lines of genera $-m$ and $m-98$. The support $L_1 \cup L_2$ being contained in a 3-dimensional family of smooth quadrics, we can choose such a quadric Q containing neither D_1 nor D_2. Then C lies on the double quadric $X = 2Q$ and

$$T(C) = \{Z_1 \cup Z_2, L_1 \cup L_2, L_1 \cup L_2\}$$

where $Z_1 \subset L_1$ has length $m+1$ and $Z_2 \subset L_2$ as length $99 - m \geq m+1$ by formula (4). For C general, Z_i can be taken to be reduced sets of points.

(b) A curve C in the family $G_{11,0,m-1}$ is a quasi-primitive 4-line supported on L of type $(0, m-1, 97-m)$ (see Remark 5 (c)) and has underlying double line of type 0 and hence genus -1. Such a double line necessarily lies on a smooth quadric surface Q [33, Remark 1.5], hence C itself lies on the double quadric $X = 2Q$. It takes some work [36, Prop. 3.1], but one finds that

$$T(C) = \{Z, 2L, 2L\},$$

where $2L$ is the double line on Q and Z consists of $98-2m$ reduced points and $m+1$ double points on $2L$, none of which are contained in L .

REMARK 7. Looking at the triples in Example 2, we note that triple in part (b) is a limit of the triples in part (a): The two lines L_1 and L_2 come together on Q to form the double line $2L$, and the sets of reduced points Z_1 and Z_2 can be brought together in this limit to form $m+1$ double points and $98-2m$ reduced points. If we could lift this flat family of triples to a flat family of curves on $X=2Q$, we would have proved the inclusion (3).

Thus we consider the map $C \mapsto T(C)=\{Z,R,P\}$, which yields a natural transformation of functors

$$H \xrightarrow{t} D$$

where H is the set of flat families of curves on $X=2F$ and D is the set of triples $\{Z,R,P\}$. The functor D is represented by a disjoint union of locally closed subschemes $D_{z,r,p}$, where $\{z,r,p\}$ are the respective Hilbert polynomials of the entries in the triple $\{Z,R,P\}$. The pre-images under t stratify the Hilbert scheme H into locally closed subschemes $H_{z,r,p}$. The map t has a nice structure over the locus of the triples in D given by a vanishing [35, Thm. 3.2]:

THEOREM 12 (Nollet and Schlesinger, 2003). *Let $V\subset D_{z,r,p}$ be the open subscheme corresponding to triples $\{Z,R,P\}$ satisfying $H^1(\mathcal{O}_R(Z+P-F))=0$. Then the map $t^{-1}(V)\to V$ is the composition of an open immersion and an affine bundle projection. In particular, if $Y\subset V$ is irreducible, then $t^{-1}(Y)$ is also irreducible (hence connected).*

EXAMPLE 3. Here are two applications of Theorem 12.

(a) In view of Remark 7, Theorem 12 will prove the inclusion (3) if the vanishing $H^1(\mathcal{O}_R(Z+P-Q))=0$ holds for both the triples in Example 2. This is easy for the triples in (a): writing $R=L_1\cup L_2$ the vanishing boils down to $H^1(\mathcal{O}_{L_i}(Z_i+1-2))=0$ for $i=1,2$, which is immediate because $\deg Z_i\geq 0$. The vanishing for family (b) uses the Cohen-Macaulay filtration (Remark 5 (c)) for the 4-line C [36, Prop. 3.1].

(b) Some of the deformations used in showing the connectedness of $H_{3,g}$ follow from Theorem 12, for example [33, Prop. 3.3].

We close this section with some open questions involving the fibres of the map $t: H\to D$. Given a triple $T=\{Z,R,P\}\in D$ on F, the fibre $t^{-1}(T)$ is the set of locally Cohen-Macaulay curves $C\subset X$ with $T(C)=T$ (there may be none). There is a bijection between such curves C and surjections $\phi:\mathcal{I}_P\otimes\mathcal{O}_R\to\mathcal{O}_R(Z-F)$ such that $\phi\circ\tau=\sigma$, where

$$\tau=(\mathcal{O}(-F)\hookrightarrow\mathcal{I}_P)\otimes\mathcal{O}_R \qquad \sigma=(\mathcal{O}_R(-Z)\hookrightarrow\mathcal{O}_R)\otimes\mathcal{O}_R(Z-F)$$

are the natural maps [35, Prop. 2.2], hence these maps can be identified with an open subset

$$U\subset \operatorname{Hom}_R(\mathcal{O}_R(-P),\mathcal{O}_R(Z-F))\cong H^0(\mathcal{O}_R(Z+P-F)).$$

PROBLEM 9. Under what conditions is the open set U non-empty? When does a given triple $T=\{Z,R,P\}$ arise from a curve $C\subset X$?

REMARK 8. Obviously a solution to Problem 9 will have applications to classifying non-reduced curves of low degree. Here are some partial results.

(a) For triple $T = \{Z, R, P\}$, the open subset U is non-empty if any of the following conditions hold [35, Remark 2.7 and Prop. 2.5]:

(1) $H^1(\mathcal{O}_R(Z + P - F)) = 0$ and $\mathcal{O}_R(Z + P - F)$ is generated by global sections.

(2) $H^1(\mathcal{O}_R(Z + P - F - H)) = 0$ for a very ample divisor H on R.

(3) $H^1(\mathcal{O}_R(P - F)) = 0$.

(b) For the double plane $X = 2H \subset \mathbb{P}^3$, the subset U is non-empty for any triple, because condition (3) above holds. Chiarli, Greco and Nagel have described the curves with fixed triple using a matrix of homogeneous polynomials over H, giving a certain "normal form" to such curves C [9].

(c) The double quadric $X = 2Q \subset \mathbb{P}^3$ is more interesting [35, Ex. 2.8]. Let $T = \{Z, R, P\}$ be a triple with Z Gorenstein of dimension zero.

(1) If $R = P$ is a smooth rational curve, then T arises from a curve with one exception: $R = P$ is a conic and Z is a reduced point.

(2) If P is ample on Q and $R \neq P$, then T arises from a curve.

(3) If $R \subset P$ are disjoint unions of rulings on Q, then T arises from a curve if and only if $Z \cap L \neq \emptyset$ for each ruling $L \subset R$.

PROBLEM 10. Answer the question implicit in part *(c)* above: Which triples on a smooth quadric in $\mathbb{P}^3$ come from a curve on the double quadric? Describe the Hilbert schemes $H_{d,g}(2Q)$.

PROBLEM 11 (Hartshorne). Which curves on a double surface $2F \subset \mathbb{P}^3$ are flat limits of curves on *smooth* surfaces? For example, the thick triple line $L^{(2)}$ on the double plane $2H$ is a flat limit of twisted cubic curves lying on smooth quadric surfaces. What is special about the curve $L^{(2)}$ or its triple $\{\emptyset, L, 2L\}$ that allow it to be such a limit?

6. Subcanonical curves

In view of Paolo Valabrega's research interests [6, 7, 13, 41], we thought it would be interesting to determine which families of curves in $H_{4,-99}$ are subcanonical. A local complete intersection curve C is α-subcanonical if $\omega_C \cong \mathcal{O}_C(\alpha)$. The following restricts our attention to just a few families in $H_{4,-99}$.

PROPOSITION 1. *Suppose that* $C \in H_{4,-99}$ *is subcanonical. Then*

(1) $\omega_C \cong \mathcal{O}_C(-50)$.

(2) *C has no smooth rational irreducible components.*

(3) *C is one of the following:*

(*a*) *A double conic.*

(*b*) *A union of two double lines.*

(*c*) *A quasi-primitive* 4*-line.*

Proof. An α-subcanonical curve of degree d and genus g satisfies $d\alpha = 2g - 2$ in general, hence $\alpha = -50$ in our case.

Suppose that C has a smooth rational component R. Then $\deg R \neq 4$ because then $C = R$ has genus $0 \neq -99$. Also $\deg R \neq 3$ because then $C = R \cup L$ (L a line) forces $p_a(C) = \deg(R \cap L) - 1 \geq -1$ is not equal to -99. Thus R is a line or a conic. We write $C = S \cup R$ and restrict the exact sequence

$$0 \to \omega_S \oplus \omega_R \to \omega_C \to \omega_{S\cap R} \to 0$$

to R. Using $\omega_C = \mathcal{O}_C(-50)$ we obtain

$$\omega_S|_R \oplus \omega_R \xrightarrow{\phi} \mathcal{O}_R(-50) \to \omega_{S\cap R} \to 0.$$

Now the sheaf $\omega_S|_R$ is torsion and ω_R is either isomorphic $\omega_R = \mathcal{O}_R(-2)$ (if R is a line) or $\mathcal{O}_R(-1)$ (if R is a conic), hence ϕ is the zero map. This proves (2) by contradiction, since the cokernel of ϕ is finitely supported.

Let $B = \operatorname{Supp} C$. Then $\deg B < 4$ (since $g < -3$) and $\deg B \neq 3$ (since then C consists of a double line and a reduced curve of degree two). Thus $\deg B = 2$ or 1 and C is either (a) a double conic, (b) a union of two double lines or (c) a multiple line by part (2). If C were a thick 4-line supported on L, then it contains the triple line with ideal $\mathcal{I}_L^2$, which has degree 3 and genus 0 (a degenerate twisted cubic curve). According to [36, Lem. 4.1], C has spectrum

$$\{-98, 0, 1^2\},$$

which is a shorthand way of saying that the function $h_C(n) = \Delta^2 h^0 \mathcal{O}_C(n)$ satisfies $h_C(-98) = 1$, $h_C(0) = 1$, $h_C(1) = 2$ and $h_C(n) = 0$ otherwise. Such a curve C cannot satisfy $\omega_C = \mathcal{O}_C(-50)$, for in this case it would not satisfy the symmetry $h_C(n) = h_C(-50 + 2 - n)$) [40, Prop. 2.15]. □

PROPOSITION 2. *There are* 18 *irreducible components of* $H_{4,-99}$ *whose general member is* (-50)*-subcanonical, as listed in Table 2.*

Proof. By Proposition 1 we need only consider (a) double conics, (b) unions of double lines, and (c) quasi-primitive 4-lines. The double conics are automatically subcanonical, for if D is the support of a double conic C, then the Cohen-Macaulay filtration is

$$0 \to \mathcal{I}_C \to \mathcal{I}_D \to \mathcal{O}_D(49) \to 0.$$

Noting that $\mathcal{O}_D(49) = \omega_D(50)$, we see that C arises by the Ferrand construction [12] and hence is subcanonical.

Table 2: Irreducible families of subcanonical curves in $H_{4,-99}$

Label from Table 1	Dimension	Spectrum
G_5 Double conics	211	$\{-49, -48, 0, 1\}$
$G_{10,49}$ Disjoint union of two double lines of genus -49	206	$\{-48^2, 0^2\}$
$G_{11,a,48-3a}$ for $0 < a \leq 16$ Quasi-primitive 4-line	$205 - 3a$	$\{-48, -48 + a, -a, 0\}$

Next consider a union $C = D_1 \cup D_2$ of double lines. If C is connected, then the support is planar and C is contained in the double plane. It follows that C is a limit of double conics by Theorem 12 or [25, Thm. 5.1], so we need only consider disjoint unions of double lines. Since a double line of genus g is $(g-1)$-subcanonical, a disjoint union of such can only be subcanonical if the double lines have the same genus, which in this case must be -49.

Now let C be a quasi-primitive 4-line of type (a, b, c) with $0 < a \leq 16$, $0 \leq b \leq 48 - 3a$ and $c = 96 - 6a - b$. This means that there are locally Cohen-Macaulay curves $L \subset D \subset W \subset C$ with quotients $\mathcal{I}_L/\mathcal{I}_D \cong \mathcal{O}_L(a)$, $\mathcal{I}_D/\mathcal{I}_W \cong \mathcal{O}_L(2a + b)$ and $\mathcal{I}_W/\mathcal{I}_C \cong \mathcal{O}_L(3a + c)$ (see Remark 5 (c)). Piecing together the exact sequences and using $a > 0$, the spectrum of C is

$$\{-3a - c, -2a - b, -a, 0\}.$$

To be (-50)-subcanonical, this sequence of integers must be symmetric about -24 [40, Prop. 2.15], which forces $b = 48 - 3a$ and $c = 48 - 3a$. It now suffices to show that the general 4-line C of type $(a, 48 - 3a, 48 - 3a)$ is subcanonical.

The exact sequence

$$0 \to \mathcal{I}_D \to \mathcal{I}_L \to \mathcal{O}_L(a) \to 0 \tag{5}$$

shows that the underlying double line $D \subset C$ arises from the Ferrand construction and is $(-a - 2)$-subcanonical, since $\mathcal{O}_L(a) \cong \omega_L(a + 2)$. In fact, D is a divisor on a smooth surface $S \subset \mathbb{P}^3$ of degree $a+2$ by [33, Rmk. 1.5]. In view of the isomorphisms $\mathcal{I}_S \cong \mathcal{O}_{\mathbb{P}^3}(-a-2)$ and $\mathcal{I}_{D,S} \otimes \mathcal{O}_D = \mathcal{O}_S(-D) \otimes \mathcal{O}_D \cong \omega_S \otimes \omega_D^{-1}$ with $\omega_S \cong \mathcal{O}_S(a-4)$ and $\omega_D \cong \mathcal{O}_D(-a - 2)$, restricting the exact sequence

$$0 \to \mathcal{I}_S \to \mathcal{I}_D \to \mathcal{I}_{D,S} \to 0$$

to D yields

$$\mathcal{O}_D(-a - 2) \xrightarrow{\tau} \mathcal{N}_D^\vee \xrightarrow{\pi} \mathcal{O}_D(2a - 2) \to 0. \tag{6}$$

Since π is a surjection of bundles on D, the kernel is a line bundle on D. Since any surjection of line bundles is an isomorphism, τ is injective and sequence (6) is short exact.

Exact sequence (5) shows that $h^0\mathcal{O}_D(m) = h^1\mathcal{I}_D(m) = 0$ for $m < -a$, hence sequence (6) yields the vanishing $H^1(\mathcal{N}_D \otimes \omega_D(m)) \perp H^0\mathcal{N}_D^\vee(-m) = 0$ for $m > 3a-2$. Therefore $\mathcal{N}_D \otimes \omega_D$ is $(3a)$-regular and so $\mathcal{N}_D \otimes \omega_D(n)$ is generated by global sections for $n \geq 3a$ by the Castelnuovo-Mumford theorem. Since $a \leq 16$, we have in particular that $\mathcal{N}_D \otimes \omega_D(50)$ is generated by global sections and we obtain a nowhere vanishing section yielding a surjection $\mathcal{I}_D \to \mathcal{N}_D^\vee \to \omega_D(50)$ whose kernel $\mathcal{I}_C$ is the ideal sheaf for a (-50)-subcanonical curve C by Ferrand's construction. Clearly C is supported on L and the sequence

$$0 \to \omega_D(50) \to \mathcal{O}_C \to \mathcal{O}_D \to 0$$

shows that the spectrum of C is $\{-48, -48+a, -a, 0\}$, so C is quasi-primitive of type $(a, 48-3a, 48-3a)$. For curves with fixed spectrum, the property of being subcanonical is open and we conclude. □

REMARK 9. Chiantini and Valabrega have given equations of such curves [6, Examples 3.1 and 3.2]. For $m, n, u > 0$ and $p \geq \max\{m, n\}$, they observe that the curve V with homogeneous ideal

$$I_V = ((x^n, y^m)^u, z^{p-n}x^n - w^{p-m}y^m = \varphi)$$

is $((1-u)p + (m+n)u - 4)$-subcanonical. Setting $4 = \deg V = mnu$, we find just a few possibilities. When $u = 1$ we obtain plane curves $(m = 4, n = 1)$ and complete intersections of two quadrics $(m = n = 2)$. More interesting are these:

(a) $m = 2$, $n = 1$ and $u = 2$. To obtain a (-50)-subcanonical curve we take $p = \deg\varphi = 52$. This is a quasi-primitive 4-line of type $(-1, 50, 52)$. It does not appear in Table 1 because such 4-lines are limits of double conics. This one is a Ferrand doubling of the plane curve with ideal (x, y^2).

(b) $m = n = 1$ and $u = 4$. To obtain a (-50)-subcanonical curve we take $p = \deg\varphi = 54$. This curve is a quasi-primitive 4-line of type $(16, 0, 0)$.

REMARK 10. Here we make a list of the families of subcanonical curves of degree four. There are none when g is even. For $g = 3$ there are plane curves and for $g = 1$ there are complete intersections of two quadrics. For odd $g < 0$ we have:

1. Double conics.
2. Disjoint unions of two lines of genus $\frac{g+1}{2}$.
3. Quasi-primitive 4-lines of type $(a, \frac{-g+3-6a}{2}, \frac{-g+3-6a}{2})$ for $0 < a \leq \frac{-g-3}{6}$ (this last family is empty for $g > -9$, as no such a exist).

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Scott NOLLET, Department of Mathematics, Texas Christian University, TCU Box 298900, Fort Worth, TX 76129, USA
e-mail: `s.nollet@tcu.edu`

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R.M. Miró-Roig[*] – R. Notari[†] – M.L. Spreafico[†]

PROPERTIES OF SOME ARTINIAN GORENSTEIN RINGS

Dedicated to Paolo Valabrega on the occasion of his 60 th birthday

Abstract. In this paper, we fix a Cohen-Macaulay ideal $I \subset R = K[x_1, \dots, x_n]$ of dimension 1 and we construct a parameter space $\mathcal{G}(I, r)$ for the family of Artinian Gorenstein ideals J with $\mathrm{reg}(J) = r$ for which I is a tight annihilating ideal. We compute the dimension of $\mathcal{G}(I, r)$ and we prove that if $r \gg 0$ (see section 5 for a precise bound) then all $J \in \mathcal{G}(I, r)$ are a basic double G-link of G on I where G is a suitable Artinian Gorenstein ideal containing I.

1. Introduction

In recent years many authors focus their attention to study Gorenstein ideals and the role that they play in various of the applications of Commutative Algebra such as Algebraic Geometry, Algebraic Combinatorics and Number Theory.

It is well known that in codimension 2 Gorenstein ideals and complete intersection ideals coincide; and in codimension 3 Gorenstein ideals are completely described from an algebraic point of view by the beautiful structure theorem of Buchsbaum and Eisenbud which allows one to associate an alternating matrix of odd order to each Gorenstein ideal of codimension 3. Unfortunately the geometric appearance of Gorenstein ideals $I \subset K[x_1, \dots, x_n]$ is less understood. For this reason, many authors have given geometric constructions of some particular families (cf. [2], [7], [9], [10] among others). In this paper, we construct a parameter space for the family of Artinian Gorenstein ideals J with fixed regularity and fixed tight annihilating ideal I and we prove that if the regularity is big enough then all these Gorenstein ideals J are obtained by basic double G-link of G on I where G is a suitable Artinian Gorenstein ideal containing I.

Next we outline the structure of the paper. Section 2 provides a brief glossary of definitions. In section 3 we recall some constructions of Gorenstein ideals and we point out some features of the constructed ideals. All of them have been successfully applied in the context of liaison to produce Gorenstein links of given ideals and to study the Gorenstein liaison classes of some particular ideals. In section 4, we first introduce the notion of tight annihilating ring R/I for an Artinian Gorenstein ring R/J of arbitrary codimension, given by A. Iarrobino and V. Kanev in [6]. Then we introduce the new definition of tight resolving ring R/I for an Artinian Gorenstein ring R/J which generalizes the other one in the codimension 3 case. We relate the Hilbert function of an Artinian Gorenstein ring R/J to the Hilbert function of a tight resolving

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[†]Members of GNSAGA of INdAM.

ring for R/J, and we compare the two notions. We prove that if a Cohen-Macaulay ring R/I of dimension 1 is a tight annihilating or resolving ring for an Artinian Gorenstein ring R/J and $f \in [R]_d$ is a regular form for I then the ideal $J' = J : fR$ is an Artinian Gorenstein ideal, J is a basic double link of J' on I if the degree d of f is not too large, and R/I is not necessarily either a tight annihilating ring or a tight resolving ring for R/J'. We end this section giving a numerical criterion to assure that R/I is also a tight resolving ring for R/J'.

Section 5 contains the main results of this paper. We fix a Cohen-Macaulay ideal $I \subset R$ of dimension 1 and we construct the parameter space $\mathcal{G}(I,r)$ for the Artinian Gorenstein ideals G with $\mathrm{reg}(G) = r$ and for which I is a tight annihilating ideal. We prove that $\mathcal{G}(I,r)$ is an open subset of an affine space of dimension $\deg(R/I)$. We also construct the parameter space $\mathcal{BDL}(I, s_I + d)$ for the family of Artinian Gorenstein ideals $L = I + fG$ with regularity $\mathrm{reg}(L) = s_I + d$ (s_I depends on the geometry of I) which are basic double G-links of G on I where G is the sum $I + I_1$ of two suitable directly linked ideals. We prove that $\mathcal{BDL}(I, s_I + d)$ is an open subset of an affine space of dimension $h_{R/I}(d)$. The main result of this paper states that if $d \geq s_I + 1$ then $\mathcal{G}(I, s_I + d) = \mathcal{BDL}(I, s_I + d)$.

2. Preliminaries and notation

Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring in the variables $x_1, \ldots, x_n$ over the field K, algebraically closed and of characteristic $\mathrm{char}(K) = 0$. We assume $\deg(x_i) = 1$, for $i = 1, \ldots, n$, and we consider R with the usual induced graduation over $\mathbb{Z}$, i.e. $R = \oplus_{n \in \mathbb{N}} [R]_n$, where $[R]_n$ contains the homogeneous polynomials of degree n.

DEFINITION 1. *Given a homogeneous ideal $I \subseteq R$, the function*

$$j \in \mathbb{Z} \to h_{R/I}(j) = \dim_K [R/I]_j$$

is the Hilbert function of the ring R/I, where $\dim_K$ *means the dimension as $K-$vector space.*

From the definition it follows that, if $I \neq R$, then $h_{R/I}(0) = 1$.
In the following, the dimension of a ring means its Krull dimension.

DEFINITION 2. *Let $I \subseteq R$ be a homogeneous ideal. If* $\dim R/I = 0$, *we say that I is an Artinian ideal, and R/I is an Artinian ring.*

Because of the noetherianity of the ring R, this definition is equivalent to the usual definition (see [1], Ch.6).

Let $I \subseteq R$ be a homogeneous ideal, and let

$$0 \to F_c \to F_{c-1} \to \cdots \to F_1 \to I \to 0 \tag{1}$$

be a minimal free resolution of I, with $F_i = \oplus_{j=1}^{n_i} R^{\beta_{ij}}(-b_{ij})$.

Following the sheaf theory, we define the regularity of an ideal and of its quotient ring.

DEFINITION 3. *Let $I \subseteq R$ be a homogeneous ideal. Then the regularity of R/I is* $\operatorname{reg}(R/I) = \max\left\{b_{ij} - i \mid i = 1, \dots, c\right\}$ *while the regularity of I is* $\operatorname{reg}(I) = \operatorname{reg}(R/I) + 1$.

Now, we define the properties of the ideals we are mainly interested in.

DEFINITION 4. *I is a Cohen-Macaulay ideal if* $c = \dim R - \dim R/I = n - \dim R/I$. *Equivalently, R/I is a Cohen-Macaulay ring.*

DEFINITION 5. *I is a Gorenstein ideal if I is a Cohen-Macaulay ideal and* $\operatorname{rank}(F_c) = \sum_{j=1}^{n_c} \beta_{cj} = 1$, *i.e. $F_c \simeq R(-t_I)$ for some integer t_I. Equivalently, R/I is a Gorenstein ring.*

We recall now the definition of regular element and some well known properties that we will use in the sequel.

DEFINITION 6. *Let $I \subseteq R$ be an ideal with* $\dim R/I = 1$, *and let $f \in [R]_d$. The element f is regular for I if $I : fR = I$.*

PROPOSITION 1. *If $f \in [R]_d$ is a regular element for a Cohen-Macaulay ideal $I \subseteq R$, then the sequence*

$$0 \to \frac{R}{I}(-d) \xrightarrow{f} \frac{R}{I} \to \frac{R}{I + fR} \to 0$$

is exact and $h_{R/I+fR}(j) = h_{R/I}(j) - h_{R/I}(j-d)$.

Now, we collect the properties of the Hilbert function needed later on in the cases $\dim R/I = 0, 1$.

PROPOSITION 2. *Let $I \subseteq R$ be a Cohen-Macaulay homogeneous ideal.*

1. *If I is Artinian, then $h_{R/I}(j) = 0$, for $j \gg 0$.*
2. *Let R/I be an Artinian Gorenstein ring of regularity* $s_I = \operatorname{reg}(R/I)$. *Let $F_c \simeq R(-t_I)$ be the last module in the minimal free resolution of I. Then*
   - *i.* $s_I = t_I - n$;
   - *ii.* $h_{R/I}(j) = h_{R/I}(s_I - j)$ *for every $j \in \mathbb{N}$, and so* $h_{R/I}(s_I) = 1$ *and* $s_I = \max\left\{j \in \mathbb{N} \mid h_{R/I}(j) \neq 0\right\}$.

The proof follows from [1], Corollaries 4.1.4 and 4.1.6.

The integer $s_I = \operatorname{reg}(R/I)$ is also called *socle degree* of the Artinian Gorenstein ring R/I.

PROPOSITION 3. *Let $I \subseteq R$ be a homogeneous Cohen-Macaulay ideal of dimension 1, and regularity $r_I = \operatorname{reg}(R/I)$. Then:*

1. $h_{R/I}(j+1) \geq h_{R/I}(j)$, *for* $j \geq 0$;
2. $h_{R/I}(r_I) = h_{R/I}(r_I + i)$ *for every* $i \in \mathbb{N}$.

The integer $h_{R/I}(r_I)$ is called the degree $\deg(R/I)$ of R/I.

The Gorenstein property gives constraints not only on the Hilbert function of a ring, but also on its minimal free resolution.

In fact, using the graded version of [5], Theorem 1.5, one can prove

PROPOSITION 4. *Let $I \subseteq R$ be a Gorenstein ideal. Then, the minimal free resolution of I is self-dual, i.e.*

1. $F_{c-j} \simeq F_j^*(-t_I)$;
2. $\delta_{c-j} : F_{c-j} \to F_{c-j-1}$ *is equal to* $\delta^*(-t_I) : F_j^*(-t_I) \to F_{j+1}^*(-t_I)$

where $F^* = \operatorname{Hom}(F, R)$ is the dual module of the free module F.

3. Some construction of Artinian Gorenstein rings

In this section we recall some well-known methods to construct homogeneous Artinian Gorenstein ideals in R and some properties that the corresponding quotient rings have.

The first method is the Buchsbaum-Eisenbud structure Theorem for codimension 3 graded Gorenstein rings ([5], Theorem 2.1).

THEOREM 1. *Let $g \geq 3$ be an odd integer, and $d_1 \leq \cdots \leq d_g$ be a sequence of positive integers; set $d = \frac{2}{g-1}(d_1 + \cdots + d_g)$ and suppose this is an integer, let $e_i = d - d_i$, and $j = d - 3$, and we suppose $1 \leq d_1, d_g \leq j + 1$ (so $e_i \geq 2$).*

Let Ψ be an alternating $g \times g$ matrix with entries from the ring R, such that the entry ψ_{ij} is homogeneous of degree $e_i - d_j$ if $e_i > d_j$ and zero otherwise (so the entries belong to the maximal ideal of R). Let Ψ_i be the $(g-1) \times (g-1)$ alternating matrix obtained by deleting the $i-$th row and column of Ψ. Then the pfaffian $\operatorname{Pf}(\Psi_i)$ is homogeneous of degree d_i. Let I be the ideal $\operatorname{Pf}(\Psi)$ generated by $\operatorname{Pf}(\Psi_i)$, $i = 1, \ldots, g$. Then I has grade (height) ≤ 3 in R. If I has grade 3, then I is a graded Gorenstein ideal of height 3, and the socle degree of R/I is $j = d - 3$.

Let λ be the column vector with entries $\lambda_i = (-1)^i \operatorname{Pf}(\Psi_i)$.

i. *Suppose I has the maximal possible grade 3. Then I has minimal free resolution*

$$0 \to R(-d) \xrightarrow{\lambda} \oplus_{i=1}^{g} R(-e_i) \xrightarrow{\Psi} \oplus_{i=1}^{g} R(-d_i) \xrightarrow{\lambda^T} I \to 0.$$

ii. *Conversely, if $I \neq R$ is a height 3 graded Gorenstein ideal of R, there is an alternating matrix Ψ as above, such that $I = \operatorname{Pf}(\Psi)$.*

No generalization of Theorem 1 is known for height $\geq$ 4 graded Gorenstein ideals.

A second method of constructing Artinian Gorenstein ideals is the following (see [4], Ex. 3.2.11):

THEOREM 2. *Let $\varphi : [R]_s \to K$ be a non-degenerate linear map. Let*

$$[I]_j = \begin{cases} [\ker\varphi : (x_1, \ldots, x_n)^{s-j}]_j & \text{if} \quad j \leq s \\ [R]_j & \text{if} \quad j > s. \end{cases}$$

Then $I = \oplus_{j\in\mathbb{Z}}[I]_j$ is an Artinian Gorenstein ideal of regularity $\operatorname{reg}(I) = s + 1$.

This construction is equivalent to Macaulay's inverse system, and allows us to construct every Artinian Gorenstein ideal with given socle degree. It is a very hard open problem to relate the linear map φ to the minimal free resolution of I or at least to the Hilbert function of R/I.

The next two methods allow one to construct Gorenstein rings of whatever dimension, but we state them only in the Artinian case.

We state the first one as a particular case of [8], Theorem 4.2.1, but it was first proved in [10].

THEOREM 3. *Let $I_1, I_2 \subseteq R$ be homogeneous Cohen-Macaulay ideals such that* $\dim R/I_1 = \dim R/I_2 = 1$. *Assume that $J = I_1 \cap I_2$ is a Gorenstein ideal such that* $\dim R/J = 1$. *Then $G = I_1 + I_2$ is an Artinian Gorenstein ideal.*

REMARK 1. (1) Two ideals I_1 and I_2 satisfying the hypotheses of the previous theorem are directly linked, and G is said the sum of directly linked Cohen-Macaulay ideals.
(2) The Gorenstein ideals arising as sum of two Cohen-Macaulay directly linked ideals were studied in various papers ([11], [12], [13], for example), and it is known that not every Gorenstein ideal can be obtained by using that construction (see [11], Example 4.1).

The second and last method is the so-called basic double G-link ([7], Lemma 4.8).

THEOREM 4. *Let $I \subseteq J \subseteq R$ be homogeneous ideals such that* $\dim R/I = 1$ *and* $\dim R/J = 0$. *Let $f \in [R]_d$ be a regular form for I. Then it holds:*

1. $\deg(I + fJ) = d\deg(I) + \deg(J)$.
2. *If I is perfect and J is unmixed, then $I + fJ$ is unmixed.*
3. $J/I \cong (I + fJ)/I(d)$.
4. *if R/I and J/I are Cohen-Macaulay and J/I has Cohen-Macaulay type 1 then J and $I + fJ$ are Artinian Gorenstein ideals.*

REMARK 2. In the same hypotheses as above, the ideal $I + fJ$ is called basic double G-link of J on I. It is known that this construction does not give every Artinian Gorenstein ideal (see [2], Example 5.13).

Now, we want to give more details on the Gorenstein ideals arising from Theorems 3 and 4. In particular, we will determine how their resolution looks like.

PROPOSITION 5. *In the same notation and hypotheses as Theorem 3, if* $s = \operatorname{reg}(R/J)$, *then*

1. $h_{R/I_2}(j) = h_{R/J}(j) + h_{R/I_1}(s-j-1) - \deg(R/I_1)$;
2. $\operatorname{reg}(R/G) = s - 1$;
3. $h_{R/G}(j) = h_{R/I_1}(j) + h_{R/I_1}(s-j-1) - \deg(R/I_1)$;
4. *if* $0 \to F_{n-1} \to \cdots \to F_1 \to I_1 \to 0$ *is a minimal free resolution of* I, *then*

$$0 \to R(-S) \to \begin{matrix} F_{n-1} \\ \oplus \\ F_1^*(-S) \end{matrix} \to \cdots \to \begin{matrix} F_1 \\ \oplus \\ F_{n-1}^*(-S) \end{matrix} \to G \to 0$$

is a free resolution of G, *not necessarily minimal, where* $S = s + n - 1$.

Proof. Recalling that $I_2 = J : I_1$, we compute the first difference of the Hilbert functions:

$$\Delta h_{R/I_2}(j) = \Delta h_{R/J}(s-j) - \Delta h_{R/I_1}(s-j)$$

$$\Delta h_{R/I_2}(j-1) = \Delta h_{R/J}(s-j+1) - \Delta h_{R/I_1}(s-j+1)$$

and so on, until

$$\Delta h_{R/I_2}(0) = \Delta h_{R/J}(s-0) - \Delta h_{R/I_1}(s-0).$$

By adding all the equations we get:

$$h_{R/I_2}(j) = h_{R/J}(s) - h_{R/J}(s-j-1) - h_{R/I_1}(s) + h_{R/I_1}(s-j-1).$$

Because of the symmetry of the function $\Delta h_{R/J}$ ($\Delta h_{R/J}(j) = \Delta h_{R/J}(s-j)$), the previous equality can be written as

$$h_{R/I_2}(j) = h_{R/J}(j) - h_{R/I_1}(s) + h_{R/I_1}(s-j-1)$$

and so the first claim is proved.

The claims (2) and (3) follow from the knowledge of the resolution of G. Then, it is enough to prove the claim (4). The resolution of G can be computed by mapping cone procedure from the short exact sequence

$$0 \to J \to I_1 \oplus I_2 \to G \to 0$$

that relates all the ideals involved in the construction of G. By using the minimal free resolutions of I_1, J and standard results from liaison theory, it is possible to compute a free resolution of I_2. From that last one, we get the claim on the free resolution of G. Also if the resolution is non minimal, it is not possible to cancel the last from the left free module in its resolution, because G is Artinian, and so we get also the result on the regularity of R/G. $\square$

PROPOSITION 6. *In the same notation and hypotheses of Theorem 4, if* $d = \deg(f)$ *then*

1. $h_{R/I+fJ}(j) = h_{R/I}(j) + h_{R/J}(j-d) - h_{R/I}(j-d)$;
2. *if* $0 \to F_{n-1} \to \cdots \to F_1 \to I \to 0$ *is a minimal free resolution of* I, *then*

$$0 \to R(-S) \to \begin{array}{c} F_{n-1} \\ \oplus \\ F_1^*(-S) \end{array} \to \cdots \to \begin{array}{c} F_1 \\ \oplus \\ F_{n-1}^*(-S) \end{array} \to J \to 0$$

is a free resolution of J, *not necessarily minimal, for some integer* S;

3. $\operatorname{reg}(R/I + fJ) = \operatorname{reg}(R/J) + d$.

Proof. At first, we have the following equality: $I \cap fJ = fI$.
The inclusion $\supseteq$ is evident. The inverse inclusion is an easy consequence of the regularity of f for I. In fact, if $fg \in I$ for some $g \in J$ then $g \in I : fR = I$ and hence $fg \in fI$.

Now, we have that the short sequence

$$0 \to I(-d) \to I \oplus J(-d) \to I + fJ \to 0$$

is exact, and so we get the claim on the Hilbert function of $R/I + fJ$.

From the proof of Lemma 4.8 in [7] we know that the two sequences

$$0 \to I \to J \to J/I \to 0$$

and

$$0 \to I \to I + fJ \to J/I(-d) \to 0$$

are exact, with both J and $I + fJ$ Artinian Gorenstein ideals.

We can choose d sufficiently large so that the generators of $I + fJ$ not in I have degree at least $2 + \operatorname{reg}(R/I)$. Then, we can apply Theorem 3.2 in [3] and we get that a minimal free resolution of $J/I(-d)$ is

$$0 \to R(-s) \to F_1^*(-s) \to \cdots \to F_{n-1}^*(-s) \to J/I(-d) \to 0$$

for some integer s. Hence, the claim on the free resolution of J follows, too. $\square$

4. A class of Artinian Gorenstein ideals

One of the most studied class of Gorenstein rings of dimension $d \geq 0$ is the class of quotients of Cohen-Macaulay rings of dimension $d+1$. Geometrically, they correspond to divisors on arithmetically Cohen-Macaulay projective schemes. In particular, it is very interesting the case when the two rings have the same Hilbert function in small degrees, and hence we recall the definition of tight annihilating ring given by Iarrobino and Kanev ([6], Definition 5.1), which describes that situation.

DEFINITION 7. *Let R/J be an Artinian Gorenstein ring. Let $I \subseteq J$ be a homogeneous ideal such that R/I is a dimension 1 Cohen-Macaulay ring. We say that R/I is a tight annihilating ring for R/J if $h_{R/J}(j) = h_{R/I}(j)$ for $j \leq \operatorname{reg}(R/I)$, and $h_{R/J}(j) \leq h_{R/I}(\operatorname{reg}(R/I)) = \deg(R/I)$, for every $j \in \mathbb{Z}$.*

REMARK 3. We know that the Hilbert function of the Artinian Gorenstein ring R/J is symmetric, while the one of the ring R/I is increasing until it reaches its maximum value $\deg(R/I)$. Then, if I is tight annihilating for J, the Hilbert function of R/J increases as the one of R/I, reaches the value $\deg(R/I)$, and after that, it takes the same value for some integers, and when it takes a different value, it can be completed by symmetry. Then, $\operatorname{reg}(R/J) \geq 2\operatorname{reg}(R/I)$.

In codimension 3, we can characterize the minimal free resolution of an Artinian Gorenstein ideal J having a tight annihilating ideal I. In fact, it holds:

PROPOSITION 7. *Let $I \subseteq R = K[x, y, z]$ be a Cohen-Macaulay homogeneous ideal such that $\dim R/I = 1$ and $h_{R/I}(1) = 3$. Let $J \supseteq I$ be an Artinian Gorenstein ideal for which I is tight annihilating. If $0 \to F_2 \to F_1 \to I \to 0$ is a minimal free resolution of I then there exists an integer $t_J \geq 3 + 2\operatorname{reg}(R/I)$ such that*

$$0 \to R(-t_J) \to \begin{array}{c} F_2 \\ \oplus \\ F_1^*(-t_J) \end{array} \to \begin{array}{c} F_1 \\ \oplus \\ F_2^*(-t_J) \end{array} \to J \to 0$$

is the minimal free resolution of J.

Proof. See [6], Theorems 5.31, 5.39, 5.46, and Remark 5.43. The proof is based on the Buchsbaum-Eisenbud structure theorem for codimension 3 Gorenstein ideals. □

This property is shared also from the ideals arising from Theorems 3 and 4. Then, we choose this last property for defining the tight resolved ring for an Artinian Gorenstein ring R/J of whatever codimension.

DEFINITION 8. *Let R/I be a graded Cohen-Macaulay ring of dimension 1 with minimal free resolution*

$$0 \to F_{n-1} \to F_{n-2} \to \cdots \to F_1 \to R \to R/I \to 0.$$

We say that R/I is a tight resolving ring for the Artinian Gorenstein graded ring R/J (or I is a tight resolving ideal for J) if the minimal free resolution of R/J is

$$0 \to R(-t_J) \to \begin{array}{c} F_{n-1} \\ \oplus \\ F_1^*(-t_J) \end{array} \to \cdots \to \begin{array}{c} F_1 \\ \oplus \\ F_{n-1}^*(-t_J) \end{array} \to R \to R/J \to 0$$

for some integer $t_J \geq n + 2\,\mathrm{reg}(R/I)$.

PROPOSITION 8. *Let R/I be a tight resolving ring for the Artinian Gorenstein graded ring R/J. Then, the Hilbert function of R/J is equal to*

$$h_{R/J}(j) = h_{R/I}(j) + h_{R/I}(\mathrm{reg}(R/J) - j) - \deg(R/I).$$

Proof. From the additivity of the Hilbert function on exact sequences, we have $h_{R/I}(j) = \sum_{i=0}^{n-1}(-1)^i \dim_K[F_i]_j$, where $F_0 = R$, and, if $F_i = \oplus_{h=1}^{n_i} R(-b_{ih})^{\beta_{ih}}$, then $\dim_K[F_i]_j = \sum_{h=1}^{n_i}\binom{j+n-1-b_{ih}}{n-1}$. Of course,

$$\begin{aligned} h_{R/J}(j) &= \sum_{i=0}^{n}(-1)^i\left[\dim_K[F_i]_j + \dim_K[F_{n-i}^*]_{j-t_J}\right] = \\ &= \sum_{i=0}^{n-1}(-1)^i \dim_K[F_i]_j + \sum_{i=1}^{n}(-1)^i \dim_K[F_{n-i}^*]_{j-t_J}. \end{aligned}$$

The sequence

$$0 \to R(-t_J) \to F_1^*(-t_J) \to \cdots \to F_{n-1}^*(-t_J)$$

is a free resolution of the canonical module of R/I and so

$$\sum_{i=1}^{n}(-1)^i \dim_K[F_{n-i}^*]_{j-t_J} = \dim_K\left[\mathrm{Ext}^{n-1}(R/I, R)\right]_{j-t_J}.$$

It is well known that

$$\dim_K\left[\mathrm{Ext}^{n-1}(R/I, R)\right]_{j-t_J} = \dim_K\left[\mathrm{Ext}^{n-2}(I, R(-n))\right]_{j+n-t_J},$$

and by Serre's duality, we have that

$$\dim_K\left[\mathrm{Ext}^{n-2}(I, R(-n))\right]_{j+n-t_J} = h^1(\mathbb{P}^{n-1}, \mathcal{I}(t_J - n - j))$$

where $\mathcal{I}$ is the ideal sheaf obtained by sheaffifing the saturated ideal I. Hence, the claim follows. □

REMARK 4. Because of the description of the Hilbert function of a dimension 1 Cohen-Macaulay graded ring R/I, we known that $h_{R/I}(k) \leq \deg(R/I)$ for every $k \in \mathbb{Z}$. Hence, $h_{R/J}(j) \leq \deg(R/I)$, for every $j \in \mathbb{Z}$.

Now, we compare the notion of tight annihilating and tight resolving ideal for an Artinian Gorenstein ideal J.

PROPOSITION 9. *Let $J \subseteq R$ be an Artinian Gorenstein ideal and let $I \subseteq J$ be a tight resolving ideal for J. Then, I is a tight annihilating ideal for J if, and only if,* $\operatorname{reg}(R/J) \geq 2\operatorname{reg}(R/I)$.

Proof. We proved in Proposition 8 above that

$$h_{R/J}(j) = h_{R/I}(j) + h_{R/I}(\operatorname{reg}(R/J) - j) - \deg(R/I).$$

It follows that $h_{R/J}(j) = h_{R/I}(j)$ for every $j \leq \operatorname{reg}(R/I)$ if, and only if, $\operatorname{reg}(R/J) - j \geq \operatorname{reg}(R/I)$ for each $j \leq \operatorname{reg}(R/I)$, i.e. $\operatorname{reg}(R/J) \geq 2\operatorname{reg}(R/I)$. □

On the other hand, in the codimension 3 case, we have that if an ideal is tight annihilating for an Artinian Gorenstein ideal J then it is tight resolving for J, too, as explained in Proposition 7. Because of the absence of a structure theorem for Gorenstein ideals in codimension ≥ 4, the best we can say is the following:

PROPOSITION 10. *Let $J \subseteq R$ be an Artinian Gorenstein ideal and let $I \subseteq J$ be a tight annihilating ideal for J. Then, if the degrees of the minimal generators of J not in I are at least $2 + \operatorname{reg}(R/I)$, then I is a tight resolving ideal for J.*

Proof. The claim is [3], Theorem 3.2. □

Now, we want to construct new Artinian Gorenstein ideals from a given one.

THEOREM 5. *Let $I \subseteq J$ be homogeneous ideals in R. Assume that R/J is an Artinian Gorenstein ring and that R/I is a dimension 1 Cohen-Macaulay ring. Let d be an integer such that $h_{R/J}(d) = h_{R/I}(d)$, and let $f \in [R]_d$ be a regular form for I. Then, $J' = J : fR$ is an Artinian Gorenstein ideal.*

Proof. At first, we prove that J' is an Artinian ideal. In fact,

$$h_{R/J}(\operatorname{reg}(R/J) + 1) = h_{R/J}(\operatorname{reg}(R/J) + i) = 0 \text{ for every } i \geq 1,$$

and this is equivalent to the equality $[J]_j = [R]_j$ for $j \geq \operatorname{reg}(R/J) + 1$.

If $g \in [R]_{j-d}$, with $j \geq \operatorname{reg}(R/J) + 1$, then $gf \in [R]_j = [J]_j$, and hence $g \in [J : fR]_{j-d} = [J']_{j-d}$, and this proves that $[J']_j = [R]_j$ for each $j \geq \operatorname{reg}(R/J) - d + 1$, i.e. J' is an Artinian ideal.

Now, we prove that J' is a Gorenstein ideal.

Let $\varphi : [R]_{\operatorname{reg}(R/J)} \to K$ be a non-degenerate $K-$linear map such that $\ker\varphi = [J]_{\operatorname{reg}(R/J)}$. We define $\psi : [R]_{\operatorname{reg}(R/J)-d} \to K$ to be the $K-$linear map such that $\psi(g) = \varphi(gf)$. According to Theorem 2, we prove that ψ is non degenerate and that

$$[\ker\psi : (x_1, \ldots, x_n)^j]_{\operatorname{reg}(R/J)-d-j} = [J']_{\operatorname{reg}(R/J)-d-j}.$$

If $\psi(g) = 0$ for every $g \in [R]_{\mathrm{reg}(R/J)-d}$, then $\varphi(gf) = 0$ for every $g \in [R]_{\mathrm{reg}(R/J)-d}$ i.e. $f \in [\ker\varphi : (x_1, \ldots, x_n)^{\mathrm{reg}(R/J)-d}]_d = [J]_d$, with $d \leq \mathrm{reg}(R/J) - \mathrm{reg}(R/I)$. But $h_{R/J}(d) = h_{R/I}(d)$ and so $[J]_d = [I]_d$. Hence, $f \in [I]_d$ and so f is not regular for I. This contradiction proves that ψ is non degenerate.

Now, $g \in [\ker\psi : (x_1, \ldots, x_n)^j]_{\mathrm{reg}(R/J)-d-j}$ if and only if $\varphi(gfh) = 0$ for every $h \in (x_1, \ldots, x_n)^j$. By definition, this means that $gf \in [J]_{\mathrm{reg}(R/J)-j}$ i.e. $g \in [J : fR]_{\mathrm{reg}(R/J)-d-j} = [J']_{\mathrm{reg}(R/J)-d-j}$.

Conversely, if $g \in [J']_{\mathrm{reg}(R/J)-d-j}$, then $gf \in \ker\varphi : (x_1, \ldots, x_n)^j$ and $\deg(gf) = \mathrm{reg}(R/J) - j$, i.e. $\varphi(gfh) = 0, \forall h \in (x_1, \ldots, x_n)^j$. From the definition of ψ, it follows that $\psi(gh) = 0$ for each $h \in (x_1, \ldots, x_n)^j$ and so $g \in [\ker\psi : (x_1, \ldots, x_n)^j]_{\mathrm{reg}(R/J)-d-j}$. □

REMARK 5. Let I, J, J' be as above. Then $I \subseteq J \subseteq J'$.

REMARK 6. If $1 \leq d \leq \mathrm{reg}(R/J) - \mathrm{reg}(R/I)$ then $h_{R/J}(d) = h_{R/I}(d)$ both in the case I is a tight annihilating ideal for J and in the case I is a tight resolving ideal for J.

Now, we give an example to show that the Hilbert function of R/J' depends on J and f and not only on J and $d = \deg(f)$.

EXAMPLE 1. Let $I \subseteq R = K[x, y, z]$ be the ideal generated by $y^3 - xz^2, x^3 - y^2z, z^3 - x^2y$. Its minimal free resolution is

$$0 \to \begin{array}{c} R(-4) \\ \oplus \\ R(-5) \end{array} \xrightarrow{A} R^3(-3) \longrightarrow I \to 0$$

where

$$A = \begin{pmatrix} z & x^2 \\ y & z^2 \\ x & y^2 \end{pmatrix}.$$

The ideal $I_1 = (x, y^2)$ is geometrically linked to I via the complete intersection ideal $(y^3 - xz^2, x^3 - y^2z)$, and the forms $f = x^6 + y^6 + z^6$ and $g = x^5y + y^5z + xz^5$ are regular for I.

Hence, $J = I + fI_1$ is an Artinian Gorenstein ideal with Hilbert function

$$h_{R/J} = (1, 3, 6, 7, 7, 7, 7, 6, 3, 1).$$

The ideals $J_1 = J : fR = (x, y^2, z^3)$ and $J_2 = J : gR = (x^2 - xy - z^2, xy - xz - yz, y^2 - 3xz - 2yz - z^2)$ are Artinian Gorenstein with different Hilbert functions. In fact, we have

$$h_{R/J_1} = (1, 2, 2, 1)$$

while

$$h_{R/J_2} = (1, 3, 3, 1)$$

and so the degree of the regular form is not enough to compute the Hilbert function of the new Artinian Gorenstein ideal.

However, if the degree d of f is not too large, then the Hilbert function of J' depends only on J and d. In fact, it holds:

PROPOSITION 11. *In the same hypotheses of Theorem 5, assume furthermore that I is a tight annihilating (resp. tight resolving) ideal for J. If $d \leq \operatorname{reg}(R/J) - 2\operatorname{reg}(R/I) + 1$, then*

$$h_{R/J'}(j) = h_{R/I}(j) + h_{R/I}(\operatorname{reg}(R/J') - j) - \deg(R/I).$$

Proof. We know that $J \subseteq J'$ and so $[J]_k \subseteq [J']_k$ for every integer k.

Let $g \in [J']_k$. By construction, $gf \in [J]_{k+d}$. If $k + d \leq \operatorname{reg}(R/J) - \operatorname{reg}(R/I)$ then $[J]_{k+d} = [I]_{k+d}$ by previous Remark 6, and so $g \in [I]_k$ because f is regular for I. Hence, $[J]_k = [J']_k$ for every k such that $k + d \leq \operatorname{reg}(R/J) - \operatorname{reg}(R/I)$ i.e. $k \leq \operatorname{reg}(R/J) - \operatorname{reg}(R/I) - d$. We proved that J' is an Artinian Gorenstein ideal and so its Hilbert function is symmetric. Then, the Hilbert function of R/J' is completely determined if the condition $k \leq \operatorname{reg}(R/J) - \operatorname{reg}(R/I) - d$ covers at least the first half of the range where the Hilbert function is non zero. The largest d for which that happens is $d = \operatorname{reg}(R/J) - 2\operatorname{reg}(R/I) + 1$ and the claim follows. □

REMARK 7. If $d = \operatorname{reg}(\frac{R}{J}) - 2\operatorname{reg}(\frac{R}{I}) + 1$, then $\operatorname{reg}(\frac{R}{J'}) = 2\operatorname{reg}(\frac{R}{I}) - 1$ and the Hilbert function of R/J' is the one of R/J after erasing its flat part, that is to say, the values where it reaches $\deg(R/I)$.

We will show how the ideals J, I and J' are related. To this aim, we need some properties which we collect in the following lemma.

LEMMA 1. *Let I, J, J' and $f \in [R]_d$ be as in Theorem 5. Then:*

(1) $I \cap fJ' = fI$;

(2) $J/I : \overline{f}R/I = J'/I$.

Proof. 1) The inclusion $I \cap fJ' \supseteq fI$ follows from the fact that $I \subseteq J'$.

Let $g \in I$ be an element such that $g = fh$, $h \in J'$. Then, $h \in I : fR = I$ and so $g \in fI$, and the other inclusion is verified.

2) It is evident that $I \subseteq J'$, because $I \subseteq J \subseteq J'$. Then, we can consider the ideals J/I, J'/I, and $\overline{f}R/I$ of R/I, where $\overline{f}$ is the class of f in R/I. We want to prove that $J/I : \overline{f}R/I = J'/I$.

Now, if $\overline{g} \in J'/I$ then $g + h \in J'$ for some $h \in I$. But $I \subseteq J'$, and so $g = (g+h) - h \in J'$. By its definition, $gf \in J$ and so $\overline{g}\,\overline{f} \in J/I$. Hence, $\overline{g} \in J/I : \overline{f}R/I$.

Conversely, if $\overline{g} \in J/I : \overline{f}R/I$, then $\overline{g}\,\overline{f} \in J/I$. Of course, there exists $h \in I$ such that $gf + h \in J$. As for the reverse inclusion, from $I \subseteq J$ we get that $gf = (gf + h) - h \in J$. By its definition, it holds that $g \in J : fR = J'$, i.e. $\overline{g} \in J'/I$. □

Now, we can prove that, if d is not too large, then J is a basic double link of J' on I.

PROPOSITION 12. *Let I, J, J' and $f \in [R]_d$ be as in Theorem 5. Moreover assume that I is either tight annihilating or tight resolving for J. Then, if $d \leq$* $\operatorname{reg}(R/J) - 2\operatorname{reg}(R/I) + 1$, *then $J = I + fJ'$.*

Proof. We know that $fJ' \subseteq J$ and so $I + fJ' \subseteq J$. Then, the equality follows if they have the same Hilbert function. The Hilbert function of $I + fJ'$ was computed in Proposition 6(1) and it is

$$h_{R/I+fJ'}(j) = h_{R/I}(j) + h_{R/J'}(j-d) - h_{R/I}(j-d).$$

By Proposition 11, we have

$$h_{R/I+fJ'}(j) = h_{R/I}(j) + h_{R/I}(\operatorname{reg}(R/J) - j) - \deg(R/I) = h_{R/J}(j)$$

and the claim follows. □

Now, we show with an example that R/I could be neither a tight annihilating ring nor a tight resolving ideal for R/J'.

EXAMPLE 2. Let $R = K[x, y, z]$ be a polynomial ring in 3 unknowns, and let $I \subseteq R$ be the ideal generated by $y^2 - xz, x^2 - yz, z^2 - xy$ whose resolution is

$$0 \to R^2(-3) \xrightarrow{A} R^3(-2) \to I \to 0$$

where

$$A = \begin{pmatrix} z & x \\ y & z \\ x & y \end{pmatrix}.$$

The Hilbert function of R/I is $h_{R/I} = (1, 3, \rightarrow)$, and so $\operatorname{reg}(R/I) = 1$.

Let $J = (y^2 - xz, x^2 - yz, z^2 - xy, xz, -xy)$ be an Artinian Gorenstein ideal, whose minimal free resolution is

$$0 \to R(-5) \to R^5(-3) \xrightarrow{B} R^5(-2) \to J \to 0$$

where

$$B = \begin{pmatrix} 0 & 0 & -x & z & x \\ 0 & 0 & 0 & y & z \\ x & 0 & 0 & x & y \\ -z & -y & -x & 0 & 0 \\ -x & -z & -y & 0 & 0 \end{pmatrix}.$$

We have $I \subseteq J$, and $t_J = 5$, $\operatorname{reg}(R/J) = 2$. It is evident that R/I is a tight annihilating ring for R/J. We can control the Hilbert function of R/J' for every $d \leq 1$.

The element $x \in [R]_1$ is general for I; in fact, $I : xR = I$. Moreover $J : xR = (y, z, x^2) = J'$ and R/I is not a tight annihilating ring for R/J', because the Hilbert function of R/J' verifies $h_{R/J'}(1) = 1 \neq h_{R/I}(1) = 3$. Moreover, the minimal free resolution of R/J' is

$$0 \to R(-4) \to \begin{array}{c} R(-2) \\ \oplus \\ R^2(-3) \end{array} \to \begin{array}{c} R^2(-1) \\ \oplus \\ R(-2) \end{array} \to R \to R/J' \to 0$$

and it does not contain the minimal free resolution of R/I as a subcomplex, and hence R/I is not a tight resolving ideal for R/J'.

Now, we compute the shape of a free resolution of R/J'.

LEMMA 2. *In the same hypotheses as above, the shape of a free resolution of R/J' is*

$$0 \to R(-t_{J'}) \to \begin{array}{c} F_{n-1} \\ \oplus \\ F_1^*(-t_{J'}) \end{array} \to \cdots \to \begin{array}{c} F_1 \\ \oplus \\ F_{n-1}^*(-t_{J'}) \end{array} \to R \to R/J' \to 0$$

where $t_{J'} = t_J - d$.

Proof. From the assumptions on the minimal free resolutions of R/I and R/J it follows the diagram

$$\begin{array}{cccccccccccc}
 & 0 & \to & F_{n-1} & \xrightarrow{\delta_{n-1}} \cdot \to & F_1 & \to & R & \to & R/I & \to 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & R(-t_J) & \to & \begin{array}{c} F_{n-1} \\ \oplus \\ F_1^*(-t_J) \end{array} & \to \cdot \to & \begin{array}{c} F_1 \\ \oplus \\ F_{n-1}^*(-t_J) \end{array} & \to & R & \to & R/J & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & R(-t_J) & \to & F_1^*(-t_J) & \to \cdot \to & F_{n-1}^*(-t_J) & \to & \mathrm{coker}(\delta_{n-1}^*) & \to & 0 &
\end{array}$$

with exact rows and split exact columns.

Hence, $0 \to \mathrm{coker}(\delta_{n-1}^*)(-t_J) \to R/I \to R/J \to 0$ is a short exact sequence. It follows that $J/I \simeq \mathrm{coker}(\delta_{n-1}^*)(-t_J)$.

But, we know that $J/I \simeq \overline{f}(J'/I)$ and that $\overline{f} : R/I(-d) \to R/I$ is an injective map. It follows that $J'/I \simeq \mathrm{coker}(\delta_{n-1}^*)(-t_J + d) = \mathrm{coker}(\delta_{n-1}^*)(-t_{J'})$. A free resolution of J'/I is then

$$0 \to R(-t_{J'}) \to F_1^*(-t_{J'}) \to \cdots \to F_{n-1}^*(-t_{J'}) \to \mathrm{coker}(\delta_{n-1}^*)(-t_{J'}) \to 0$$

and so

$$0 \to R(-t_{J'}) \to \begin{array}{c} F_{n-1} \\ \oplus \\ F_1^*(-t_{J'}) \end{array} \to \cdots \to \begin{array}{c} F_1 \\ \oplus \\ F_{n-1}^*(-t_{J'}) \end{array} \to R \to R/J' \to 0$$

is a (eventually non minimal) free resolution of R/J'. □

Now, we give a numerical criterion to guarantee that I is a tight resolving ideal for J', too.

PROPOSITION 13. *Let I, J, J' be as above. If $d \leq \operatorname{reg}(R/J) - 2\operatorname{reg}(R/I)$ then R/I is a tight resolving ring for R/J'.*

Proof. At first, we check that $h_{R/J'}(j) = h_{R/I}(j)$ for $j \leq \operatorname{reg}(R/I)$.

Thanks to Proposition 11, we verify that $h_{R/I}(t_{J'} - j - n) = \deg(R/I)$ for every $j \leq \operatorname{reg}(R/I)$, i.e. that $t_J - d - n - j \geq \operatorname{reg}(R/I)$ for every $j \leq \operatorname{reg}(R/I)$. By hypothesis, $\operatorname{reg}(R/J) - d \geq 2\operatorname{reg}(R/I)$, and so the previous inequality can be written as $2\operatorname{reg}(R/I) - j \geq \operatorname{reg}(R/I)$ that holds for every $j \leq \operatorname{reg}(R/I)$.

Now, we want to check that the resolution of R/J' we computed is minimal. The resolution is obtained by mapping cone, and so we have to check that no entry of a matrix representing $F^*_{n-j-1}(-t_J) \to F_j$ has degree zero. Using the notation stated in Section 2, we have that $F_j = \oplus_{k=1}^{n_j} R^{\beta_{jk}}(-b_{jk})$ and $F^*_{n-j-1}(-t_J) = \oplus_{h=1}^{n_{n-j-1}} R^{\beta_{n-j-1,h}}(-t_J + d - b_{n-j-1,h})$, and so we have to prove that $t_J - d - b_{n-j-1,h} - b_{jk} > 0$ for every h and k. By substituting the regularity of R/J we get

$$\begin{aligned} t_J - d - b_{n-j-1,h} - b_{jk} &= n + \operatorname{reg}(R/J) - d - b_{n-j-1,h} - b_{jk} \geq \\ &\geq n + 2\operatorname{reg}(R/I) - b_{n-j-1,h} - b_{jk} \geq \\ &\geq n - (n - j - 1) - j = 1 \end{aligned} \tag{2}$$

where the first inequality follows from our hypothesis on d and the second one from the definition of regularity. Then, no cancellation can be performed and the resolution is minimal. □

REMARK 8. The regularity $\operatorname{reg}(R/J')$ of R/J' is equal to $\operatorname{reg}(R/J') = \operatorname{reg}(R/J) - d$. If $d \leq \operatorname{reg}(R/J) - 2\operatorname{reg}(R/I)$ then $\operatorname{reg}(R/J') \geq 2\operatorname{reg}(R/I)$ and so R/I is a tight annihilating ring for R/J' by Proposition 9.

Before ending the section, we compute the residual of J with respect to J'.

PROPOSITION 14. *Let J be an Artinian Gorenstein ideal, let $I \subseteq J$ be a tight resolving ideal for J and let $J' = J : fR$ where $f \in [R]_d$ is a regular form for I, with $d \leq \operatorname{reg}(R/J) - \operatorname{reg}(R/I)$. Then, $J : J' = J + fR$.*

Proof. The inclusion $J \subseteq J'$, observed in Remark 5, induces a map of complexes between the minimal free resolution of J and J':

$$\begin{array}{ccccccccccccc} 0 & \to & R(-t_J) & \to & H_{n-1} & \to & \dots & \to & H_1 & \to & J & \to & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \to & R(-t_{J'}) & \to & G_{n-1} & \to & \dots & \to & G_1 & \to & J' & \to & 0 \end{array}$$

and so we get the equality $J : J' = J + gR$ where $g \in [R]_{t_J - t_{J'} = d}$ represents the last map $R(-t_J) \to R(-t_{J'})$ (see [2]). By the definition of J' we get $fJ' \subseteq J$. Hence, $f \in J : J' = J + gR$.

If $f \in J$, then $f \in I$, because of its degree, and this is not possible being f regular for I. It follows that $f \notin J$ and so $f = ag + h, a \in K - \{0\}, h \in J$, i.e. $J : J' = J + fR$. □

5. The main result

In this section, chosen a dimension 1 Cohen-Macaulay ring R/I that contains an ideal G/I such that R/G is an Artinian Gorenstein ring, we want to compare the ideals which are basic double G-links of G on I with the Artinian Gorenstein ideals containing I for which I is a tight annihilating ideal.

With this in mind, we construct an Artinian Gorenstein ideal G as the sum of I and an ideal I_1 which is geometrically linked to I (see Theorem 3).

CONSTRUCTION 1. Let J be a Gorenstein ideal that verifies the following conditions

1. $J \subseteq I$;
2. $\dim R/J = 1$;
3. if $I_1 = J : I$ then $J = I \cap I_1$;
4. the minimal generators of J can be choosen among the minimal ones of I.

Then, the ideal $G = I + I_1$ is an Artinian Gorenstein ideal, which is the sum of two directly linked Cohen-Macaulay ideals.

Of course, there are dimension 1 Cohen-Macaulay rings for which the construction does not work, and that depends on the geometry of the schemes defined by those rings, as the following example shows.

EXAMPLE 3. Let $\mathbb{P}^2 = \mathrm{Proj}(K[x, y, z])$, and let X, Y, Z be three 0-dimensional schemes of degree 11 defined by the ideals $I_X = (x^3 - y^2z, z^4 - xy^3, y^5 - x^2z^3)$, $I_Y = (x^3, -xy^3, y^5 - x^2z^3)$ and $I_Z = (x^3, -xy^3, y^5)$, respectively. The three schemes have the same Hilbert function

$$h_X = h_Y = h_Z = (1, 3, 6, 9, 11, \rightarrow).$$

The minimal free resolution of I_X is

$$0 \to R^2(-6) \xrightarrow{A} R(-3) \oplus R(-4) \oplus R(-5) \to I_X \to 0$$

where

$$A = \begin{pmatrix} y^3 & z^3 \\ x^2 & y^2 \\ z & x \end{pmatrix}.$$

The first two generators of I_X form a regular sequence, and so $J = (x^3 - y^2z, z^4 - xy^3)$ is a complete intersection ideal. The ideal $I_1 = J_1 : I_X = (x, z)$ has degree 1 and $J_1 = I_X \cap I_1$.

The minimal free resolution of I_Y is

$$0 \to R^2(-6) \xrightarrow{B} R(-3) \oplus R(-4) \oplus R(-5) \to I_Y \to 0$$

where

$$B = \begin{pmatrix} y^3 & z^3 \\ x^2 & y^2 \\ 0 & x \end{pmatrix}.$$

It is evident that, if $F \in [I_Y]_4$, then x^3, F is not a regular sequence because $(x^3, F) \subseteq (x)$. Hence, the minimal degrees of two generators of I_Y that form a regular sequence are 3, 5, and the corresponding ideal J has degree 15.

The geometrical reason for that behavior is that Y contains a degree 5 subscheme Y' contained in a line: $I_{Y'} = (x, y^5) \supseteq I_Y$.

The minimal free resolution of I_Z is

$$0 \to R^2(-6) \xrightarrow{C} R(-3) \oplus R(-4) \oplus R(-5) \to I_Z \to 0$$

where

$$C = \begin{pmatrix} y^3 & 0 \\ x^2 & y^2 \\ 0 & x \end{pmatrix}.$$

Z is supported on the point $A(0:0:1)$ with $I_A = (x, y)$. If $J_1 \subseteq I_Z$ is a complete intersection with generators of degrees 3, 5, then $J_1 = (x^3, xy^3 l + y^5)$ for some $l \in [R]_1$.

We have $J_1 : I_Z = (x^2, y^2+xl) = I_1$ and $I_Z \cap I_1 = (x^3, y^3(y^2+xl), xy^3) \subset J$ (in fact, $xy^3 = xy(y^2+xl) - yl(x^2)$) and hence no complete intersection ideal J gives a geometric link of Z with another scheme. This happens because Z is not locally Gorenstein.

REMARK 9. We were informed by A. Iarrobino that M. Boij, in a talk at Northeastern University, proved that there exists an Artinian Gorenstein ideal $J \supseteq I$ with I tight annihilating for J if, and only if, the ring R/I is locally Gorenstein, i.e. every localization of R/I at a minimal prime is a Gorenstein ring.

It is natural to look for an ideal J of *minimal socle degree* to construct the ideal I_1. Then, we define

DEFINITION 9. *Let I be a Cohen-Macaulay ideal such that R/I has dimension 1 and is locally Gorenstein. We set*

$$s_I = \min\{\operatorname{reg}(R/J) \mid J \text{ verifies the hypotheses of Construction 1}\}.$$

If we write a minimal free resolution of an ideal J of minimal socle degree we have

$$0 \to R(-s_I - n + 1) \to Q_{n-2} \to \cdots \to Q_1 \to J \to 0.$$

As we showed in Example 3, the integer s_I depends on the geometry of the ring R/I.

Let J be a Gorenstein ideal fulfilling all the assumptions of Construction 1, and verifying $\operatorname{reg}(R/J) = s_I$. Then, we fix now and forever, the ideal $G = I + I_1$, where $I_1 = J : I$. A free resolution of G was computed in Proposition 6(2).

Now, we want to construct a parameter space for the family of the Artinian Gorenstein ideals obtained by basic double G-link from G on I, that are $I + fG = I + fI_1$, for some $f \in R$ regular for I.

The key property to construct this parameter space is the following:

PROPOSITION 15. *Let $f_1, f_2 \in [R]_d$ be two elements, regular for I. Then $I + f_1 I_1 = I + f_2 I_1$ if, and only if, $f_1 = f_2 \mod (I)$.*

Proof. First, assume $f_1 = f_2 \mod (I)$. Let $g \in I + f_2 I_1$ be a form. Then, there exist $p \in I$ and $q \in I_1$ such that $g = p + f_2 q$. By assumption, there exists $h \in I$ such that $f_2 = f_1 + h$. Hence, $g = p + f_1 q + hq = (p + hq) + f_1 q = p_1 + f_1 q \in I + f_1 I_1$ because $p_1 = p + hq \in I$.

Vice versa, if $L_1 = I + f_1 I_1$, $L_2 = I + f_2 I_1$ and $L_1 = L_2$ then, $L_1 : G = L_2 : G$. By Proposition 14, $I + f_1 R = I + f_2 R$ and then $f_1 - f_2 \in I$. □

We are able to construct the parameter space for the family of the Artinian Gorenstein ideals which are basic double G-links of G on I.

THEOREM 6. *Let $I \subseteq R$ be a dimension 1, locally Gorenstein, Cohen-Macaulay ideal. Let*

$$\mathcal{BDL}(I, s_I + d) = \{L = I + fI_1 | \deg(f) = d,\ f \text{ regular for } I\}$$

be the family of the Artinian Gorenstein ideals that are basic double G-links of G on I, of regularity $\operatorname{reg}(L) = s_I + d$. Then, $\mathcal{BDL}(I, s_I + d)$ is parametrized by an open subset of an affine space of dimension $h_{R/I}(d)$.

Proof. Every ideal $L \in \mathcal{BDL}(I, s_I + d)$ corresponds to the choice of $f \in [R]_d$ such that $I : fR = I$, that is an open condition.

By Proposition 15, the Artinian Gorenstein ideals $L_1 = I + f_1 I_1$ and $L_2 = I + f_2 I_1$ are equal if, and only if, $f_1 - f_2 \in I$.

Hence, the natural parameter space for $\mathcal{BDL}(I, s_I + d)$ is the open subset W of the affine space $[R/I]_d$ corresponding to the forms that are regular for I.

By definition, $\dim W = \dim_K [R/I]_d = h_{R/I}(d)$. □

Now, following [6], we construct the parameter space for the Gorenstein ideals for which I is a tight annihilating ideal.

THEOREM 7. *Let $I \subseteq R$ be a dimension 1, locally Gorenstein, Cohen-Macaulay ideal. Let*

$$\mathcal{G}(I, r) = \{L | I \text{ is a tight annihilating ideal for } L, \operatorname{reg}(L) = r\}$$

be the family of Artinian Gorenstein ideals L containing I as tight annihilating ideal, of regularity $\operatorname{reg}(L) = r$. *Then, the parameter space of $\mathcal{G}(I, r)$ is an open subset of an affine space of dimension* $\deg(R/I)$.

Proof. The statement in the case of codimension 3 is part of the more general Theorem 5.31 in [6]. But the proof holds verbatim in our hypotheses. □

Now, we can state our main result.

THEOREM 8. *Let $I \subseteq R$ be a dimension* 1, *locally Gorenstein, Cohen-Macaulay ideal. Let $d \geq s_I + 1$ be an integer. Then $\mathcal{G}(I, s_I + d) = \mathcal{BDL}(I, s_I + d)$.*

Proof. If $d \geq s_I + 1$, then $h_{R/L}(j) = h_{R/I}(j)\ \forall j \leq s_I$. Moreover, by using arguments as in the proof of Proposition 13, we can prove that a minimal free resolution of L is

$$0 \to R(-t) \to \begin{matrix} F_{n-1} \\ \oplus \\ F_1^*(-t) \end{matrix} \to \cdots \to \begin{matrix} F_1 \\ \oplus \\ F_{n-1}^*(-t) \end{matrix} \to L \to 0,$$

where $t = s_I + n + d - 1$, and so I is a tight annihilating ideal for every $L \in \mathcal{BDL}(I, s_I + d)$. Hence, $\mathcal{BDL}(I, s_I + d) \subseteq \mathcal{G}(I, s_I + d)$. Moreover, both $\mathcal{BDL}(I, s_I + d)$ and $\mathcal{G}(I, s_I + d)$ are parametrized by open subsets of affine spaces of dimension $\deg(R/I) = h_{R/I}(d)$. Now, to prove that $\mathcal{BDL}(I, s_I + d) = \mathcal{G}(I, s_I + d)$, it is enough to prove that $\mathcal{BDL}(I, s_I + d)$ is closed in $\mathcal{G}(I, s_I + d)$.

If $L \in \mathcal{G}(I, s_I + d)$, and it is the flat limit of a 1-parameter flat family of ideals in $\mathcal{BDL}(I, s_I + d)$, then there exists a 1-parameter family of polynomials $f_t \in [R]_d$ such that $L_t = I + f_t I_1 \to L$ for $t \to 0$.

It is evident that $I + f_t I_1 \to I + f_0 I_1 = L$, for $t \to 0$, with $f_0 \in [R]_d$.

If f_0 is not regular for I, then there exists a minimal homogeneous prime ideal $\mathcal{P} \in R/I$ that contains $f_0 R + I$, and so $\dim \frac{R}{I + f_0 I_1} \neq 0$. But this is a contradiction because L is Artinian, and so f_0 is regular for I.

Hence, L is a basic double G-link of G on I, and so we get the claim. □

The same argument as above proves also the following

PROPOSITION 16. *Let I, J be as above, and let $1 \leq d \leq s_I$ be an integer. Then $\mathcal{BDL}(I, s_I + d)$ is a quasi projective subscheme of $\mathcal{G}(I, s_I + d)$ of codimension* $\deg(R/I) - h_{R/I}(d)$.

In $R = K[x, y, z]$, it is known that the first half of the Hilbert function of the Artinian Gorenstein ring R/L is admissible as Hilbert function of a dimension 1 Cohen-Macaulay ring R/I. Moreover, if $h_{R/L}(j) = s$ for at least 3 consecutive integers, then there exists a dimension 1 Cohen-Macaulay ring R/I of degree $\deg(R/I) = s$ that is a tight annihilating ring for L. In higher codimension, we could not find an analogous numerical condition to guarantee the existence of a tight annihilating ring for L.

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Rosa Maria MIRÓ-ROIG, Departamento de Algebra y Geometria, Universidad de Barcelona, E-08007 Barcelona, ESPAÑA
e-mail: `miro@ub.edu`

Roberto NOTARI, Maria Luisa SPREAFICO, Dipartimento di Matematica, Politecnico di Torino, I-10129 Torino, ITALIA
e-mail: `roberto.notari@polito.it, maria.spreafico@polito.it`

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N. Mohan Kumar – A.P. Rao – G.V. Ravindra

FOUR-BY-FOUR PFAFFIANS

Dedicated to Paolo Valabrega on the occasion of his 60th birthday

Abstract. This paper shows that the general hypersurface of degree ≥ 6 in projective four space cannot support an indecomposable rank two vector bundle which is Arithmetically Cohen-Macaulay and four generated. Equivalently, the defining polynomial of the hypersurface is not the Pfaffian of a four by four minimal skew-symmetric matrix.

1. Introduction

In this note, we study indecomposable rank two bundles E on a smooth hypersurface X in $\mathbf{P}^4$ which are Arithmetically Cohen-Macaulay. The existence of such a bundle on X is equivalent to X being the Pfaffian of a minimal skew-symmetric matrix of size $2k \times 2k$, with $k \geq 2$. The general hypersurface of degree ≤ 5 in $\mathbf{P}^4$ is known to be Pfaffian ([1], [2], [6]) and the general sextic in $\mathbf{P}^4$ is known to be not Pfaffian ([4]). One should expect the result of [4] to extend to all general hypersurfaces of degree ≥ 6. (Indeed the analogous statement for hypersurfaces in $\mathbf{P}^5$ was established in [8], see also [5].) However, in this note we offer a partial result towards that conclusion. We show that the general hypersurface in $\mathbf{P}^4$ of degree ≥ 6 is not the Pfaffian of a 4×4 skew-symmetric matrix. For a hypersurface of degree r to be the Pfaffian of a $2k \times 2k$ skew-symmetric matrix, we must have $2 \leq k \leq r$. It is quite easy to show by a dimension count that the general hypersurface of degree $r \geq 6$ in $\mathbf{P}^4$ is not the Pfaffian of a $2r \times 2r$ skew-symmetric matrix of linear forms. Thus, this note addresses the lower extreme of the range for k.

2. Reductions

Let X be a smooth hypersurface in $\mathbf{P}^4$ of degree $r \geq 2$. A rank two vector bundle E on X will be called Arithmetically Cohen-Macaulay (or ACM) if $\oplus_{k\in\mathbb{Z}} H^i(X, E(k))$ equals 0 for $i = 1, 2$. Since $\mathrm{Pic}(X)$ equals $\mathbb{Z}$, with generator $\mathcal{O}_X(1)$, the first Chern class $c_1(E)$ can be treated as an integer t. The bundle E has a minimal resolution over $\mathbf{P}^4$ of the form

$$0 \to L_1 \xrightarrow{\phi} L_0 \to E \to 0,$$

where L_0, L_1 are sums of line bundles. By using the isomorphism of E and $E^\vee(t)$, we obtain (see [2]) that $L_1 \cong L_0^\vee(t-r)$ and the matrix ϕ (of homogeneous polynomials) can be chosen as skew-symmetric. In particular, L_0 has even rank and the defining polynomial of X is the Pfaffian of this matrix. The case where ϕ is two by two is just the case where E is decomposable. The next case is where ϕ is a four by four minimal matrix. These correspond to ACM bundles E with four global sections (in possibly different degrees) which generate it.

Our goal is to show that the generic hypersurface of degree $r \geq 6$ in $\mathbf{P}^4$ does not support an indecomposable rank two ACM bundle which is four generated, or equivalently, that such a hypersurface does not have the Pfaffian of a four by four minimal matrix as its defining polynomial.

So fix a degree $r \geq 6$. Let us assume that E is a rank two ACM bundle which is four generated and which has been normalized so that its first Chern class t equals 0 or -1. If $L_0 = \oplus_{i=1}^4 \mathcal{O}_{\mathbf{P}}(a_i)$ with $a_1 \geq a_2 \geq a_3 \geq a_4$, the resolution for E is given by

$$\bigoplus_{i=1}^{4} \mathcal{O}_{\mathbf{P}}(t - a_i - r) \xrightarrow{\phi} \bigoplus_{i=1}^{4} \mathcal{O}_{\mathbf{P}}(a_i).$$

Write the matrix of ϕ as

$$\phi = \begin{bmatrix} 0 & A & B & C \\ -A & 0 & D & E \\ -B & -D & 0 & F \\ -C & -E & -F & 0 \end{bmatrix}.$$

Since X is smooth with equation $AF - BE + CD = 0$, the homogeneous entries A, B, C, D, E, F are all non-zero and have no common zero on $\mathbf{P}^4$.

LEMMA 1. *For fixed r and t (normalized), there are only finitely many possibilities for* (a_1, a_2, a_3, a_4).

Proof. Let a, b, c, d, e, f denote the degrees of the poynomials A, B, C, D, E, F. Since the Pfaffian of the matrix is $AF - BE + CD$, the degree of each matrix entry is bounded between 1 and $r - 1$. $a = a_1 + a_2 + (r - t)$, $b = a_1 + a_3 + (r - t)$ etc. Thus if $i \neq j$, $0 < a_i + a_j + r - t < r$ while $\sum a_i = -r + 2t$. From the inequality, regardless of the sign of a_1, the other three values a_2, a_3, a_4 are < 0. But again using the inequality, their pairwise sums are $> -r + t$, hence there are only finitely many choices for them. Lastly, a_1 depends on the remaining quantities. □

It suffices therefore to fix $r \geq 6$, $t = 0$ or -1 and a four-tuple (a_1, a_2, a_3, a_4) and show that there is no ACM bundle on the general hypersurface of degree r which has a resolution given by a matrix ϕ of the type $(a_1, a_2, a_3, a_4), t$.

From the inequalities on a_i, we obtain the inequalities

$$0 < a \leq b \leq c, d \leq e \leq f < r.$$

We do no harm by rewriting the matrix ϕ with the letters C and D interchanged to assume without loss of generality that $c \leq d$.

PROPOSITION 1. *Let X be a smooth hypersurface of degree ≥ 3 in $\mathbf{P}^4$ supporting an ACM bundle E of type $(a_1 \geq a_2 \geq a_3 \geq a_4), t$. The degrees of the entries of ϕ can be arranged (without loss of generality) as:*

$$a \leq b \leq c \leq d \leq e \leq f.$$

Then X will contain a curve Y which is the complete intersection of hypersurfaces of the three lowest degrees in the arrangement and a curve Z which is the complete intersection of hypersurfaces of the three highest degrees in the arrangement.

Proof. Consider the ideals (A, B, C) and (D, E, F). Since the equation of X is $AF - BE + CD$, these ideals give subschemes of X. Take for example (A, B, C). If the variety Y it defines has a surface component, this gives a divisor on X. As Pic(X)= $\mathbb{Z}$, there is a hypersurface $S = 0$ in $\mathbf{P}^4$ inducing this divisor. Now at a point in $\mathbf{P}^4$ where $S = D = E = F = 0$, all six polynomials $A, \ldots, F$ vanish, making a multiple point for X. Hence, X being smooth, Y must be a curve on X. Thus (A, B, C) defines a complete intersection curve on X. $\square$

To make our notations non-vacuous, we will assume that at least one smooth hypersurface exists of a fixed degree $r \geq 6$ with an ACM bundle of type $(a_1 \geq a_2 \geq a_3 \geq a_4), t$. Let $\mathcal{F}_{(a,b,c);r}$ denote the Hilbert flag scheme that parametrizes all inclusions $Y \subset X \subset \mathbf{P}^4$ where X is a hypersurface of degree r and Y is a complete intersection curve lying on X which is cut out by three hypersurfaces of degrees a, b, c. Our discussion above produces points in $\mathcal{F}_{(a,b,c);r}$ and $\mathcal{F}_{(d,e,f);r}$.

Let $\mathcal{H}_r$ denote the Hilbert scheme of all hypersurfaces in $\mathbf{P}^4$ of degree r and let $\mathcal{H}_{a,b,c}$ denote the Hilbert scheme of all curves in $\mathbf{P}^4$ with the same Hilbert polynomial as the complete intersection of three hypersurfaces of degrees a, b and c. Following J. Kleppe ([7]), the Zariski tangent spaces of these three schemes are related as follows: corresponding to the projections

$$\begin{array}{ccc} \mathcal{F}_{(a,b,c);r} & \xrightarrow{p_2} & \mathcal{H}_{a,b,c} \\ \downarrow p_1 & & \\ \mathcal{H}_r & & \end{array}$$

if T is the tangent space at the point $Y \overset{i}{\hookrightarrow} X \subset \mathbf{P}^4$ of $\mathcal{F}_{(a,b,c);r}$, there is a Cartesian diagram

$$\begin{array}{ccc} T & \xrightarrow{p_2} & H^0(Y, \mathcal{N}_{Y/\mathbf{P}}) \\ \downarrow p_1 & & \downarrow \alpha \\ H^0(X, \mathcal{N}_{X/\mathbf{P}}) & \xrightarrow{\beta} & H^0(Y, i^*\mathcal{N}_{X/\mathbf{P}}) \end{array}$$

of vector spaces.

Hence $p_1 : T \to H^0(X, \mathcal{N}_{X/\mathbf{P}})$ is onto if and only if $\alpha : H^0(Y, \mathcal{N}_{Y/\mathbf{P}}) \to H^0(Y, i^*\mathcal{N}_{X/\mathbf{P}})$ is onto. The map α is easy to describe. It is the map given as

$$H^0(Y, \mathcal{O}_Y(a) \oplus \mathcal{O}_Y(b) \oplus \mathcal{O}_Y(c)) \xrightarrow{[F,-E,D]} H^0(Y, \mathcal{O}_Y(r)).$$

Hence

PROPOSITION 2. *Choose general forms A, B, C, D, E, F of degrees a, b, c, d, e, f and let Y be the curve defined by $A = B = C = 0$. If the map*

$$H^0(Y, \mathcal{O}_Y(a) \oplus \mathcal{O}_Y(b) \oplus \mathcal{O}_Y(c)) \xrightarrow{[F,-E,D]} H^0(Y, \mathcal{O}_Y(r))$$

is not onto, then the general hypersurface of degree r does not support a rank two ACM bundle of type $(a_1, a_2, a_3, a_4), t$.

Proof. Consider a general Pfaffian hypersurface X of equation $AF - BE + CD = 0$ where A, B, C, D, E, F are chosen generally. Such an X contains such a Y and X is in the image of p_1. By our hypothesis, $p_1 : T \to H^0(X, \mathcal{N}_{X/\mathbf{P}})$ is not onto and (in characteristic zero) it follows that $p_1 : \mathcal{F}_{(a,b,c);r} \to \mathcal{H}_r$ is not dominant. Since all hypersurfaces X supporting such a rank two ACM bundle are in the image of p_1, we are done. □

REMARK 1. Note that the last proposition can also be applied to the situation where Y is replaced by the curve Z given by $D = E = F = 0$, with the map given by $[A, -B, C]$, with a similar statement.

3. Calculations

We are given general forms A, B, C, D, E, F of degrees a, b, c, d, e, f where $a + f = b + e = c + d = r$ and where without loss of generality, by interchanging C and D we may assume that $1 \le a \le b \le c \le d \le e \le f < r$. Assume that $r \ge 6$. We will show that if Y is the curve $A = B = C = 0$ or if Z is the curve $D = E = F = 0$, depending on the conditions on a, b, c, d, e, f, either $H^0(\mathcal{N}_{Y/\mathbf{P}}) \xrightarrow{[F,-E,D]} H^0(\mathcal{O}_Y(r))$ or $H^0(\mathcal{N}_{Z/\mathbf{P}}) \xrightarrow{[A,-B,C]} H^0(\mathcal{O}_Z(r))$ is not onto. This will prove the desired result.

3.1. Case 1

$b \ge 3,\ c \ge a + 1,\ 2a + b < r - 2$.
In $\mathbf{P}^5$ (or in 6 variables) consider the homogeneous complete intersection ideal

$$I = (X_0^a, X_1^b, X_2^c, X_3^{r-c}, X_4^{r-b}, X_5^{r-a} - X_2^{c-a-1} X_3^{r-c-a-1} X_4^{a+2})$$

in the polynomial ring S_5 on $X_0, \ldots, X_5$. Viewed as a module over S_4 (the polynomial ring on $X_0, \ldots, X_4$), $M = S_5/I$ decomposes as a direct sum

$$M = N(0) \oplus N(1)X_5 \oplus N(2)X_5^2 \oplus \cdots \oplus N(r - a - 1)X_5^{r-a-1},$$

where the $N(i)$ are graded S_4 modules. Consider the multiplication map $X_5 : M_{r-1} \to M_r$ from the $(r - 1)$-st to the r-th graded pieces of M. We claim it is injective and not surjective.

Indeed, any element m in the kernel is of the form nX_5^{r-a-1} where n is a homogeneous element in $N(r - a - 1)$ of degree a. Since $X_5 \cdot m = n \cdot X_5^{r-a} \equiv n \cdot X_2^{c-a-1} X_3^{r-c-a-1} X_4^{a+2} \equiv 0 \mod (X_0^a, X_1^b, X_2^c, X_3^{r-c}, X_4^{r-b})$ we may assume that n itself is represented by a monomial in $X_0, \ldots, X_4$ of degree a. Our inequalities have been chosen so that even in the case where n is represented by X_4^a, the exponents of X_4 in the product is $a + a + 2$ which is less than $r - b$. Thus n and hence the kernel must be 0.

On the other hand, the element $X_0^{a-1}X_1^2X_2^{c-a-1}X_3^{r-c-a-1}X_4^{a+1}$ in M_r lies in its first summand $N(0)_r$. In order to be in the image of multiplication by X_5, this element must be a multiple of $X_2^{c-a-1}X_3^{r-c-a-1}X_4^{a+2}$. By inspecting the factor in X_4, this is clearly not the case. So the multiplication map is not surjective.

Hence dim M_{r-1} < dim M_r. Now the Hilbert function of a complete intersection ideal like I depends only on the degrees of the generators. Hence, for any complete intersection ideal I' in S_5 with generators of the same degrees, for the corresponding module $M' = S_5/I'$, dim M'_{r-1} < dim M'_r.

Now coming back to our general six forms A, B, C, D, E, F in S_4, of the same degrees as the generators of the ideal I above. Since they include a regular sequence on $\mathbf{P}^4$, we can lift these polynomials to forms A', B', C', D', E', F' in S_5 which give a complete intersection ideal I' in S_5.

The module $\bar{M} = S_4/(A, B, C, D, E, F)$ is the cokernel of the map

$$X_5 : M'(-1) \to M'.$$

By our argument above, we conclude that $\bar{M}_r \neq 0$.

Lastly, the map $H^0(\mathcal{O}_Y(a) \oplus \mathcal{O}_Y(b) \oplus \mathcal{O}_Y(c)) \xrightarrow{[F,-E,D]} H^0(\mathcal{O}_Y(r))$ has cokernel precisely $\bar{M}_r$ which is not zero, and hence the map is not onto.

3.2. Case 2

$b \leq 2$.
Since the forms are general, the curve Y given by $A = B = C = 0$ is a smooth complete intersection curve, with $\omega_Y \cong \mathcal{O}_Y(a + b + c - 5)$. Since $a + b \leq 4$, $\mathcal{O}_Y(c)$ is nonspecial.

1. Suppose $\mathcal{O}_Y(a)$ is nonspecial. Then all three of $\mathcal{O}_Y(a), \mathcal{O}_Y(b), \mathcal{O}_Y(c)$ are nonspecial. Hence $h^0(\mathcal{N}_{Y/\mathbf{P}}) = (a+b+c)\delta+3(1-g)$ where $\delta = abc$ is the degree of Y and g is the genus. Also $h^0(\mathcal{O}_Y(r)) = r\delta+1-g+h^1(\mathcal{O}_Y(r)) \geq r\delta+1-g$. To show that $h^0(\mathcal{N}_{Y/\mathbf{P}}) < h^0(\mathcal{O}_Y(r))$, it is enough to show that

$$(a + b + c)\delta + 3(1 - g) < r\delta + 1 - g.$$

Since $2g - 2 = (a + b + c - 5)\delta$, this inequality becomes $5\delta < r\delta$ which is true as $r \geq 6$.

2. Suppose $\mathcal{O}_Y(a)$ is special (so $b + c \geq 5$), but $\mathcal{O}_Y(b)$ is nonspecial. By Clifford's theorem, $h^0(\mathcal{O}_Y(a)) \leq \frac{1}{2}a\delta + 1$. In this case $h^0(\mathcal{N}_{Y/\mathbf{P}}) < h^0(\mathcal{O}_Y(r))$ will be true provided that

$$\frac{1}{2}a\delta + 1 + (b + c)\delta + 2(1 - g) < r\delta + (1 - g)$$

or $r > \frac{b+c}{2} + \frac{1}{\delta} + \frac{5}{2}$.
Since $c \leq \frac{r}{2}$ and $b \leq 2$, this is achieved if

$r > \frac{2+r/2}{2} + \frac{1}{\delta} + \frac{5}{2}$ which is the same as $r > \frac{14}{3} + \frac{4}{3\delta}$.
But $c \geq 3$, so $\delta \geq 3$, hence the last inequality is true as $r \geq 6$.

3. Suppose both $\mathcal{O}_Y(a)$ and $\mathcal{O}_Y(b)$ are special. Hence $a + c \geq 5$. Using Clifford's theorem, in this case $h^0(\mathcal{N}_{Y/\mathbf{P}}) < h^0(\mathcal{O}_Y(r))$ will be true provided that
$\frac{1}{2}(a+b)\delta + 2 + c\delta + (1-g) < r\delta + (1-g)$.
This becomes $r > \frac{1}{2}(a+b) + \frac{2}{\delta} + c$. Using $c \leq \frac{r}{2}$, $a + b \leq 4$, and $\delta \geq 3$, this is again true when $r \geq 6$.

3.3. Case 3

$c < a + 1$.
In this case $a = b = c$ and $r \geq 2a$. Using the sequence

$$0 \to \mathcal{I}_Y(a) \to \mathcal{O}_{\mathbf{P}}(a) \to \mathcal{O}_Y(a) \to 0,$$

we get $h^0(\mathcal{N}_{Y/\mathbf{P}}) = 3h^0(\mathcal{O}_Y(a)) = 3[\binom{a+4}{4} - 3]$ while $h^0(\mathcal{O}_Y(r)) \geq h^0(\mathcal{O}_Y(2a)) = \binom{2a+4}{4} - 3\binom{a+4}{4} + 3$. Hence the inequality $h^0(\mathcal{N}_{Y/\mathbf{P}}) < h^0(\mathcal{O}_Y(r))$ will be true provided $\binom{2a+4}{4} > 6\binom{a+4}{4} - 12$. The reader may verify that it reduces to $10a^4 + 20a^3 - 70a^2 - 200a + 7(4!) > 0$ and the last inequality is true when $a \geq 3$. Thus we have settled this case when $r \geq 6$ and $a \geq 3$. If $r \geq 6$ and a (and hence b) ≤ 2, we are back in the previous case.

3.4. Case 4

$2a + b \geq r - 2$ and $r \geq 82$.
For this case, we will study the curve Z given by $D = E = F = 0$ (of degrees $r - c, r - b, r - a$) and consider the inequality $h^0(\mathcal{N}_{Z/\mathbf{P}}) < h^0(\mathcal{O}_Z(r))$.
Since $a, b, c \leq \frac{r}{2}$, $2a+2 \geq r-b \geq \frac{r}{2}$, hence $a \geq \frac{r}{4} - 1$. Also $b \geq a$ and $2a+b \geq r-2$, hence $b \geq \frac{r}{3} - \frac{2}{3}$. Likewise, $c \geq \frac{r}{3} - \frac{2}{3}$.

Now $h^0(\mathcal{O}_Z(r-a)) = h^0(\mathcal{O}_{\mathbf{P}}(r-a)) - h^0(\mathcal{I}_Z(r-a)) \leq \binom{r-a+4}{4} - 1$ etc., hence

$$h^0(\mathcal{N}_{Z/\mathbf{P}}) \leq \binom{r-a+4}{4} + \binom{r-b+4}{4} + \binom{r-c+4}{4} - 3 \leq \binom{\frac{3r}{4}+5}{4} + 2\binom{\frac{2r}{3}+\frac{14}{3}}{4} - 3$$

or $h^0(\mathcal{N}_{Z/\mathbf{P}}) \leq G(r)$, where $G(r)$ is the last expression.

Looking at the Koszul resolution for $\mathcal{O}_Z(r)$, since $a + b + c \leq \frac{3r}{2} < 2r$, the last term in the resolution has no global sections. Hence $h^0(\mathcal{O}_Z(r)) \geq h^0(\mathcal{O}_{\mathbf{P}}(r)) - [h^0(\mathcal{O}_{\mathbf{P}}(a)) + h^0(\mathcal{O}_{\mathbf{P}}(b)) + h^0(\mathcal{O}_{\mathbf{P}}(c))] \geq \binom{r+4}{4} - \binom{a+4}{4} - \binom{b+4}{4} - \binom{c+4}{4} \geq \binom{r+4}{4} - 3\binom{\frac{r}{2}+4}{4}$, or $h^0(\mathcal{O}_Z(r)) \geq F(r)$, where $F(r)$ is the last expression.

The reader may verify that $G(r) < F(r)$ for $r \geq 82$.

3.5. Case 5

$6 \leq r \leq 81$, $2a + b \geq r - 2$, $b \geq 3$, $c \geq a + 1$.
We still have $\frac{r}{4} - 1 \leq a \leq \frac{r}{2}$, $\frac{r}{3} - \frac{2}{3} \leq b, c \leq \frac{r}{2}$. For the curve Y given by $A = B =$

$C = 0$, we can explicitly compute $h^0(\mathcal{O}_Y(k))$ for any k using the Koszul resolution for $\mathcal{O}_Y(k)$. Hence both terms in the inequality $h^0(\mathcal{N}_{Y/\mathbf{P}}) < h^0(\mathcal{O}_Y(r))$ can be computed for all allowable values of a, b, c, r using a computer program like Maple and the inequality can be verified. We will leave it to the reader to verify this claim.

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N. MOHAN KUMAR, G.V. RAVINDRA, Department of Mathematics, Washington University in St. Louis, St. Louis, Missouri, 63130, USA
e-mail: `kumar@wustl.edu, ravindra@math.wustl.edu`

A.P. RAO, Department of Mathematics, University of Missouri-St. Louis, St. Louis, Missouri, 63121, USA,
e-mail: `rao@arch.umsl.edu`

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G. Restuccia

SYMMETRIC ALGEBRAS OF FINITELY GENERATED GRADED MODULES AND s-SEQUENCES

Dedicated to Paolo Valabrega on the occasion of his 60 th birthday

Abstract. We study properties of the symmetric algebra of finitely generated graded modules M on a Noetherian ring R, generated by s-sequences. For these modules we investigate the Eisenbud-Goto conjecture. If $R = K[X_1, \dots, X_n]$ is a polynomial ring over a field K and M has linear syzygies, we consider the jacobian dual module of M in order to describe the Rees algebra of M.

1. Introduction

The aim of this paper is to study an interesting class of finitely generated modules M on a Noetherian ring R for which the initial ideal of the presentation ideal J of their symmetric algebra is very simple. More precisely, with respect to a special order on the variables that correspond to the generators of the module M, we have a good expression for the initial ideal of J. This area was investigated in [7], where the authors computed some algebraic invariants of $\mathrm{Sym}_R(M)$ or their bounds in terms of special ideals of the ring R. The theory gives definitive results if R is the polynomial ring in m variables on a field K of any characteristic by using the Gröbner basis theory (in the following K always denotes a field). Here we would like to study an application of previous results essentially in two directions. We have many areas of applications and this is only the starting point of investigation via s-sequences. The first is to test the Eisenbud-Goto conjecture (EGC) for the symmetric algebra of a module M generated by an s-sequence. After we have given formulations in this case, we begin to work in this direction. For regular sequences of forms in the polynomial ring (which are strong s-sequences) we prove the (EGC). If M has linear syzygies on the polynomial ring $R = K[X_1, \dots, X_m]$, a nice construction of [12] leads to the jacobian dual module N of M. N is a finitely generated module on the ring $Q = K[Y_1, \dots, Y_n]$ and we have the isomorphism $\mathrm{Sym}_R(M) = \mathrm{Sym}_Q(N)$. Then it is interesting to ask, when M is generated by an s-sequence, if N is generated by an s-sequence and viceversa, and this is the second area. In this context it is possible to describe the Rees algebra of the module M as a quotient of the symmetric algebra by its torsion submodule. In particular, in section 1 we give the definition of s-sequence introduced in [7], we recall some known results and we formulate the Eisenbud-Goto conjecture for the symmetric algebra of a finitely generated graded module M generated by an s-sequence on a Noetherian ring in different ways. In particular, if $R = K[X_1, \dots, X_m]$ is the polynomial ring, we give it in terms of the annihilator ideals of the s-sequence. As an application, we verify (EGC) for a regular sequence of forms of R.
In section 2 we introduce the jacobian dual of a module M on a polynomial ring

$R = K[X_1, \dots, X_m]$. This module can be defined if the presentation matrix of M has linear entries in the variables X_i and its interest appears in many fields of commutative algebra. If this module N over the polynomial ring $K[Y_1, \dots, Y_n]$ is generated by a strong s-sequence, we obtain that the torsion submodule of $\mathrm{Sym}_R(M)$ coincides with the first annihilator ideal of the s-sequence generating N and it is an ideal of Q. The Rees algebra of M, as a quotient of $\mathrm{Sym}_R(M)$ by its torsion submodule, can be computed. As an example, we consider a monomial ideal with linear syzygies not generated by an s-sequence, whose jacobian dual is generated by an s-sequence. The idea is to address our interest to many computations in this direction.

2. Preliminaries

Let R be any Noetherian ring and M a finitely generated R-module with generators $f_1, \dots, f_n$. If we consider a presentation of M

$$R^m \xrightarrow{f} R^n \to M \to 0,$$

then f is represented by an $n \times m$ matrix (a_{ij}) with entries in R, $1 \le i \le n$, $1 \le j \le m$. The symmetric algebra of M on R, $\mathrm{Sym}_R(M) = \oplus_{i \ge 0} S_i(M)$, where, for each i, $S_i(M)$ is the component of degree i of $\mathrm{Sym}_R(M)$, has a presentation:

$$0 \to J \to \mathrm{Sym}_R(R^n) \to \mathrm{Sym}_R(M) \to 0$$

and $\mathrm{Sym}_R(R^n) \simeq R[Y_1, \dots, Y_n]$ is the polynomial ring on R in the variables Y_j, J is the relation ideal of $\mathrm{Sym}_R(M)$, $J = (g_1, \dots, g_m)$, with g_i form of degree 1 in the Y_j, $g_i = \sum_{j=1}^{n} a_{ij} Y_j$, for $i = 1, \dots, m$, then $\mathrm{Sym}_R(M) \simeq R[Y_1, \dots, Y_n]/J$.
The main problem is how to compute standard algebraic invariants of the graded algebra $\mathrm{Sym}_R(M)$ such as the dimension $\dim(\mathrm{Sym}_R(M))$, the multiplicity $\mathrm{e}(\mathrm{Sym}_R(M))$, the depth $\mathrm{depth}(\mathrm{Sym}_R(M))$ with respect to the graded maximal ideal $\mathrm{Sym}_R(M)^+ = \oplus_{i>0} S_i(M)$, the regularity $\mathrm{reg}(M)$, in terms of the corresponding invariants of special quotients of the ring R.
The first three invariants are classical. For the last invariant, we recall that $\mathrm{reg}(M)$ is the Castelnuovo-Mumford regularity of the graded module M. Its importance is briefly indicated in Eisenbud-Goto theorem which is an interesting description of regularity in terms of the graded Betti numbers of M ([2]).
They show that $\mathrm{reg}(M)$, when M is a graded finite R-module, where $R = K[X_1, \dots, X_m]$, measures the "complexity" of the minimal free resolution of M as an S-module. Therefore regularity plays an important role in many fields of commutative algebra.
More precisely, if we consider a graded minimal free resolution of M on S

$$0 \to F_\ell \to \cdots \to F_1 \to F_0 \to M \to 0,$$

if b_i is the maximum degree of the generators of the free module F_i, then

$$\mathrm{reg}(M) = \sup\{b_i - i, i \ge 0\}.$$

In other words, $\operatorname{reg}(M)$ is the smallest integer m such that for every j, the j-th syzygy module of M is generated in degree $\leq m + j$ (equivalently, $\operatorname{reg}(M) = \sup\{\beta_{i,i+j} \neq 0, \text{ for some } i\}$, where $\beta_{i,\ell}$ are the graded Betti numbers of M).
If R is not a polynomial ring, the regularity of M can be infinite. A nice area of investigation in commutative algebra is the study of graded homogeneous algebras A generated in the same degree such that $\operatorname{reg}_A(A/m^+) = 0$ as an A-module and m^+ is the maximal graded ideal of A.

A computation of the previous invariants can be obtained for a finitely generated R-module that is generated by an s-sequence $f_1, \dots, f_n$ in the sense of [7]. Consider the presentation of $\operatorname{Sym}_R(M)$

$$\operatorname{Sym}_R(M) = R[Y_1, \dots, Y_n]/J.$$

The ideal $\operatorname{Sym}_R(M)^+$ is generated by the residue classes of the Y_i that are called f_i^*, because the variables Y_i correspond to the generators of the module $M = Rf_1 + \cdots + Rf_n$ in the presentation of $\operatorname{Sym}_R(M)$. For every $i = 1, \dots, n$, we set $M_{i-1} = Rf_1 + \cdots + Rf_{i-1}$ and let $I_i = M_{i-1} :_R f_i$ be the colon ideal. We set $I_0 = (0)$ for convenience. Since $M_i/M_{i-1} \simeq R/I_i$, so I_i is the annihilator of the cyclic module R/I_i, I_i is called an annihilator ideal of the sequence $f_1, \dots, f_i$.

Consider the polynomial ring $R[Y_1, \dots, Y_n]$ and let $<$ be a monomial order on the monomials of $R[Y_1, \dots, Y_n]$ in the variables Y_i such that

$$Y_1 < Y_2 < \cdots < Y_n.$$

We call $<$ an admissible order.
With respect to this term order, if $f = \sum a_\alpha \underline{Y}^\alpha$, $\underline{Y}^\alpha = Y_1^{\alpha_1} \cdots Y_n^{\alpha_n}$, $\alpha \in \mathbb{N}^n$, we put $\operatorname{in}_< f = a_\alpha \underline{Y}^\alpha$, where $\underline{Y}^\alpha$ is the largest monomial in f such that $a_\alpha \neq 0$.
If we assign degree 1 to each variable Y_i and degree 0 to the elements of R, we have the following facts:

1) J is a graded ideal,

2) the natural epimorphism $S \to \operatorname{Sym}_R(M)$ is a graded homomorphism of graded algebras on R, S is a graded ring and $\operatorname{Sym}_R(M)$ is a graded ring.

DEFINITION 1. *The sequence $f_1, \dots, f_n$ is an s-sequence for M if*

$$(I_1Y_1, I_2Y_2, \dots, I_nY_n) = in_< J.$$

If $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n$, the sequence is a strong s-sequence.

EXAMPLE 1. Any d-sequence of elements $a_1, \dots, a_n$ in R is a strong s-sequence, with respect to the reverse lexicographic order on the Y_i, with $Y_1 < Y_2 < \cdots < Y_n$ ([7], Cor. 3.3).
As a consequence regular sequences, proper sequences are strong s-sequences, since they are d-sequences ([10]).
If $I = (a_1, \dots, a_n)$, we have $\operatorname{Sym}_R(I) = \mathcal{R}_R(I)$, the Rees algebra of I.

If $R = K[X_1, \dots, X_m]$ we can use the Gröbner basis theory and Buchberger's algorithm to compute $\mathrm{in}_< J$.
If $R = K[X_1, \dots, X_m]$, then $\mathrm{Sym}_R(M) = K[X_1, \dots, X_m, Y_1, \dots, Y_n]/J$. We can introduce a term order on

$$S = K[X_1, \dots, X_m, Y_1, \dots, Y_n]$$

such that $Y_1 < Y_2 < \dots < Y_n$ and $X_i < Y_i$ for any i.
For example $X_1 < X_2 < \dots < X_m < Y_1 < Y_2 < \dots < Y_n$ is such a term order.
If G is a Gröbner basis for $J \subset K[X_1, \dots, X_m, Y_1, \dots, Y_n]$, we have $\mathrm{in}_< J = (\mathrm{in}_< G) = (\mathrm{in}_< f,\ f \in J)$ and if the elements of G are linear in the $Y_i s$, it follows that $f_1, \dots, f_n$ is an s-sequence for M.

REMARK 1. If $R = K[X_1, \dots, X_m]$, from the theory of Gröbner basis, if $f_1, \dots, f_n$ is an s-sequence with respect to any admissible term order $<$, then $f_1, \dots, f_n$ is an s-sequence for another admissible term order, too.

THEOREM 1 ([7]). *Suppose R is a standard graded algebra, M is a graded R-module which is generated by the homogeneous s-sequence $f_1, \dots, f_n$, where all f_i have the same degree, $I_1, \dots, I_n$ are the annihilator ideals of the sequence $f_1, \dots, f_n$. Then:*

i) $\dim \mathrm{Sym}_R(M) = \max_{\substack{0 \le r \le n \\ 1 \le r_1 \le \dots \le r_n \le n}} \{\dim R/(I_{r_1} + \dots + I_{r_n}) + r\}$

ii) $e(\mathrm{Sym}_R(M)) = \sum_{\substack{0 \le r \le n \\ 1 \le r_1 \le \dots \le r_n \le n}} e(R/(I_{r_1} + \dots + I_{r_n}))$,
where $\dim R/(I_{r_1} + \dots + I_{r_n}) = d - r$, $d = \dim \mathrm{Sym}_R(M)$.

For a strong s-sequence we have:

$$d = \dim \mathrm{Sym}_R(M) = \max_{0 \le r \le n} \{\dim R/I_r + r\},$$

$$e(\mathrm{Sym}_R(M)) = \sum_{\substack{0 \le r \le n \\ \dim R/I_r = d - r}} e(R/I_r).$$

THEOREM 2 ([7]). *If $R = K[X_1, \dots, X_m]$ and M is generated by elements of the same degree, which are a strong s-sequence, then*

1) $\mathrm{reg}(\mathrm{Sym}_R(M)) \le \max\{\mathrm{reg}(R/I_i), i = 1, \dots, n\} + 1$,

2) $\mathrm{depth}(\mathrm{Sym}_R(M)) \ge \min\{\mathrm{depth}(R/I_i) + i, i = 0, \dots, n\}$.

The notion of s-sequence can be useful essentially:

1) to test some conjectures for graded modules M generated by s-sequences, "via" conjectures about annihilator ideals of M, in particular we are interested to test

the Eisenbud-Goto conjecture (EGC) for $\mathrm{Sym}_R(M)$, when M has generators of the same degree and the regularity is the ordinary regularity.

The (EGC), that involves all invariants of $\mathrm{Sym}_R(M)$, can be more easily verified if M is generated by an s-sequence.

2) to describe the Rees algebra of the R-module M, $R = K[X_1, \dots, X_m]$,

$$\mathcal{R}_R(M) = \mathrm{Sym}_R(M)/(\mathrm{Sym}_R(M))_0$$

and to test (EGC) in the case the matrix of the relations of M is linear in the variables $X_1, \dots, X_m$. In this situation in fact we have a nice construction that collects many cases of ideals and modules (in particular those ones with linear resolution): the jacobian dual module N.
If N is the Jacobian dual of M, then natural questions arise:

i) When the jacobian dual N of M is generated by an s-sequence?

ii) If it is the case, does $\mathrm{Sym}_Q(N)$ verify (EGC)?

3. Eisenbud-Goto conjecture

There are several conjectures to connect the measures of the complexity of an algebra. One of the most important is the following:

CONJECTURE 2. (**EGC**) If A is a standard graded domain on a field K then

$$\mathrm{reg}(A) \leq e(A) - \mathrm{codim}(A),$$

where $\mathrm{codim}(A) = \mathrm{emb}\,\mathrm{dim}(A) - \mathrm{dim}(A)$.

If A is Cohen-Macaulay, the conjecture is true and we have equality ([2]).
We will establish the (EGC) for symmetric algebras of finitely generated graded module M generated by s-sequences.
We consider different formulations of the conjecture.

1) (**EGC1**) Eisenbud-Goto conjecture for the symmetric algebra of a module M on a standard graded algebra R, generated on R by a strong s-sequence of elements of the same degree, and such that $\mathrm{Sym}_R(M)$ is a domain

$$\mathrm{reg}(\mathrm{Sym}_R(M)) \leq e(\mathrm{Sym}_R(M)) - \mathrm{codim}(\mathrm{Sym}_R(M))$$

2) (**EGC2**) If $R = K[X_1, \dots, X_m]$, M a graded R-module generated by a strong s-sequence of elements of the same degree, Eisenbud-Goto conjecture for the symmetric algebra of M in terms of the annihilator ideals of the strong s-sequence generating M is

$$\max\{\mathrm{reg}(R/I_i) : i = 1, \dots, n\} + 1 \leq \sum_{i=1}^{n} e(R/I_i) - (n+m) + \max_{0 \leq i \leq n}\{\mathrm{dim}(R/I_i) + i\}$$

3) (**EGC3**) Eisenbud-Goto conjecture for any annihilator prime ideal of a strong s-sequence of the same degree > 1, generating a graded R-module M, $R = K[X_1, \dots, X_m]$, or R standard graded algebra that is a domain

(**EGC**$_i$) $\operatorname{reg}(R/I_i) \leq e(R/I_i) - \operatorname{codim}(R/I_i)$, for $i = 1, \dots, n$, $\dim R/I_i = d - i$.
(**EGC'**$_i$) $\operatorname{reg}(R/I_i) \leq e(R/I_i) - m + \dim(R/I_i)$, $i = 1, \dots, n$, $\dim R/I_i = d - i$.
In 2) and 3) $d = \dim \operatorname{Sym}_R(M)$.

4) The same conjecture formulated for the Rees algebra $\mathcal{R}(M)$, when $\mathcal{R}(M) = \operatorname{Sym}_R(M)$, R a standard graded domain and M generated on R by a strong s-sequence of elements of the same degree, becomes:
(**EGC1'**) $\operatorname{reg}(\mathcal{R}(M)) \leq e(\mathcal{R}(M)) - \operatorname{codim}(\mathcal{R}(M))$.
If $R = K[X_1, \dots, X_m]$, M a graded finitely generated R-module generated by an s-sequence of elements of the same degree, Eisenbud-Goto conjecture of $\mathcal{R}(M) = \operatorname{Sym}_R(M)$ in terms of annihilator ideals of the strong s-sequence generating M (for example, M is an ideal of $R = K[X_1, \dots, X_m]$ generated by a d-sequence of elements of R):

(**EGC2'**) $\max\{\operatorname{reg}(R/I_i) : i = 1, \dots, n\} \leq \sum_{i=1}^{n} e(R/I_i) - n$
(**EGC3'**) $\operatorname{reg}(R/I_i) \leq e(R/I_i) - m + \dim(R/I_i)$

Some implications:
(**EGC2**) $\Rightarrow$ (**EGC1**)
If $R = K[X_1, \dots, X_m]$, by Theorem 1 and Theorem 2, we have:

$$\begin{aligned}\operatorname{reg}(\operatorname{Sym}_R(M)) &\leq \max\{\operatorname{reg} R/I_i : i = 1, \dots, n\} + 1 \\ &\leq \sum_{i=1}^{n} e(R/I_i) - (n+m) - \max_{0 \leq i \leq n}\{\dim(R/I_i + i)\} \\ &\leq e(\operatorname{Sym}_R(M)) - \operatorname{codim}(\operatorname{Sym}_R(M))\end{aligned}$$

(**EGC'**$_i$) $\Rightarrow$ (**EGC**$_i$), for any ideal I_i, $i = 1, \dots, n$.

$$\operatorname{reg}(R/I_i) \leq e(R/I_i) - m + \dim(R/I_i) \leq e(R/I_i) - \operatorname{codim}(R/I_i).$$

(**EGC'**$_i$) for every $i = 1, \dots, n \Rightarrow$ (**EGC2**)

Since (**EGC'**$_i$) is true for every i, $i = 1, \dots, n$, we have:

$$\begin{aligned}&\max\{\operatorname{reg} R/I_i : i = 1, \dots, n\} \\ &\quad \leq \sum_{i=1}^{n} e(R/I_i) - (n+m) - (n-2)m + \dim(R/I_s) + \sum_{\substack{i=1 \\ i \neq s}}^{n} \dim(R/I_i)\end{aligned}$$

$$\leq \sum_{i=1}^{n} e(R/I_i) - (n+m) + \sum_{\substack{i=1 \\ i\neq s}}^{n} (\dim(R/I_i) - m) + \dim R/I_s$$

$$\leq \sum_{i=1}^{n} e(R/I_i) - (n+m) + \dim R/I_s + s - 1$$

since $n \leq m$.

If s is the integer such that $\dim(R/I_s) + s = \max_{1\leq i\leq n}\{\dim(R/I_i) + i\}$, then we have

$$\max\{\operatorname{reg} R/I_i : i = 1, \dots, n\}$$

$$\leq \sum_{i=1}^{n} e(R/I_i) - (n+m) - (n-2)m + \dim(R/I_s) + \sum_{\substack{i=1 \\ i\neq s}}^{n} \dim(R/I_i)$$

$$\leq \sum_{i=1}^{n} e(R/I_i) - (n+m) + \sum_{\substack{i=1 \\ i\neq s}}^{n} (\dim(R/I_i) - m) + \dim R/I_s$$

$$\leq \sum_{i=1}^{n} e(R/I_i) - (n+m) + \dim R/I_s + s - 1$$

since $\dim R/I_i - m \leq 0$, $i = 1, \dots, n$.

In order to state the (EGC) for the jacobian dual module N of M, we need some facts on N. As a consequence we will give the formulation of (EGC) for N in the next section.

EXAMPLE 2 (Regular sequences). Let R be a Noetherian ring and let $f_1, \dots, f_n$ be a regular sequence of elements of R. Then $f_1, \dots, f_n$ is a strong s-sequence with respect to any reverse lexicographic order on the variables $Y_1, \dots, Y_n$ such that $Y_1 < Y_2 < \dots < Y_n$ with annihilator ideals $I_1 = (0)$, $I_2 = (f_1), \dots, I_n = (f_1, \dots, f_{n-1})$ and $\operatorname{in}_< J = ((f_1)Y_2, \dots, (f_1, \dots, f_{n-1})Y_n)$.
In fact a regular sequence is a d-sequence, hence the assertion follows.

THEOREM 3. *Let R be a Noetherian ring and let $f_1, \dots, f_n$ be a regular sequence. Let $I = (f_1, \dots, f_n)$ and $\operatorname{Sym}_R(I) = R[Y_1, \dots, Y_n]/J$. Then we have:*

1) *J is minimally generated by the elements $g_{ij} = f_iY_j - f_jY_i$, $1 \leq i < j \leq n$.*
2) *If $R = K[X_1, \dots, X_m]$, the set $\{g_{ij}, 1 \leq i < j \leq n\}$ is a Gröbner basis with respect to any reverse lexicographic order on the Y_j and such that $Y_1 < \dots < Y_n$.*

Proof. 1) Put $g_{ij} = f_i Y_j - f_j Y_i$, $i < j$ and suppose that $\{g_{ij}\}_{1 \le i < j \le n}$ is not a minimal system of generators of J. Then

$$f_i Y_j - f_j Y_i = \sum_{(\rho,k) \neq (i,j)} h_{\rho k} g_{\rho k},$$

fore some i, j, $1 \le i < j \le n$.

Hence

$$f_i = \sum_{\rho > j} h_{j\rho} f_\rho + \sum_{\rho < j} h_{\rho j} f_\rho$$

and this is a contradiction.

2) We have to consider the S-couples:

i) $S(g_{ij}, g_{ik})$, $j \neq k$

ii) $S(g_{ij}, g_{kj})$, $i \neq k$

iii) $S(g_{ij}, g_{k\rho})$, $i \neq k$, $j \neq \rho$

For i), $S(g_{ij}, g_{ik}) = Y_i(-f_k Y_j + f_j Y_k)$.
For ii), $S(g_{ij}, g_{kj}) = f_j(-f_i Y_k + f_k Y_i)$.
For iii), $S(g_{ij}, g_{k\rho}) = -f_k Y_\rho g_{ij} - f_j Y_i g_{k\rho}$.
So, by Buchberger's criterion we get $\text{in}_<(J) = (\text{in}_< g_{ij})$. □

Now, let $R = K[X_1, \ldots, X_m]$ be a polynomial ring and let I be an ideal of R generated by an R-sequence $f_1, \ldots, f_n$ of homogeneous elements.

Case I: $f_1, \ldots, f_n$ have the same degree a.

PROPOSITION 1. $\text{reg}(\text{Sym}_R(I)) \le (n-1)(a-1) + 1$.

Proof. Since $f_1, \ldots, f_n$ is a strong s-sequence, then we can apply the formula

$$\text{reg}(\text{Sym}_R(I)) \le \max\{\text{reg}(R/I_i), 1 \le i \le n\} + 1,$$

where $I_0 = I_1 = (0)$, $I_2 = (f_1), \ldots, I_n = (f_1, \ldots, f_{n-1})$. The result follows by the Koszul resolution for the annihilator ideals I_i, $2 \le i \le n$. □

PROPOSITION 2. *Let* $I = (f_1, \ldots, f_n) \subset R = K[X_1, \ldots, X_m]$ *be generated by a regular sequence of forms of the same degree* a. *Then* (EGC2') *is true.*

Proof. We have to prove that

$$\max\{\text{reg}(R/I_i), 1 \le i \le n\} \le \sum_{1 \le i \le n} e(R/I_i) - n$$

that is

$$(n-1)a-(n-1)\le\sum_{i=1}^{n}a^{i-1}-n$$
$$(n-1)a\le a+a^2+\cdots+a^{n-1}.$$

The assertion follows. □

PROPOSITION 3. *Let $I=(f_1,\dots,f_n)\subset R=K[X_1,\dots,X_m]$ be generated by a regular sequence of forms of the same degree $a\ge 2$. Then* (EGC'$_i$) *is true for i, $1\le i\le n$, such that the annihilator ideal I_i is a prime ideal.*

Proof. We have to prove that $\mathrm{reg}(R/I_i)\le e(R/I_i)-i+1$, i.e. $(i-1)a-(i-1)\le a^{i-1}-i+1$ and *(EGC'$_i$)* is true. □

REMARK 2.

1) For $n>1$, in Proposition 1 we have in fact equality. The result can follow from the resolution of the algebra $\mathrm{Sym}_R(I)=\mathcal{R}(I)$, by employing the Eagon-Northcott complex. Let $S=K[X_1,\dots,X_m;Y_1,\dots,Y_n]$ and let F and G be finitely generated free graded S-modules of rank 2 and m respectively. Consider a graded homomorphism of degree zero $g:G\to F$, g represented by the matrix

$$\begin{pmatrix} f_1 & \dots & f_n\\ Y_1 & \dots & Y_n\end{pmatrix}.$$

We can write $g:S(-a)^n\to S^2$ and we consider the Koszul complex arising from g.

$$K(g):0\to\bigwedge^{n}G\otimes S(F)(-n)\to\bigwedge^{n-1}G\otimes S(F)(-n+1)\to\cdots$$
$$\cdots\to G\otimes S(F)(-1)\to S(F)\to 0,$$

where $S(F)=\mathrm{Sym}_S(F)=S[\underline{T}]=S[T_1,T_2]$ and the differential

$$\delta:\bigwedge^{i}G\otimes S(F)(-i)\to\bigwedge^{i-1}G\otimes S(F)(-i+1)$$

is defined by

$$\delta(t_1\wedge t_2\wedge\ldots\wedge t_i\otimes f(\underline{T}))=\sum_{j=1}^{i}(-1)^j g(t_j)t_1\wedge t_2\wedge\ldots\wedge\widehat{t_j}\wedge\ldots\wedge t_i\otimes f(\underline{T}).$$

Since $ht(J)=n-1$, $\dim\mathcal{R}(I)=m+1=n+m-ht(J)$ and J is perfect.
The complex

$$D_0(g):0\to\Big(\bigwedge^{0}G\otimes S_{n-2}(F)\Big)^*\to\big(G\otimes S_{n-3}(F)(-1)\big)^*\to\cdots$$
$$\to\Big(\bigwedge^{n-3}G\otimes S_1(F)(-n+3)\Big)^*\to\Big(\bigwedge^{n-2}G\otimes S_0(F)(-n+2)\Big)^*\to S\to 0,$$

resolves S/J ([7], (2.16)).
Since any generator of J has degree $a+1$, the shift in the place 1 is $-a$, that is a is the shift of the generators of the module $(\wedge^{n-2}G \otimes S_0(F))^*$. Finally, the complex above is the dual of a Koszul complex. Hence:

$$\operatorname{reg}(S/J) = (n-1)a - (n-1) + 1.$$

2) If $f_1, \dots, f_n$ is a regular sequence of n forms of degree $a \geq 2$, $\operatorname{Sym}_R(I) = \mathcal{R}(I)$, $I = (f_1, \dots, f_n)$ and $\operatorname{reg}(R) \leq \operatorname{reg}\mathcal{R}(I) \leq \max\{\operatorname{reg} R + 1, \operatorname{reg} R + n(a-1)\}$ ([5], Corollary 2.6).
For $m = 1$, $\operatorname{reg} R = 0$, $\operatorname{reg}\mathcal{R}(I) = 0$. The assertion follows and (EGC) is true.
For $m > 1$, $\operatorname{reg} R = 0$ and $0 \leq \operatorname{reg}\mathcal{R}(I) \leq \max\{1, n(a-1)\} = n(a-1)$.
For $n \geq 3$, what is needed is $n(a-1) \leq \sum_{i=1}^{n} a^{i-1} - n + 1$, $na \leq \sum_{i=1}^{n} a^{i-1} + 1$. If we write $na = a + (a+a) + (n-3)a$, we have $na \leq a + a^2 + \sum_{i=4}^{n} a^{i-1}$ and (EGC) is true.
Case II: $f_1, \dots, f_n$ are forms of different degrees $d_1, \dots, d_n$, $d_1 \leq d_2 \leq \dots \leq d_n$.
Consider R as a graded ring by assigning to each variable X_i degree 0. Then $S = R[Y_1, \dots, Y_n]$ is a graded ring if we assign to each variable Y_i degree 1. Let $<$ be a monomial order on the monomials in $Y_1, \dots, Y_n$ such that $Y_1 < Y_2 < \cdots < Y_n$. Since $f_1, \dots, f_n$ is a regular sequence, it is a strong s-sequence and $\operatorname{in}_< J = (I_1Y_1, \dots, I_nY_n)$, $I_i = (f_1, \dots, f_{i-1})$ for $i = 1, \dots, n$.
As a consequence

$$\begin{aligned}\operatorname{reg}\mathcal{R}(I) &= \operatorname{reg} R[Y_1, \dots, Y_n]/J \leq \operatorname{reg} R[Y_1, \dots, Y_n]/\operatorname{in}_< J = \\ &= \operatorname{reg} R[Y_1, \dots, Y_n]/(I_1Y_1, \dots, I_nY_n) \leq \max\{\operatorname{reg}(R/I_i), 1 \leq i \leq n\} + 1.\end{aligned}$$

But the last regularity is 0, since all matrices have entries of degree 0 in the minimal graded resolution of any annihilator ideal I_i. We need $\mathcal{R}(I)$ is a standard graded algebra for the formulation of (EGC).

4. Jacobian dual

Let $R = K[X_1, \dots, X_s]$ be a polynomial ring and let E be a finitely generated R-module with presentation :

$$R^m \xrightarrow{\phi} R^n \to E \to 0$$

where the entries of the $n \times m$ matrix $A = (a_{ij})$ that represents ϕ are homogeneous linear forms.
The equations of the symmetric algebra of E, $\operatorname{Sym}_R(E) = S(E)$ are

$$f_j = \sum_{i=1}^{n} a_{ij} Y_i \qquad j = 1, \dots, m.$$

There is a naive duality for $S(E)$, obtained from rewriting the equations f_j in the X_i's variables.

$$f_j = \sum_{i=1}^{n} a_{ij} Y_i = \sum_{i=1}^{s} b_{ij} X_i \qquad j = 1, \dots, m$$

and $B = (b_{ij})$ is an $s \times m$ matrix of homogeneous linear forms in the Y_i's variables.

We have:

$$A^t \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = B^t \begin{pmatrix} X_1 \\ \vdots \\ X_s \end{pmatrix} = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}.$$

Now we put $Q = K[Y_1, \dots, Y_n]$ and consider the cokernel N of the map

$$Q^m \overset{\Psi}{\to} Q^s \to N \to 0,$$

where Ψ is the map represented by B.
N defines the Jacobian dual module of E ([12], [14]).

EXAMPLE 3. We can write the relation $f = (X_1 - 2X_2)Y_1 + (X_1 + X_2)Y_2 + X_3Y_3$ as $f = (Y_1 + Y_2)X_1 + (-2Y_1 + Y_2)X_2 + Y_3X_3$.

REMARK 3. $\mathrm{Sym}_R(E) \cong \mathrm{Sym}_Q(N)$.

EXAMPLE 4. Suppose that $A \cong B$, in the sense that the two matrices A and B have the same elements under the substitution $X_i \longrightarrow Y_i$, $n = s$. Then $R \cong Q$ and $E \cong N$.

There is a nice situation that will be interesting in the following.
Let $R = K[X_1, \dots, X_n]$, $I = m_+ = (X_1, \dots, X_n)$, $\mathrm{Sym}_R(m_+) = \mathcal{R}(m_+) = K[X_1, \dots, X_n; Y_1, \dots, Y_n]/J$, where J is generated by the binomials $X_iY_j - X_jY_i$, $1 \le i < j \le n$, the 2×2-minors of the $2 \times n$ matrix

$$\begin{pmatrix} X_1 & X_2 & \dots & X_n \\ Y_1 & Y_2 & \dots & Y_n \end{pmatrix}.$$

The binomials in the X_i's give the dual matrix B of the relation matrix A of m_+ under the substitution $X_i \to Y_j$, $i, j = 1, \dots, n$.
Notice that the set of binomials is an universal Gröbner basis for the ideal J and this implies m_+ is generated by an s-sequence linear in the Y_i's and linear in the X_i's, too.

Another example is given by $m_i = (X_1, \dots, X_i)$, $i < n$.

$$\mathrm{Sym}_R(m_i) = K[X_1, \dots, X_n; Y_1, \dots, Y_i]/J_i$$

where J_i is generated by the binomials $X_\ell Y_s - X_s Y_\ell$, $1 \le \ell < s \le i$, the 2×2 minors of the matrix

$$\begin{pmatrix} X_1 & X_2 & \dots & X_i \\ Y_1 & Y_2 & \dots & Y_i \end{pmatrix}.$$

Put $S = K[Y_1, \ldots, Y_i]$, $\mathrm{Sym}_R(m_i) = \mathrm{Sym}_S(N)$ and $X_1, \ldots, X_i$ is an s-sequence (it is a regular sequence) for m_i and the sequence of 1-forms $x_1^*, \ldots, x_i^*$ is an s-sequence for the jacobian dual N of m_i, where $x_1, \ldots, x_n$ are the residue classes of $X_1, \ldots, X_n$ in $S[X_1, \ldots, X_n]/J_i$.

PROPOSITION 4. *Let $R = K[X_1, \ldots, X_m]$ be a polynomial ring and M a graded R-module generated by forms $f_1, \ldots, f_n$ of the same degree. Suppose that the relation ideal J of $\mathrm{Sym}_R(M)$ is generated by forms that are linear in both sets of variables $\underline{X}$ and $\underline{Y}$, and let N be the jacobian dual of M generated by $x_1, \ldots, x_m$, where $x_1^*, \ldots, x_m^*$ are the images of the elements $X_1, \ldots, X_m$ in the ring $K[X_1, \ldots, X_m; Y_1, \ldots, Y_n]/J$.*
Suppose J has a Gröbner basis linear in the $\underline{X}$ and $\underline{Y}$ variables with respect to the reverse lexicographic order on all variables and to the two orders of variables $X_m > \ldots > X_1 > Y_n > \ldots > Y_1$ and $Y_n > \ldots > Y_1 > X_m > \ldots > X_1$.
Then M is generated by an s-sequence if and only if N is generated by an s-sequence.

Proof. It is a consequence of the previous facts. □

REMARK 4. The strong case concerns J with a universal Gröbner basis that is linear in the $\underline{X}$ and $\underline{Y}$ variables with respect to any permutation of variables.

REMARK 5. If we know the Gröbner basis of J that is linear in the variables $\underline{X}$ and $\underline{Y}$, with respect to the reverse lexicographic order and to the two orders of variables $X_m > \ldots > X_1 > Y_n > \ldots > Y_1$ and $Y_n > \ldots > Y_1 > X_m > \ldots > X_1$, then we can write the annihilator ideals of the sequences $f_1, \ldots, f_n$ and $x_1, \ldots, x_m$ by using lemma 3.3 of [9].
The theorem gives the annihilator ideals for the s-sequence generating the jacobian dual N of M, but the proof can be repeated to have the annihilator ideals of M.

EXAMPLE 5. Let $I = (X^2, Y^2, XY)$ that is generated by an s-sequence ([7], Examples 1.5(1)). The jacobian dual N of I is generated by an s-sequence, too, but the relation ideal J has a Gröbner basis linear in the X_i's, but not linear in the Y_i's ([7]).

Now consider $\mathrm{Sym}_R(M) = R[Y_1, \ldots, Y_n]/J \cong Q[X_1, \ldots, X_m]/J = \mathrm{Sym}_Q(N)$. Let $x_1^*, \ldots, x_m^*$ be the images of $X_1, \ldots, X_m$ mod J that we can consider as the generators of N (we denote by $x_1, \ldots, x_m$ the generators of N).

We recall some propositions:

PROPOSITION 5. *Let $I \subset R$ be an ideal generated by $f_1, \ldots, f_n$. Then the following conditions are equivalent:*

1) $f_1, \ldots, f_n$ *is a d-sequence;*

2) $(0 : f_1) \cap I = 0$ *and* $f_2, \ldots, f_n$ *is a d-sequence in* $R/(f_1)$.

Proof. [7], Lemma 3.1. □

PROPOSITION 6. *Let M be an R-module generated by $f_1, \dots, f_n$. Then the following conditions are equivalent:*

1) *$f_1, \dots, f_n$ is a strong s-sequence with respect to the lexicographic order induced by $Y_n > Y_{n-1} > \dots > Y_1$;*
2) *$f_1^*, \dots, f_n^*$ is a d-sequence in $\mathrm{Sym}_R(M)$.*

Proof. [7], Theorem 3.2. □

PROPOSITION 7. *Let M be a finitely generated R-module, and let R be a domain. Then $\mathrm{Sym}_R(M)$ is a domain if and only if $(\mathrm{Sym}_R(M))_0 = 0$, where $(\mathrm{Sym}_R(M))_0 \subseteq \mathrm{Sym}_R(M)$ is the torsion submodule of M ([13]).*

THEOREM 4. *Suppose N is generated by a strong s-sequence $x_1, \dots, x_m$. Then we have*

1. *$x_1^*, \dots, x_m^*$ is a d-sequence in $\mathrm{Sym}_Q(N)$;*
2. *the ideal $(0 : x_1^*)$ is generated by elements of Q;*
3. *if $(0 : x_1^*)$ is a prime ideal then*

$$\mathrm{Sym}_Q(N)/(0 : x_1^*) \cong \mathcal{R}(M).$$

Proof. 1) If N is generated by a strong s-sequence, then $x_1^*, \dots, x_m^*$ is a d-sequence in $\mathrm{Sym}_R(N)$ (by Proposition 6).
Then $(0 : x_1^*) \cap (x_1^*, \dots, x_m^*) = (0)$ and $(0 : x_1^*)$ is generated by polynomials in $Y_1, \dots, Y_n$ and we have 2).
3) Suppose $(0 : x_1^*)$ a prime ideal of $Q = K[Y_1, \dots, Y_n]$. So $\mathrm{Sym}_Q(N)/(0 : x_1^*) \cong \mathrm{Sym}_R(M)/(0 : x_1^*)$ is a domain, then $(0 : x_1^*) \subseteq (\mathrm{Sym}_R(M))_0$, where $(\mathrm{Sym}_R(M))_0$ is the torsion submodule of $\mathrm{Sym}_R(M)$. Since R is a domain, $(0 : x_1^*) = (\mathrm{Sym}_R(M))_0$ (by Proposition 7), then $\mathrm{Sym}_R(M)/(0 : x_1^*) \cong \mathcal{R}(M)$, the Rees algebra of M ([3]). □

EXAMPLE 6.

$$M = I = (X_1^2, X_2^2, X_1X_2) \subset R = K[X_1, X_2], \;\; X_2 > X_1$$

$$R^2 \xrightarrow{\varphi} R^3 \to I \to 0$$

$$A = (a_{ij}) = \begin{pmatrix} X_2 & 0 \\ 0 & X_1 \\ -X_1 & -X_2 \end{pmatrix}$$

$J = (f_1, f_2)$, where

$$f_1 = X_2Y_1 - X_1Y_3 = Y_1X_2 - Y_3X_1$$

$$f_2 = X_1Y_2 - X_2Y_3 = Y_3X_2 - Y_2X_1$$

The sequence X_1^2, X_2^2, X_1X_2 is a strong s-sequence for the ideal I ([7], Ex. 1.5(1)). Consider

$$B = (b_{ij}) = \begin{pmatrix} -Y_3 & Y_2 \\ Y_1 & -Y_3 \end{pmatrix}.$$

If $S = K[Y_1, Y_2, Y_3]$, the jacobian dual module N of I is

$$0 \longrightarrow S^2 \xrightarrow{\psi} S^2 \longrightarrow N \longrightarrow 0$$

$S(f_1, f_2) = Y_3 f_1 + Y_1 f_2 = (-Y_3^2 + Y_1Y_2)X_1 = f_3$.
Then a Gröbner basis w.r.t. $X_2 > X_1 > Y_3 > Y_2 > Y_1$ is $\{f_1, f_2, f_3\}$,

$$\text{in}_< J = ((Y_1, Y_3)X_2, (Y_3^2 - Y_1Y_2)X_1).$$

$I_0^* = (0)$, $I_1^* = (Y_3^2 - Y_1Y_2)$, $I_2^* = (Y_1, Y_3)$, and since $I_1^* \subset I_2^*$, x_1, x_2 is a strong s-sequence for N.
From $f_3 = (-Y_3^2 + Y_1Y_2)X_1$, we have: $(0 : x_1^*) = (Y_1Y_2 - Y_3^2)$. In order to prove $(Y_3^2 - Y_1Y_2)$ is a prime ideal in $\text{Sym}_Q(N)$ we remark that $J' = (f_1, f_2, Y_3^2 - Y_1Y_2)$ is a prime ideal in $\text{Sym}_Q(N)$ if and only if $(Y_1Y_2 - Y_3^2)$ is a prime ideal in $\text{Sym}_Q(N)$. But J' is the ideal generated by the 2×2-minors of the generic matrix

$$\begin{pmatrix} X_1 & Y_3 & Y_1 \\ X_2 & Y_2 & Y_3 \end{pmatrix}.$$

Then the assertion follows and $(Y_1Y_2 - Y_3^2)$ is a prime ideal in $\text{Sym}_Q(N)$ and

$$\begin{aligned} \text{Sym}_Q(N)/(0 : x_1^*) &\cong \text{Sym}_R((X_1^2, X_2^2, X_1X_2))/(0 : x_1^*) \cong \mathcal{R}(I) \cong \\ &\cong R[Y_1, Y_2, Y_3]/(-Y_3X_1 + Y_1X_2, -Y_2X_1 + Y_3X_2, Y_1Y_2 - {Y_3}^2). \end{aligned}$$

EXAMPLE 7 (Monomial square-free matroidal ideals). Now we follow the notations used in [11, page 130].
Let I be a monomial ideal of $K[x_1, \dots, x_n]$ with the minimal set of generators $G(I) = \{x^{J_1}, \dots, x^{J_t}\}$, where $x^J = x_1^{j_1} \cdots x_n^{j_n}$, $J = (j_1, \dots, j_n)$ and $i = (0, 0, \dots, 1, \dots, 0)$. We set $|J| = j_1 + \cdots + j_n$.
We can associate the vector $\sum_{i=1}^t a_i \otimes x^{J_i}$ to a syzygy of I $\sum_{i=1}^t a_i x^{J_i}$, where $\otimes$ means $\otimes_K$.
For any monomial order on $K[x_1, \dots, x_n]$, we will say that

$$x_i \otimes x^J < x_k \otimes x^k \text{ if } x_i x^J < x_k x^k.$$

If $u \in G(I)$, we put $\nu_i(u) = j_i$, if $x_i^{j_i}$ appears in the monomial u.
Now, let I be a monomial ideal for which all generators have the same degree. I is matroidal if it satisfies the following exchange property ([24]):
For all $u, v \in G(I)$ and all i with $\nu_i(u) > \nu_i(v)$, there exists an integer j with $\nu_j(v) > \nu_j(u)$, such that $x_j(u/x_i) \in G(I)$.

THEOREM 5. *Let I be a matroidal square-free ideal with generators $x^{J_1}, \dots, x^{J_N}$ of the same degree. Then a minimal set of generators for the first syzygies of I has the form*

$$x_j \otimes x^{J_i} - x_t \otimes x^{J_\ell}, \qquad j + J_i = t + J_\ell, \qquad x^{J_i}, x^{J_\ell} \in G(I)$$

where $j < t$, t integer such that if $J_i = (a_1, \dots, a_n)$, $a_k = b_k$, $k = t+1, \dots, n$ and such that $b_j > a_j$, for some $x^{J_k} \in G(I)$, $J_k = (b_1, \dots, b_n)$.

Proof. In the reverse lexicographic order we can suppose that $x^{J_1} > x^{J_2} > \cdots > x^{J_N}$. Let $x^{J_i} < x^{J_k}$. Then there exists an integer t such that $a_m = b_m$ for $m = t+1, \dots, n$, $J_i = (a_1, \dots, a_n)$, $J_k = (b_1, \dots, b_n)$ and $a_t > b_t$. Hence there exists an integer j with $b_j > a_j$ such that $u' = x_j(x^{J_i}/x_t) \in G(I)$. Thus there is a syzygy of the form $x_j \otimes x^{J_i} - x_t \otimes x^{J_\ell}$, $x^{J_\ell} \in G(I)$. □

EXAMPLE 8.

$$I = (x_1, x_2)(x_3, x_4) = (x_1x_3, x_1x_4, x_2x_3, x_2x_4)$$

In the reverse lexicographic order and for $x_4 > x_3 > x_2 > x_1$

$$x_4x_2 > x_3x_2 > x_4x_1 > x_3x_1.$$

We consider the mapping $Y_1 \to x^{J_4} = x_3x_1$, $Y_2 \to x^{J_3} = x_4x_1$, $Y_3 \to x^{J_2} = x_3x_2$, $Y_4 \to x^{J_1} = x_4x_2$.
The syzygies are:

$$\begin{aligned}
x_1 \otimes x^{J_2} - x_2 \otimes x^{J_4} &\longrightarrow f_1 = x_2Y_1 - x_1Y_3;\\
x_1 \otimes x^{J_1} - x_2 \otimes x^{J_3} &\longrightarrow f_2 = x_2Y_2 - x_1Y_4;\\
x_3 \otimes x^{J_3} - x_4 \otimes x^{J_4} &\longrightarrow f_3 = x_4Y_1 - x_3Y_2;\\
x_3 \otimes x^{J_1} - x_4 \otimes x^{J_2} &\longrightarrow f_4 = x_4Y_3 - x_3Y_4.
\end{aligned}$$

The relations matrix of I is

$$\begin{pmatrix} x_2 & 0 & x_4 & 0 \\ 0 & x_2 & -x_3 & 0 \\ -x_1 & 0 & 0 & x_4 \\ 0 & -x_1 & 0 & -x_3 \end{pmatrix}$$

and the dual matrix is

$$\begin{pmatrix} -Y_3 & -Y_4 & 0 & 0 \\ Y_1 & Y_2 & 0 & 0 \\ 0 & 0 & -Y_2 & -Y_4 \\ 0 & 0 & Y_1 & Y_3 \end{pmatrix}$$

Consider the order $x_4 > \cdots > x_1 > Y_4 > \cdots > Y_1$, J has a Gröbner basis linear in the x_i variables. In fact $J = (f_1, f_2, f_3, f_4)$ and a Gröbner basis of J is

$G = \{f_1, f_2, f_3, f_4, f_5, f_6\}$ with $f_5 = x_4(Y_2Y_3 - Y_1Y_4)$, $f_6 = x_2(Y_2Y_3 - Y_1Y_4)$, $\mathrm{in}_< J = ((Y_3, Y_4)x_1, (Y_3Y_2)x_2, (Y_2, Y_4)x_3, (Y_3Y_2)x_4)$, $I_1^* = (Y_3, Y_4)$, $I_2^* = (Y_3Y_2)$, $I_3^* = (Y_2, Y_4)$, $I_4^* = (Y_3Y_2)$.
The jacobian dual N of I is generated by an s-sequence $x_1^*, \ldots, x_4^*$ that is not a strong s-sequence.
The torsion submodule of $\mathrm{Sym}_R(I)$ can be read in the dual matrix:

$$0 : x_1^* = \left(\begin{vmatrix} -Y_1 & Y_2 \\ Y_3 & Y_4 \end{vmatrix} \right) = (Y_1Y_4 - Y_3Y_2).$$

REMARK 6. The ideal I is not generated by an s-sequence. The Gröbner basis of J is not linear in the variables $Y_1, \ldots, Y_4$ for any admissible order such that $Y_4 > Y_3 > Y_2 > Y_1$. Then we are forced in this case to study the invariants of $\mathrm{Sym}_Q(N) \cong \mathrm{Sym}_R(M)$ "via" the jacobian dual.
The computation of the annihilator ideals of the s-sequence generating N can be done by the lemma 3.2 of [9]. Moreover this lemma can be used to compute the annhilator ideals of a generating s-sequence of M, changing the variables x_i's with the Y_j's and for a term order on all variables x_i's and Y_j's, that is admissible for x_i and for Y_j, for example $x_n > x_{n-1} > \ldots > x_1 > Y_n > Y_{n-1} > \ldots > Y_1$ or $Y_n > Y_{n-1} > \ldots > Y_1 > x_n > x_{n-1} > \ldots > x_1$.

In general, it is possible that M is not generated by an s-sequence and N is generated by an s-sequence. If this is the case (Ex. 8), we can obtain the Rees algebra of M by the quotient $\mathrm{Sym}_R(M)/I_1{}^*$, $I_1{}^* = 0 : x_1{}^* = (\mathrm{Sym}_R(M))_0$, where $I_1{}^*$ is a prime ideal. Then (EGC) can be true for $\mathcal{R}(M)$ that is a domain,

$$(\mathrm{EGC1}^*) \quad \mathrm{reg}(\mathcal{R}(M)) \leq e(\mathcal{R}(M)) - \mathrm{codim}(\mathcal{R}(M)).$$

Moreover we have, if $R = K[X_1, \ldots, X_m]$, M a graded finitely generated R-module, N the jacobian dual on $Q = K[Y_1, \ldots, Y_n]$, generated by an s-sequence of elements of Q of the same degree, Eisenbud-Goto conjecture for the symmetric algebra $\mathrm{Sym}_Q(N)$, in terms of the annihilator ideals $I_1^*, \ldots, I_m^*$ of Q.
$(\mathrm{EGC2}^*)$ $\max\{\mathrm{reg}(Q/I_i^*) : i = 1, \ldots, m\} + 1 \leq \sum_{i=1}^m e(Q/I_i^*) - (n+m) + \max_{0 \leq i \leq m}\{\dim(Q/I_i^*) + i\}$
(EGC_i^*) $\mathrm{reg}(Q/I_i^*) \leq e(Q/I_i^*) - \mathrm{codim}(Q/I_i^*)$, for $i = 1, \ldots, m$.
$(\mathrm{EGC}_i^{*\prime})$ $\mathrm{reg}(Q/I_i^*) \leq e(Q/I_i^*) - n + \dim(Q/I_i^*)$, for $i = 1, \ldots, m$.

EXAMPLE 9. $I = (X^2, Y^2, XY)$, $\mathrm{Sym}_R(I)$ verifies the inequality of (EGC2), $I_0 = I_1 = (0)$, $I_2 = (X^2)$, $I_3 = (X, Y)$.

$$\mathrm{reg}(R/I_2) = 1, \ \mathrm{reg}(R/I_3) = 0, \ e(R/I_2) = 2, \ e(R/I_3) = 1$$

then:

$$\max\{\mathrm{reg}(R/I_i), i = 1, 2, 3\} < \sum_{i=1}^{3} e(R/I_i) - 2$$

is true. For the jacobian dual: $I_1^* = (Y_3^2 - Y_1Y_2)$, $I_2^* = (Y_1, Y_3)$

$$\mathrm{reg}(Q/I_1^*) = 1,\ \mathrm{reg}(Q/I_2^*) = 0,\ e(Q/I_1^*) = 2,\ e(Q/I_2^*) = 1$$

then the inequality:

$$\max\{\mathrm{reg}(Q/I_i^*), i = 1, 2\} < \sum_{i=0}^{2} e(Q/I_i^*) - 2 + 1$$

is verified.

REMARK 7. Let M be a graded module on $R = K[X_1, \ldots, X_m]$, let N be the jacobian dual of M. Suppose that $x_1, \ldots, x_m$ is a strong s-sequence for N and $(0 : x_1^*)$ is a prime ideal of Q. Then $x_1^*, \ldots, x_m^*$ is a d-sequence in $\mathcal{R}(M) = \mathrm{Sym}_Q(N)/(0 : x_1^*)$.

Our attention actually applies to prove the conjecture via the annihilator ideals of the jacobian dual of large classes of monomial ideals with linear resolution.

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Gaetana RESTUCCIA, Dipartimento di Matematica, Università di Messina, Contrada Papardo, salita Sperone 31, 98166 Messina, ITALIA
e-mail: grest@dipmat.unime.it

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