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PROPERTIES OF SOME ARTINIAN GORENSTEIN RINGS

Dedicated to Paolo Valabrega on the occasion of his 60 th birthday

Abstract. In this paper, we fix a Cohen-Macaulay ideal $I \subset R = K[x_1, \dots, x_n]$ of dimension 1 and we construct a parameter space $\mathcal{G}(I, r)$ for the family of Artinian Gorenstein ideals J with $\mathrm{reg}(J) = r$ for which I is a tight annihilating ideal. We compute the dimension of $\mathcal{G}(I, r)$ and we prove that if $r \gg 0$ (see section 5 for a precise bound) then all $J \in \mathcal{G}(I, r)$ are a basic double G-link of G on I where G is a suitable Artinian Gorenstein ideal containing I.

1. Introduction

In recent years many authors focus their attention to study Gorenstein ideals and the role that they play in various of the applications of Commutative Algebra such as Algebraic Geometry, Algebraic Combinatorics and Number Theory.

It is well known that in codimension 2 Gorenstein ideals and complete intersection ideals coincide; and in codimension 3 Gorenstein ideals are completely described from an algebraic point of view by the beautiful structure theorem of Buchsbaum and Eisenbud which allows one to associate an alternating matrix of odd order to each Gorenstein ideal of codimension 3. Unfortunately the geometric appearance of Gorenstein ideals $I \subset K[x_1, \dots, x_n]$ is less understood. For this reason, many authors have given geometric constructions of some particular families (cf. [2], [7], [9], [10] among others). In this paper, we construct a parameter space for the family of Artinian Gorenstein ideals J with fixed regularity and fixed tight annihilating ideal I and we prove that if the regularity is big enough then all these Gorenstein ideals J are obtained by basic double G-link of G on I where G is a suitable Artinian Gorenstein ideal containing I.

Next we outline the structure of the paper. Section 2 provides a brief glossary of definitions. In section 3 we recall some constructions of Gorenstein ideals and we point out some features of the constructed ideals. All of them have been successfully applied in the context of liaison to produce Gorenstein links of given ideals and to study the Gorenstein liaison classes of some particular ideals. In section 4, we first introduce the notion of tight annihilating ring R/I for an Artinian Gorenstein ring R/J of arbitrary codimension, given by A. Iarrobino and V. Kanev in [6]. Then we introduce the new definition of tight resolving ring R/I for an Artinian Gorenstein ring R/J which generalizes the other one in the codimension 3 case. We relate the Hilbert function of an Artinian Gorenstein ring R/J to the Hilbert function of a tight resolving

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ring for R/J, and we compare the two notions. We prove that if a Cohen-Macaulay ring R/I of dimension 1 is a tight annihilating or resolving ring for an Artinian Gorenstein ring R/J and $f \in [R]_d$ is a regular form for I then the ideal $J' = J : fR$ is an Artinian Gorenstein ideal, J is a basic double link of J' on I if the degree d of f is not too large, and R/I is not necessarily either a tight annihilating ring or a tight resolving ring for R/J'. We end this section giving a numerical criterion to assure that R/I is also a tight resolving ring for R/J'.

Section 5 contains the main results of this paper. We fix a Cohen-Macaulay ideal $I \subset R$ of dimension 1 and we construct the parameter space $\mathcal{G}(I, r)$ for the Artinian Gorenstein ideals G with $\mathrm{reg}(G) = r$ and for which I is a tight annihilating ideal. We prove that $\mathcal{G}(I, r)$ is an open subset of an affine space of dimension $\deg(R/I)$. We also construct the parameter space $\mathcal{BDL}(I, s_I + d)$ for the family of Artinian Gorenstein ideals $L = I + fG$ with regularity $\mathrm{reg}(L) = s_I + d$ (s_I depends on the geometry of I) which are basic double G-links of G on I where G is the sum $I + I_1$ of two suitable directly linked ideals. We prove that $\mathcal{BDL}(I, s_I + d)$ is an open subset of an affine space of dimension $h_{R/I}(d)$. The main result of this paper states that if $d \geq s_I + 1$ then $\mathcal{G}(I, s_I + d) = \mathcal{BDL}(I, s_I + d)$.

2. Preliminaries and notation

Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring in the variables $x_1, \ldots, x_n$ over the field K, algebraically closed and of characteristic $\mathrm{char}(K) = 0$. We assume $\deg(x_i) = 1$, for $i = 1, \ldots, n$, and we consider R with the usual induced graduation over $\mathbb{Z}$, i.e. $R = \oplus_{n\in\mathbb{N}}[R]_n$, where $[R]_n$ contains the homogeneous polynomials of degree n.

DEFINITION 1. *Given a homogeneous ideal $I \subseteq R$, the function*

$$j \in \mathbb{Z} \to h_{R/I}(j) = \dim_K [R/I]_j$$

is the Hilbert function of the ring R/I, where $\dim_K$ *means the dimension as $K-$vector space.*

From the definition it follows that, if $I \neq R$, then $h_{R/I}(0) = 1$.
In the following, the dimension of a ring means its Krull dimension.

DEFINITION 2. *Let $I \subseteq R$ be a homogeneous ideal. If* $\dim R/I = 0$, *we say that I is an Artinian ideal, and R/I is an Artinian ring.*

Because of the noetherianity of the ring R, this definition is equivalent to the usual definition (see [1], Ch.6).

Let $I \subseteq R$ be a homogeneous ideal, and let

$$0 \to F_c \to F_{c-1} \to \cdots \to F_1 \to I \to 0 \tag{1}$$

be a minimal free resolution of I, with $F_i = \oplus_{j=1}^{n_i} R^{\beta_{ij}}(-b_{ij})$.

Following the sheaf theory, we define the regularity of an ideal and of its quotient ring.

DEFINITION 3. *Let $I \subseteq R$ be a homogeneous ideal. Then the regularity of R/I is $\operatorname{reg}(R/I) = \max\{b_{ij} - i \mid i = 1, \ldots, c\}$ while the regularity of I is $\operatorname{reg}(I) = \operatorname{reg}(R/I) + 1$.*

Now, we define the properties of the ideals we are mainly interested in.

DEFINITION 4. *I is a Cohen-Macaulay ideal if $c = \dim R - \dim R/I = n - \dim R/I$. Equivalently, R/I is a Cohen-Macaulay ring.*

DEFINITION 5. *I is a Gorenstein ideal if I is a Cohen-Macaulay ideal and $\operatorname{rank}(F_c) = \sum_{j=1}^{n_c} \beta_{cj} = 1$, i.e. $F_c \simeq R(-t_I)$ for some integer t_I. Equivalently, R/I is a Gorenstein ring.*

We recall now the definition of regular element and some well known properties that we will use in the sequel.

DEFINITION 6. *Let $I \subseteq R$ be an ideal with $\dim R/I = 1$, and let $f \in [R]_d$. The element f is regular for I if $I : fR = I$.*

PROPOSITION 1. *If $f \in [R]_d$ is a regular element for a Cohen-Macaulay ideal $I \subseteq R$, then the sequence*

$$0 \to \frac{R}{I}(-d) \xrightarrow{f} \frac{R}{I} \to \frac{R}{I + fR} \to 0$$

is exact and $h_{R/I+fR}(j) = h_{R/I}(j) - h_{R/I}(j-d)$.

Now, we collect the properties of the Hilbert function needed later on in the cases $\dim R/I = 0, 1$.

PROPOSITION 2. *Let $I \subseteq R$ be a Cohen-Macaulay homogeneous ideal.*

1. *If I is Artinian, then $h_{R/I}(j) = 0$, for $j \gg 0$.*
2. *Let R/I be an Artinian Gorenstein ring of regularity $s_I = \operatorname{reg}(R/I)$. Let $F_c \simeq R(-t_I)$ be the last module in the minimal free resolution of I. Then*
   - *i. $s_I = t_I - n$;*
   - *ii. $h_{R/I}(j) = h_{R/I}(s_I - j)$ for every $j \in \mathbb{N}$, and so $h_{R/I}(s_I) = 1$ and $s_I = \max\{j \in \mathbb{N} \mid h_{R/I}(j) \neq 0\}$.*

The proof follows from [1], Corollaries 4.1.4 and 4.1.6.

The integer $s_I = \operatorname{reg}(R/I)$ is also called *socle degree* of the Artinian Gorenstein ring R/I.

PROPOSITION 3. *Let $I \subseteq R$ be a homogeneous Cohen-Macaulay ideal of dimension 1, and regularity $r_I = \operatorname{reg}(R/I)$. Then:*

1. $h_{R/I}(j+1) \geq h_{R/I}(j)$, *for* $j \geq 0$;
2. $h_{R/I}(r_I) = h_{R/I}(r_I + i)$ *for every* $i \in \mathbb{N}$.

The integer $h_{R/I}(r_I)$ is called the degree $\deg(R/I)$ of R/I.

The Gorenstein property gives constraints not only on the Hilbert function of a ring, but also on its minimal free resolution.

In fact, using the graded version of [5], Theorem 1.5, one can prove

PROPOSITION 4. *Let $I \subseteq R$ be a Gorenstein ideal. Then, the minimal free resolution of I is self-dual, i.e.*

1. $F_{c-j} \simeq F_j^*(-t_I)$;
2. $\delta_{c-j} : F_{c-j} \to F_{c-j-1}$ *is equal to* $\delta^*(-t_I) : F_j^*(-t_I) \to F_{j+1}^*(-t_I)$

where $F^* = \operatorname{Hom}(F, R)$ is the dual module of the free module F.

3. Some construction of Artinian Gorenstein rings

In this section we recall some well-known methods to construct homogeneous Artinian Gorenstein ideals in R and some properties that the corresponding quotient rings have.

The first method is the Buchsbaum-Eisenbud structure Theorem for codimension 3 graded Gorenstein rings ([5], Theorem 2.1).

THEOREM 1. *Let $g \geq 3$ be an odd integer, and $d_1 \leq \cdots \leq d_g$ be a sequence of positive integers; set $d = \frac{2}{g-1}(d_1 + \cdots + d_g)$ and suppose this is an integer, let $e_i = d - d_i$, and $j = d - 3$, and we suppose $1 \leq d_1, d_g \leq j + 1$ (so $e_i \geq 2$).*

Let Ψ be an alternating $g \times g$ matrix with entries from the ring R, such that the entry ψ_{ij} is homogeneous of degree $e_i - d_j$ if $e_i > d_j$ and zero otherwise (so the entries belong to the maximal ideal of R). Let Ψ_i be the $(g-1) \times (g-1)$ alternating matrix obtained by deleting the $i-$th row and column of Ψ. Then the pfaffian $\operatorname{Pf}(\Psi_i)$ is homogeneous of degree d_i. Let I be the ideal $\operatorname{Pf}(\Psi)$ generated by $\operatorname{Pf}(\Psi_i)$, $i = 1, \ldots, g$. Then I has grade (height) ≤ 3 in R. If I has grade 3, then I is a graded Gorenstein ideal of height 3, and the socle degree of R/I is $j = d - 3$.

Let λ be the column vector with entries $\lambda_i = (-1)^i \operatorname{Pf}(\Psi_i)$.

i. *Suppose I has the maximal possible grade 3. Then I has minimal free resolution*

$$0 \to R(-d) \xrightarrow{\lambda} \oplus_{i=1}^{g} R(-e_i) \xrightarrow{\Psi} \oplus_{i=1}^{g} R(-d_i) \xrightarrow{\lambda^T} I \to 0.$$

ii. *Conversely, if $I \neq R$ is a height 3 graded Gorenstein ideal of R, there is an alternating matrix Ψ as above, such that $I = \operatorname{Pf}(\Psi)$.*

No generalization of Theorem 1 is known for height $\geq$ 4 graded Gorenstein ideals.

A second method of constructing Artinian Gorenstein ideals is the following (see [4], Ex. 3.2.11):

THEOREM 2. *Let $\varphi : [R]_s \to K$ be a non-degenerate linear map. Let*

$$[I]_j = \begin{cases} [\ker\varphi : (x_1, \dots, x_n)^{s-j}]_j & \text{if} \quad j \leq s \\ [R]_j & \text{if} \quad j > s. \end{cases}$$

Then $I = \oplus_{j\in\mathbb{Z}}[I]_j$ is an Artinian Gorenstein ideal of regularity $\operatorname{reg}(I) = s + 1$.

This construction is equivalent to Macaulay's inverse system, and allows us to construct every Artinian Gorenstein ideal with given socle degree. It is a very hard open problem to relate the linear map φ to the minimal free resolution of I or at least to the Hilbert function of R/I.

The next two methods allow one to construct Gorenstein rings of whatever dimension, but we state them only in the Artinian case.

We state the first one as a particular case of [8], Theorem 4.2.1, but it was first proved in [10].

THEOREM 3. *Let $I_1, I_2 \subseteq R$ be homogeneous Cohen-Macaulay ideals such that* $\dim R/I_1 = \dim R/I_2 = 1$. *Assume that $J = I_1 \cap I_2$ is a Gorenstein ideal such that* $\dim R/J = 1$. *Then $G = I_1 + I_2$ is an Artinian Gorenstein ideal.*

REMARK 1. (1) Two ideals I_1 and I_2 satisfying the hypotheses of the previous theorem are directly linked, and G is said the sum of directly linked Cohen-Macaulay ideals.
(2) The Gorenstein ideals arising as sum of two Cohen-Macaulay directly linked ideals were studied in various papers ([11], [12], [13], for example), and it is known that not every Gorenstein ideal can be obtained by using that construction (see [11], Example 4.1).

The second and last method is the so-called basic double G-link ([7], Lemma 4.8).

THEOREM 4. *Let $I \subseteq J \subseteq R$ be homogeneous ideals such that* $\dim R/I = 1$ *and* $\dim R/J = 0$. *Let $f \in [R]_d$ be a regular form for I. Then it holds:*

1. $\deg(I + fJ) = d\deg(I) + \deg(J)$.
2. *If I is perfect and J is unmixed, then $I + fJ$ is unmixed.*
3. $J/I \cong (I + fJ)/I(d)$.
4. *if R/I and J/I are Cohen-Macaulay and J/I has Cohen-Macaulay type 1 then J and $I + fJ$ are Artinian Gorenstein ideals.*

REMARK 2. In the same hypotheses as above, the ideal $I + fJ$ is called basic double G-link of J on I. It is known that this construction does not give every Artinian Gorenstein ideal (see [2], Example 5.13).

Now, we want to give more details on the Gorenstein ideals arising from Theorems 3 and 4. In particular, we will determine how their resolution looks like.

PROPOSITION 5. *In the same notation and hypotheses as Theorem 3, if* $s = \mathrm{reg}(R/J)$, *then*

1. $h_{R/I_2}(j) = h_{R/J}(j) + h_{R/I_1}(s - j - 1) - \deg(R/I_1)$;
2. $\mathrm{reg}(R/G) = s - 1$;
3. $h_{R/G}(j) = h_{R/I_1}(j) + h_{R/I_1}(s - j - 1) - \deg(R/I_1)$;
4. *if* $0 \to F_{n-1} \to \cdots \to F_1 \to I_1 \to 0$ *is a minimal free resolution of* I, *then*

$$0 \to R(-S) \to \begin{matrix} F_{n-1} \\ \oplus \\ F_1^*(-S) \end{matrix} \to \cdots \to \begin{matrix} F_1 \\ \oplus \\ F_{n-1}^*(-S) \end{matrix} \to G \to 0$$

is a free resolution of G, *not necessarily minimal, where* $S = s + n - 1$.

Proof. Recalling that $I_2 = J : I_1$, we compute the first difference of the Hilbert functions:

$$\Delta h_{R/I_2}(j) = \Delta h_{R/J}(s - j) - \Delta h_{R/I_1}(s - j)$$

$$\Delta h_{R/I_2}(j - 1) = \Delta h_{R/J}(s - j + 1) - \Delta h_{R/I_1}(s - j + 1)$$

and so on, until

$$\Delta h_{R/I_2}(0) = \Delta h_{R/J}(s - 0) - \Delta h_{R/I_1}(s - 0).$$

By adding all the equations we get:

$$h_{R/I_2}(j) = h_{R/J}(s) - h_{R/J}(s - j - 1) - h_{R/I_1}(s) + h_{R/I_1}(s - j - 1).$$

Because of the symmetry of the function $\Delta h_{R/J}$ ($\Delta h_{R/J}(j) = \Delta h_{R/J}(s - j)$), the previous equality can be written as

$$h_{R/I_2}(j) = h_{R/J}(j) - h_{R/I_1}(s) + h_{R/I_1}(s - j - 1)$$

and so the first claim is proved.

The claims (2) and (3) follow from the knowledge of the resolution of G. Then, it is enough to prove the claim (4). The resolution of G can be computed by mapping cone procedure from the short exact sequence

$$0 \to J \to I_1 \oplus I_2 \to G \to 0$$

that relates all the ideals involved in the construction of G. By using the minimal free resolutions of I_1, J and standard results from liaison theory, it is possible to compute a free resolution of I_2. From that last one, we get the claim on the free resolution of G. Also if the resolution is non minimal, it is not possible to cancel the last from the left free module in its resolution, because G is Artinian, and so we get also the result on the regularity of R/G. □

PROPOSITION 6. *In the same notation and hypotheses of Theorem 4, if* $d = \deg(f)$ *then*

1. $h_{R/I+fJ}(j) = h_{R/I}(j) + h_{R/J}(j-d) - h_{R/I}(j-d)$;
2. *if* $0 \to F_{n-1} \to \cdots \to F_1 \to I \to 0$ *is a minimal free resolution of* I, *then*

$$0 \to R(-S) \to \begin{matrix} F_{n-1} \\ \oplus \\ F_1^*(-S) \end{matrix} \to \cdots \to \begin{matrix} F_1 \\ \oplus \\ F_{n-1}^*(-S) \end{matrix} \to J \to 0$$

   is a free resolution of J, *not necessarily minimal, for some integer* S;

3. $\operatorname{reg}(R/I + fJ) = \operatorname{reg}(R/J) + d$.

Proof. At first, we have the following equality: $I \cap fJ = fI$.
The inclusion $\supseteq$ is evident. The inverse inclusion is an easy consequence of the regularity of f for I. In fact, if $fg \in I$ for some $g \in J$ then $g \in I : fR = I$ and hence $fg \in fI$.

Now, we have that the short sequence

$$0 \to I(-d) \to I \oplus J(-d) \to I + fJ \to 0$$

is exact, and so we get the claim on the Hilbert function of $R/I + fJ$.

From the proof of Lemma 4.8 in [7] we know that the two sequences

$$0 \to I \to J \to J/I \to 0$$

and

$$0 \to I \to I + fJ \to J/I(-d) \to 0$$

are exact, with both J and $I + fJ$ Artinian Gorenstein ideals.

We can choose d sufficiently large so that the generators of $I + fJ$ not in I have degree at least $2 + \operatorname{reg}(R/I)$. Then, we can apply Theorem 3.2 in [3] and we get that a minimal free resolution of $J/I(-d)$ is

$$0 \to R(-s) \to F_1^*(-s) \to \cdots \to F_{n-1}^*(-s) \to J/I(-d) \to 0$$

for some integer s. Hence, the claim on the free resolution of J follows, too. □

4. A class of Artinian Gorenstein ideals

One of the most studied class of Gorenstein rings of dimension $d \geq 0$ is the class of quotients of Cohen-Macaulay rings of dimension $d+1$. Geometrically, they correspond to divisors on arithmetically Cohen-Macaulay projective schemes. In particular, it is very interesting the case when the two rings have the same Hilbert function in small degrees, and hence we recall the definition of tight annihilating ring given by Iarrobino and Kanev ([6], Definition 5.1), which describes that situation.

DEFINITION 7. *Let R/J be an Artinian Gorenstein ring. Let $I \subseteq J$ be a homogeneous ideal such that R/I is a dimension 1 Cohen-Macaulay ring. We say that R/I is a tight annihilating ring for R/J if $h_{R/J}(j) = h_{R/I}(j)$ for $j \leq \operatorname{reg}(R/I)$, and $h_{R/J}(j) \leq h_{R/I}(\operatorname{reg}(R/I)) = \deg(R/I)$, for every $j \in \mathbb{Z}$.*

REMARK 3. We know that the Hilbert function of the Artinian Gorenstein ring R/J is symmetric, while the one of the ring R/I is increasing until it reaches its maximum value $\deg(R/I)$. Then, if I is tight annihilating for J, the Hilbert function of R/J increases as the one of R/I, reaches the value $\deg(R/I)$, and after that, it takes the same value for some integers, and when it takes a different value, it can be completed by symmetry. Then, $\operatorname{reg}(R/J) \geq 2\operatorname{reg}(R/I)$.

In codimension 3, we can characterize the minimal free resolution of an Artinian Gorenstein ideal J having a tight annihilating ideal I. In fact, it holds:

PROPOSITION 7. *Let $I \subseteq R = K[x, y, z]$ be a Cohen-Macaulay homogeneous ideal such that $\dim R/I = 1$ and $h_{R/I}(1) = 3$. Let $J \supseteq I$ be an Artinian Gorenstein ideal for which I is tight annihilating. If $0 \to F_2 \to F_1 \to I \to 0$ is a minimal free resolution of I then there exists an integer $t_J \geq 3 + 2\operatorname{reg}(R/I)$ such that*

$$0 \to R(-t_J) \to \begin{matrix} F_2 \\ \oplus \\ F_1^*(-t_J) \end{matrix} \to \begin{matrix} F_1 \\ \oplus \\ F_2^*(-t_J) \end{matrix} \to J \to 0$$

is the minimal free resolution of J.

Proof. See [6], Theorems 5.31, 5.39, 5.46, and Remark 5.43. The proof is based on the Buchsbaum-Eisenbud structure theorem for codimension 3 Gorenstein ideals. □

This property is shared also from the ideals arising from Theorems 3 and 4. Then, we choose this last property for defining the tight resolved ring for an Artinian Gorenstein ring R/J of whatever codimension.

DEFINITION 8. *Let R/I be a graded Cohen-Macaulay ring of dimension 1 with minimal free resolution*

$$0 \to F_{n-1} \to F_{n-2} \to \cdots \to F_1 \to R \to R/I \to 0.$$

We say that R/I is a tight resolving ring for the Artinian Gorenstein graded ring R/J (or I is a tight resolving ideal for J) if the minimal free resolution of R/J is

$$0 \to R(-t_J) \to \begin{matrix} F_{n-1} \\ \oplus \\ F_1^*(-t_J) \end{matrix} \to \cdots \to \begin{matrix} F_1 \\ \oplus \\ F_{n-1}^*(-t_J) \end{matrix} \to R \to R/J \to 0$$

for some integer $t_J \geq n + 2\operatorname{reg}(R/I)$.

PROPOSITION 8. *Let R/I be a tight resolving ring for the Artinian Gorenstein graded ring R/J. Then, the Hilbert function of R/J is equal to*

$$h_{R/J}(j) = h_{R/I}(j) + h_{R/I}(\operatorname{reg}(R/J) - j) - \deg(R/I).$$

Proof. From the additivity of the Hilbert function on exact sequences, we have $h_{R/I}(j) = \sum_{i=0}^{n-1}(-1)^i \dim_K[F_i]_j$, where $F_0 = R$, and, if $F_i = \oplus_{h=1}^{n_i} R(-b_{ih})^{\beta_{ih}}$, then $\dim_K[F_i]_j = \sum_{h=1}^{n_i} \binom{j+n-1-b_{ih}}{n-1}$. Of course,

$$\begin{aligned} h_{R/J}(j) &= \sum_{i=0}^{n}(-1)^i \left[\dim_K[F_i]_j + \dim_K[F_{n-i}^*]_{j-t_J}\right] = \\ &= \sum_{i=0}^{n-1}(-1)^i \dim_K[F_i]_j + \sum_{i=1}^{n}(-1)^i \dim_K[F_{n-i}^*]_{j-t_J}. \end{aligned}$$

The sequence

$$0 \to R(-t_J) \to F_1^*(-t_J) \to \cdots \to F_{n-1}^*(-t_J)$$

is a free resolution of the canonical module of R/I and so

$$\sum_{i=1}^{n}(-1)^i \dim_K[F_{n-i}^*]_{j-t_J} = \dim_K \left[\operatorname{Ext}^{n-1}(R/I, R)\right]_{j-t_J}.$$

It is well known that

$$\dim_K \left[\operatorname{Ext}^{n-1}(R/I, R)\right]_{j-t_J} = \dim_K \left[\operatorname{Ext}^{n-2}(I, R(-n))\right]_{j+n-t_J},$$

and by Serre's duality, we have that

$$\dim_K \left[\operatorname{Ext}^{n-2}(I, R(-n))\right]_{j+n-t_J} = h^1(\mathbb{P}^{n-1}, \mathcal{I}(t_J - n - j))$$

where $\mathcal{I}$ is the ideal sheaf obtained by sheaffifing the saturated ideal I. Hence, the claim follows. □

REMARK 4. Because of the description of the Hilbert function of a dimension 1 Cohen-Macaulay graded ring R/I, we known that $h_{R/I}(k) \leq \deg(R/I)$ for every $k \in \mathbb{Z}$. Hence, $h_{R/J}(j) \leq \deg(R/I)$, for every $j \in \mathbb{Z}$.

Now, we compare the notion of tight annihilating and tight resolving ideal for an Artinian Gorenstein ideal J.

PROPOSITION 9. *Let $J \subseteq R$ be an Artinian Gorenstein ideal and let $I \subseteq J$ be a tight resolving ideal for J. Then, I is a tight annihilating ideal for J if, and only if,* $\operatorname{reg}(R/J) \geq 2\operatorname{reg}(R/I)$.

Proof. We proved in Proposition 8 above that

$$h_{R/J}(j) = h_{R/I}(j) + h_{R/I}(\operatorname{reg}(R/J) - j) - \deg(R/I).$$

It follows that $h_{R/J}(j) = h_{R/I}(j)$ for every $j \leq \operatorname{reg}(R/I)$ if, and only if, $\operatorname{reg}(R/J) - j \geq \operatorname{reg}(R/I)$ for each $j \leq \operatorname{reg}(R/I)$, i.e. $\operatorname{reg}(R/J) \geq 2\operatorname{reg}(R/I)$. □

On the other hand, in the codimension 3 case, we have that if an ideal is tight annihilating for an Artinian Gorenstein ideal J then it is tight resolving for J, too, as explained in Proposition 7. Because of the absence of a structure theorem for Gorenstein ideals in codimension ≥ 4, the best we can say is the following:

PROPOSITION 10. *Let $J \subseteq R$ be an Artinian Gorenstein ideal and let $I \subseteq J$ be a tight annihilating ideal for J. Then, if the degrees of the minimal generators of J not in I are at least* $2 + \operatorname{reg}(R/I)$, *then I is a tight resolving ideal for J.*

Proof. The claim is [3], Theorem 3.2. □

Now, we want to construct new Artinian Gorenstein ideals from a given one.

THEOREM 5. *Let $I \subseteq J$ be homogeneous ideals in R. Assume that R/J is an Artinian Gorenstein ring and that R/I is a dimension 1 Cohen-Macaulay ring. Let d be an integer such that $h_{R/J}(d) = h_{R/I}(d)$, and let $f \in [R]_d$ be a regular form for I. Then, $J' = J : fR$ is an Artinian Gorenstein ideal.*

Proof. At first, we prove that J' is an Artinian ideal. In fact,

$$h_{R/J}(\operatorname{reg}(R/J) + 1) = h_{R/J}(\operatorname{reg}(R/J) + i) = 0 \text{ for every } i \geq 1,$$

and this is equivalent to the equality $[J]_j = [R]_j$ for $j \geq \operatorname{reg}(R/J) + 1$.

If $g \in [R]_{j-d}$, with $j \geq \operatorname{reg}(R/J) + 1$, then $gf \in [R]_j = [J]_j$, and hence $g \in [J : fR]_{j-d} = [J']_{j-d}$, and this proves that $[J']_j = [R]_j$ for each $j \geq \operatorname{reg}(R/J) - d + 1$, i.e. J' is an Artinian ideal.

Now, we prove that J' is a Gorenstein ideal.

Let $\varphi : [R]_{\operatorname{reg}(R/J)} \to K$ be a non-degenerate $K-$linear map such that $\ker\varphi = [J]_{\operatorname{reg}(R/J)}$. We define $\psi : [R]_{\operatorname{reg}(R/J)-d} \to K$ to be the $K-$linear map such that $\psi(g) = \varphi(gf)$. According to Theorem 2, we prove that ψ is non degenerate and that

$$[\ker\psi : (x_1, \ldots, x_n)^j]_{\operatorname{reg}(R/J)-d-j} = [J']_{\operatorname{reg}(R/J)-d-j}.$$

If $\psi(g) = 0$ for every $g \in [R]_{\mathrm{reg}(R/J)-d}$, then $\varphi(gf) = 0$ for every $g \in [R]_{\mathrm{reg}(R/J)-d}$ i.e. $f \in [\ker\varphi : (x_1, \ldots, x_n)^{\mathrm{reg}(R/J)-d}]_d = [J]_d$, with $d \leq \mathrm{reg}(R/J) - \mathrm{reg}(R/I)$. But $h_{R/J}(d) = h_{R/I}(d)$ and so $[J]_d = [I]_d$. Hence, $f \in [I]_d$ and so f is not regular for I. This contradiction proves that ψ is non degenerate.

Now, $g \in [\ker\psi : (x_1, \ldots, x_n)^j]_{\mathrm{reg}(R/J)-d-j}$ if and only if $\varphi(gfh) = 0$ for every $h \in (x_1, \ldots, x_n)^j$. By definition, this means that $gf \in [J]_{\mathrm{reg}(R/J)-j}$ i.e. $g \in [J : fR]_{\mathrm{reg}(R/J)-d-j} = [J']_{\mathrm{reg}(R/J)-d-j}$.

Conversely, if $g \in [J']_{\mathrm{reg}(R/J)-d-j}$, then $gf \in \ker\varphi : (x_1, \ldots, x_n)^j$ and $\deg(gf) = \mathrm{reg}(R/J) - j$, i.e. $\varphi(gfh) = 0, \forall h \in (x_1, \ldots, x_n)^j$. From the definition of ψ, it follows that $\psi(gh) = 0$ for each $h \in (x_1, \ldots, x_n)^j$ and so $g \in [\ker\psi : (x_1, \ldots, x_n)^j]_{\mathrm{reg}(R/J)-d-j}$. □

REMARK 5. Let I, J, J' be as above. Then $I \subseteq J \subseteq J'$.

REMARK 6. If $1 \leq d \leq \mathrm{reg}(R/J) - \mathrm{reg}(R/I)$ then $h_{R/J}(d) = h_{R/I}(d)$ both in the case I is a tight annihilating ideal for J and in the case I is a tight resolving ideal for J.

Now, we give an example to show that the Hilbert function of R/J' depends on J and f and not only on J and $d = \deg(f)$.

EXAMPLE 1. Let $I \subseteq R = K[x, y, z]$ be the ideal generated by $y^3 - xz^2, x^3 - y^2z, z^3 - x^2y$. Its minimal free resolution is

$$0 \to \begin{matrix} R(-4) \\ \oplus \\ R(-5) \end{matrix} \xrightarrow{A} R^3(-3) \longrightarrow I \to 0$$

where

$$A = \begin{pmatrix} z & x^2 \\ y & z^2 \\ x & y^2 \end{pmatrix}.$$

The ideal $I_1 = (x, y^2)$ is geometrically linked to I via the complete intersection ideal $(y^3 - xz^2, x^3 - y^2z)$, and the forms $f = x^6 + y^6 + z^6$ and $g = x^5y + y^5z + xz^5$ are regular for I.

Hence, $J = I + fI_1$ is an Artinian Gorenstein ideal with Hilbert function

$$h_{R/J} = (1, 3, 6, 7, 7, 7, 7, 6, 3, 1).$$

The ideals $J_1 = J : fR = (x, y^2, z^3)$ and $J_2 = J : gR = (x^2 - xy - z^2, xy - xz - yz, y^2 - 3xz - 2yz - z^2)$ are Artinian Gorenstein with different Hilbert functions. In fact, we have

$$h_{R/J_1} = (1, 2, 2, 1)$$

while

$$h_{R/J_2} = (1, 3, 3, 1)$$

and so the degree of the regular form is not enough to compute the Hilbert function of the new Artinian Gorenstein ideal.

However, if the degree d of f is not too large, then the Hilbert function of J' depends only on J and d. In fact, it holds:

PROPOSITION 11. *In the same hypotheses of Theorem 5, assume furthermore that I is a tight annihilating (resp. tight resolving) ideal for J. If $d \leq \operatorname{reg}(R/J) - 2\operatorname{reg}(R/I) + 1$, then*

$$h_{R/J'}(j) = h_{R/I}(j) + h_{R/I}(\operatorname{reg}(R/J') - j) - \deg(R/I).$$

Proof. We know that $J \subseteq J'$ and so $[J]_k \subseteq [J']_k$ for every integer k.

Let $g \in [J']_k$. By construction, $gf \in [J]_{k+d}$. If $k + d \leq \operatorname{reg}(R/J) - \operatorname{reg}(R/I)$ then $[J]_{k+d} = [I]_{k+d}$ by previous Remark 6, and so $g \in [I]_k$ because f is regular for I. Hence, $[J]_k = [J']_k$ for every k such that $k + d \leq \operatorname{reg}(R/J) - \operatorname{reg}(R/I)$ i.e. $k \leq \operatorname{reg}(R/J) - \operatorname{reg}(R/I) - d$. We proved that J' is an Artinian Gorenstein ideal and so its Hilbert function is symmetric. Then, the Hilbert function of R/J' is completely determined if the condition $k \leq \operatorname{reg}(R/J) - \operatorname{reg}(R/I) - d$ covers at least the first half of the range where the Hilbert function is non zero. The largest d for which that happens is $d = \operatorname{reg}(R/J) - 2\operatorname{reg}(R/I) + 1$ and the claim follows. □

REMARK 7. If $d = \operatorname{reg}(\frac{R}{J}) - 2\operatorname{reg}(\frac{R}{I}) + 1$, then $\operatorname{reg}(\frac{R}{J'}) = 2\operatorname{reg}(\frac{R}{I}) - 1$ and the Hilbert function of R/J' is the one of R/J after erasing its flat part, that is to say, the values where it reaches $\deg(R/I)$.

We will show how the ideals J, I and J' are related. To this aim, we need some properties which we collect in the following lemma.

LEMMA 1. *Let I, J, J' and $f \in [R]_d$ be as in Theorem 5. Then:*

(1) $I \cap fJ' = fI$;

(2) $J/I : \overline{f}R/I = J'/I$.

Proof. 1) The inclusion $I \cap fJ' \supseteq fI$ follows from the fact that $I \subseteq J'$.

Let $g \in I$ be an element such that $g = fh$, $h \in J'$. Then, $h \in I : fR = I$ and so $g \in fI$, and the other inclusion is verified.

2) It is evident that $I \subseteq J'$, because $I \subseteq J \subseteq J'$. Then, we can consider the ideals J/I, J'/I, and $\overline{f}R/I$ of R/I, where $\overline{f}$ is the class of f in R/I. We want to prove that $J/I : \overline{f}R/I = J'/I$.

Now, if $\overline{g} \in J'/I$ then $g + h \in J'$ for some $h \in I$. But $I \subseteq J'$, and so $g = (g+h) - h \in J'$. By its definition, $gf \in J$ and so $\overline{g}\overline{f} \in J/I$. Hence, $\overline{g} \in J/I : \overline{f}R/I$.

Conversely, if $\overline{g} \in J/I : \overline{f}R/I$, then $\overline{g}\overline{f} \in J/I$. Of course, there exists $h \in I$ such that $gf + h \in J$. As for the reverse inclusion, from $I \subseteq J$ we get that $gf = (gf + h) - h \in J$. By its definition, it holds that $g \in J : fR = J'$, i.e. $\overline{g} \in J'/I$. □

Now, we can prove that, if d is not too large, then J is a basic double link of J' on I.

PROPOSITION 12. *Let I, J, J' and $f \in [R]_d$ be as in Theorem 5. Moreover assume that I is either tight annihilating or tight resolving for J. Then, if $d \leq \operatorname{reg}(R/J) - 2\operatorname{reg}(R/I) + 1$, then $J = I + fJ'$.*

Proof. We know that $fJ' \subseteq J$ and so $I + fJ' \subseteq J$. Then, the equality follows if they have the same Hilbert function. The Hilbert function of $I + fJ'$ was computed in Proposition 6(1) and it is

$$h_{R/I+fJ'}(j) = h_{R/I}(j) + h_{R/J'}(j-d) - h_{R/I}(j-d).$$

By Proposition 11, we have

$$h_{R/I+fJ'}(j) = h_{R/I}(j) + h_{R/I}(\operatorname{reg}(R/J) - j) - \deg(R/I) = h_{R/J}(j)$$

and the claim follows. □

Now, we show with an example that R/I could be neither a tight annihilating ring nor a tight resolving ideal for R/J'.

EXAMPLE 2. Let $R = K[x, y, z]$ be a polynomial ring in 3 unknowns, and let $I \subseteq R$ be the ideal generated by $y^2 - xz, x^2 - yz, z^2 - xy$ whose resolution is

$$0 \to R^2(-3) \xrightarrow{A} R^3(-2) \to I \to 0$$

where

$$A = \begin{pmatrix} z & x \\ y & z \\ x & y \end{pmatrix}.$$

The Hilbert function of R/I is $h_{R/I} = (1, 3, \rightarrow)$, and so $\operatorname{reg}(R/I) = 1$.

Let $J = (y^2 - xz, x^2 - yz, z^2 - xy, xz, -xy)$ be an Artinian Gorenstein ideal, whose minimal free resolution is

$$0 \to R(-5) \to R^5(-3) \xrightarrow{B} R^5(-2) \to J \to 0$$

where

$$B = \begin{pmatrix} 0 & 0 & -x & z & x \\ 0 & 0 & 0 & y & z \\ x & 0 & 0 & x & y \\ -z & -y & -x & 0 & 0 \\ -x & -z & -y & 0 & 0 \end{pmatrix}.$$

We have $I \subseteq J$, and $t_J = 5$, $\operatorname{reg}(R/J) = 2$. It is evident that R/I is a tight annihilating ring for R/J. We can control the Hilbert function of R/J' for every $d \leq 1$.

The element $x \in [R]_1$ is general for I; in fact, $I : xR = I$. Moreover $J : xR = (y, z, x^2) = J'$ and R/I is not a tight annihilating ring for R/J', because the Hilbert function of R/J' verifies $h_{R/J'}(1) = 1 \neq h_{R/I}(1) = 3$. Moreover, the minimal free resolution of R/J' is

$$0 \to R(-4) \to \begin{array}{c} R(-2) \\ \oplus \\ R^2(-3) \end{array} \to \begin{array}{c} R^2(-1) \\ \oplus \\ R(-2) \end{array} \to R \to R/J' \to 0$$

and it does not contain the minimal free resolution of R/I as a subcomplex, and hence R/I is not a tight resolving ideal for R/J'.

Now, we compute the shape of a free resolution of R/J'.

LEMMA 2. *In the same hypotheses as above, the shape of a free resolution of R/J' is*

$$0 \to R(-t_{J'}) \to \begin{array}{c} F_{n-1} \\ \oplus \\ F_1^*(-t_{J'}) \end{array} \to \cdots \to \begin{array}{c} F_1 \\ \oplus \\ F_{n-1}^*(-t_{J'}) \end{array} \to R \to R/J' \to 0$$

where $t_{J'} = t_J - d$.

Proof. From the assumptions on the minimal free resolutions of R/I and R/J it follows the diagram

$$\begin{array}{ccccccccccccc}
 & & 0 & \to & F_{n-1} & \stackrel{\delta_{n-1}}{\longrightarrow} \cdot \to & F_1 & \to & R & \to & R/I & \to 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & & R(-t_J) & \to & \begin{array}{c} F_{n-1} \\ \oplus \\ F_1^*(-t_J) \end{array} & \to \cdot \to & \begin{array}{c} F_1 \\ \oplus \\ F_{n-1}^*(-t_J) \end{array} & \to & R & \to & R/J & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & & R(-t_J) & \to & F_1^*(-t_J) & \to \cdot \to & F_{n-1}^*(-t_J) & \to & \text{coker}(\delta_{n-1}^*) & \to & 0 &
\end{array}$$

with exact rows and split exact columns.

Hence, $0 \to \text{coker}(\delta_{n-1}^*)(-t_J) \to R/I \to R/J \to 0$ is a short exact sequence. It follows that $J/I \simeq \text{coker}(\delta_{n-1}^*)(-t_J)$.

But, we know that $J/I \simeq \overline{f}(J'/I)$ and that $\overline{f} : R/I(-d) \to R/I$ is an injective map. It follows that $J'/I \simeq \text{coker}(\delta_{n-1}^*)(-t_J + d) = \text{coker}(\delta_{n-1}^*)(-t_{J'})$. A free resolution of J'/I is then

$$0 \to R(-t_{J'}) \to F_1^*(-t_{J'}) \to \cdots \to F_{n-1}^*(-t_{J'}) \to \text{coker}(\delta_{n-1}^*)(-t_{J'}) \to 0$$

and so

$$0 \to R(-t_{J'}) \to \begin{array}{c} F_{n-1} \\ \oplus \\ F_1^*(-t_{J'}) \end{array} \to \cdots \to \begin{array}{c} F_1 \\ \oplus \\ F_{n-1}^*(-t_{J'}) \end{array} \to R \to R/J' \to 0$$

is a (eventually non minimal) free resolution of R/J'. □

Now, we give a numerical criterion to guarantee that I is a tight resolving ideal for J', too.

PROPOSITION 13. *Let I, J, J' be as above. If $d \le \operatorname{reg}(R/J) - 2\operatorname{reg}(R/I)$ then R/I is a tight resolving ring for R/J'.*

Proof. At first, we check that $h_{R/J'}(j) = h_{R/I}(j)$ for $j \le \operatorname{reg}(R/I)$.

Thanks to Proposition 11, we verify that $h_{R/I}(t_{J'} - j - n) = \deg(R/I)$ for every $j \le \operatorname{reg}(R/I)$, i.e. that $t_J - d - n - j \ge \operatorname{reg}(R/I)$ for every $j \le \operatorname{reg}(R/I)$. By hypothesis, $\operatorname{reg}(R/J) - d \ge 2\operatorname{reg}(R/I)$, and so the previous inequality can be written as $2\operatorname{reg}(R/I) - j \ge \operatorname{reg}(R/I)$ that holds for every $j \le \operatorname{reg}(R/I)$.

Now, we want to check that the resolution of R/J' we computed is minimal. The resolution is obtained by mapping cone, and so we have to check that no entry of a matrix representing $F^*_{n-j-1}(-t_J) \to F_j$ has degree zero. Using the notation stated in Section 2, we have that $F_j = \oplus_{k=1}^{n_j} R^{\beta_{jk}}(-b_{jk})$ and $F^*_{n-j-1}(-t_J) = \oplus_{h=1}^{n_{n-j-1}} R^{\beta_{n-j-1,h}}(-t_J + d - b_{n-j-1,h})$, and so we have to prove that $t_J - d - b_{n-j-1,h} - b_{jk} > 0$ for every h and k. By substituting the regularity of R/J we get

$$\begin{aligned} t_J - d - b_{n-j-1,h} - b_{jk} &= n + \operatorname{reg}(R/J) - d - b_{n-j-1,h} - b_{jk} \ge \\ &\ge n + 2\operatorname{reg}(R/I) - b_{n-j-1,h} - b_{jk} \ge \\ &\ge n - (n-j-1) - j = 1 \end{aligned} \tag{2}$$

where the first inequality follows from our hypothesis on d and the second one from the definition of regularity. Then, no cancellation can be performed and the resolution is minimal. □

REMARK 8. The regularity $\operatorname{reg}(R/J')$ of R/J' is equal to $\operatorname{reg}(R/J') = \operatorname{reg}(R/J) - d$. If $d \le \operatorname{reg}(R/J) - 2\operatorname{reg}(R/I)$ then $\operatorname{reg}(R/J') \ge 2\operatorname{reg}(R/I)$ and so R/I is a tight annihilating ring for R/J' by Proposition 9.

Before ending the section, we compute the residual of J with respect to J'.

PROPOSITION 14. *Let J be an Artinian Gorenstein ideal, let $I \subseteq J$ be a tight resolving ideal for J and let $J' = J : fR$ where $f \in [R]_d$ is a regular form for I, with $d \le \operatorname{reg}(R/J) - \operatorname{reg}(R/I)$. Then, $J : J' = J + fR$.*

Proof. The inclusion $J \subseteq J'$, observed in Remark 5, induces a map of complexes between the minimal free resolution of J and J':

$$\begin{array}{ccccccccccccc} 0 & \to & R(-t_J) & \to & H_{n-1} & \to & \dots & \to & H_1 & \to & J & \to & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \to & R(-t_{J'}) & \to & G_{n-1} & \to & \dots & \to & G_1 & \to & J' & \to & 0 \end{array}$$

and so we get the equality $J : J' = J + gR$ where $g \in [R]_{t_J - t_{J'} = d}$ represents the last map $R(-t_J) \to R(-t_{J'})$ (see [2]). By the definition of J' we get $fJ' \subseteq J$. Hence, $f \in J : J' = J + gR$.

If $f \in J$, then $f \in I$, because of its degree, and this is not possible being f regular for I. It follows that $f \notin J$ and so $f = ag + h, a \in K - \{0\}, h \in J$, i.e. $J : J' = J + fR$. □

5. The main result

In this section, chosen a dimension 1 Cohen-Macaulay ring R/I that contains an ideal G/I such that R/G is an Artinian Gorenstein ring, we want to compare the ideals which are basic double G-links of G on I with the Artinian Gorenstein ideals containing I for which I is a tight annihilating ideal.

With this in mind, we construct an Artinian Gorenstein ideal G as the sum of I and an ideal I_1 which is geometrically linked to I (see Theorem 3).

CONSTRUCTION 1. Let J be a Gorenstein ideal that verifies the following conditions

1. $J \subseteq I$;
2. $\dim R/J = 1$;
3. if $I_1 = J : I$ then $J = I \cap I_1$;
4. the minimal generators of J can be choosen among the minimal ones of I.

Then, the ideal $G = I + I_1$ is an Artinian Gorenstein ideal, which is the sum of two directly linked Cohen-Macaulay ideals.

Of course, there are dimension 1 Cohen-Macaulay rings for which the construction does not work, and that depends on the geometry of the schemes defined by those rings, as the following example shows.

EXAMPLE 3. Let $\mathbb{P}^2 = \text{Proj}(K[x, y, z])$, and let X, Y, Z be three 0-dimensional schemes of degree 11 defined by the ideals $I_X = (x^3 - y^2z, z^4 - xy^3, y^5 - x^2z^3)$, $I_Y = (x^3, -xy^3, y^5 - x^2z^3)$ and $I_Z = (x^3, -xy^3, y^5)$, respectively. The three schemes have the same Hilbert function

$$h_X = h_Y = h_Z = (1, 3, 6, 9, 11, \rightarrow).$$

The minimal free resolution of I_X is

$$0 \to R^2(-6) \xrightarrow{A} R(-3) \oplus R(-4) \oplus R(-5) \to I_X \to 0$$

where

$$A = \begin{pmatrix} y^3 & z^3 \\ x^2 & y^2 \\ z & x \end{pmatrix}.$$

The first two generators of I_X form a regular sequence, and so $J = (x^3 - y^2z, z^4 - xy^3)$ is a complete intersection ideal. The ideal $I_1 = J_1 : I_X = (x, z)$ has degree 1 and $J_1 = I_X \cap I_1$.

The minimal free resolution of I_Y is

$$0 \to R^2(-6) \xrightarrow{B} R(-3) \oplus R(-4) \oplus R(-5) \to I_Y \to 0$$

where

$$B = \begin{pmatrix} y^3 & z^3 \\ x^2 & y^2 \\ 0 & x \end{pmatrix}.$$

It is evident that, if $F \in [I_Y]_4$, then x^3, F is not a regular sequence because $(x^3, F) \subseteq (x)$. Hence, the minimal degrees of two generators of I_Y that form a regular sequence are $3, 5$, and the corresponding ideal J has degree 15.

The geometrical reason for that behavior is that Y contains a degree 5 subscheme Y' contained in a line: $I_{Y'} = (x, y^5) \supseteq I_Y$.

The minimal free resolution of I_Z is

$$0 \to R^2(-6) \xrightarrow{C} R(-3) \oplus R(-4) \oplus R(-5) \to I_Z \to 0$$

where

$$C = \begin{pmatrix} y^3 & 0 \\ x^2 & y^2 \\ 0 & x \end{pmatrix}.$$

Z is supported on the point $A(0 : 0 : 1)$ with $I_A = (x, y)$. If $J_1 \subseteq I_Z$ is a complete intersection with generators of degrees $3, 5$, then $J_1 = (x^3, xy^3 l + y^5)$ for some $l \in [R]_1$.

We have $J_1 : I_Z = (x^2, y^2 + xl) = I_1$ and $I_Z \cap I_1 = (x^3, y^3(y^2 + xl), xy^3) \subset J$ (in fact, $xy^3 = xy(y^2 + xl) - yl(x^2)$) and hence no complete intersection ideal J gives a geometric link of Z with another scheme. This happens because Z is not locally Gorenstein.

REMARK 9. We were informed by A. Iarrobino that M. Boij, in a talk at Northeastern University, proved that there exists an Artinian Gorenstein ideal $J \supseteq I$ with I tight annihilating for J if, and only if, the ring R/I is locally Gorenstein, i.e. every localization of R/I at a minimal prime is a Gorenstein ring.

It is natural to look for an ideal J of *minimal socle degree* to construct the ideal I_1. Then, we define

DEFINITION 9. *Let I be a Cohen-Macaulay ideal such that R/I has dimension 1 and is locally Gorenstein. We set*

$$s_I = \min\{\operatorname{reg}(R/J) \mid J \textit{ verifies the hypotheses of Construction 1}\}.$$

If we write a minimal free resolution of an ideal J of minimal socle degree we have

$$0 \to R(-s_I - n + 1) \to Q_{n-2} \to \cdots \to Q_1 \to J \to 0.$$

As we showed in Example 3, the integer s_I depends on the geometry of the ring R/I.

Let J be a Gorenstein ideal fulfilling all the assumptions of Construction 1, and verifying $\operatorname{reg}(R/J) = s_I$. Then, we fix now and forever, the ideal $G = I + I_1$, where $I_1 = J : I$. A free resolution of G was computed in Proposition 6(2).

Now, we want to construct a parameter space for the family of the Artinian Gorenstein ideals obtained by basic double G-link from G on I, that are $I + fG = I + fI_1$, for some $f \in R$ regular for I.

The key property to construct this parameter space is the following:

PROPOSITION 15. *Let $f_1, f_2 \in [R]_d$ be two elements, regular for I. Then $I + f_1 I_1 = I + f_2 I_1$ if, and only if, $f_1 = f_2 \mod (I)$.*

Proof. First, assume $f_1 = f_2 \mod (I)$. Let $g \in I + f_2 I_1$ be a form. Then, there exist $p \in I$ and $q \in I_1$ such that $g = p + f_2 q$. By assumption, there exists $h \in I$ such that $f_2 = f_1 + h$. Hence, $g = p + f_1 q + hq = (p + hq) + f_1 q = p_1 + f_1 q \in I + f_1 I_1$ because $p_1 = p + hq \in I$.

Vice versa, if $L_1 = I + f_1 I_1$, $L_2 = I + f_2 I_1$ and $L_1 = L_2$ then, $L_1 : G = L_2 : G$. By Proposition 14, $I + f_1 R = I + f_2 R$ and then $f_1 - f_2 \in I$. □

We are able to construct the parameter space for the family of the Artinian Gorenstein ideals which are basic double G-links of G on I.

THEOREM 6. *Let $I \subseteq R$ be a dimension 1, locally Gorenstein, Cohen-Macaulay ideal. Let*

$$\mathcal{BDL}(I, s_I + d) = \{L = I + fI_1 \,|\, \deg(f) = d,\ f \text{ regular for } I\}$$

be the family of the Artinian Gorenstein ideals that are basic double G-links of G on I, of regularity $\operatorname{reg}(L) = s_I + d$. Then, $\mathcal{BDL}(I, s_I + d)$ is parametrized by an open subset of an affine space of dimension $h_{R/I}(d)$.

Proof. Every ideal $L \in \mathcal{BDL}(I, s_I + d)$ corresponds to the choice of $f \in [R]_d$ such that $I : fR = I$, that is an open condition.

By Proposition 15, the Artinian Gorenstein ideals $L_1 = I + f_1 I_1$ and $L_2 = I + f_2 I_1$ are equal if, and only if, $f_1 - f_2 \in I$.

Hence, the natural parameter space for $\mathcal{BDL}(I, s_I + d)$ is the open subset W of the affine space $[R/I]_d$ corresponding to the forms that are regular for I.

By definition, $\dim W = \dim_K [R/I]_d = h_{R/I}(d)$. □

Now, following [6], we construct the parameter space for the Gorenstein ideals for which I is a tight annihilating ideal.

THEOREM 7. *Let $I \subseteq R$ be a dimension 1, locally Gorenstein, Cohen-Macaulay ideal. Let*

$$\mathcal{G}(I, r) = \{L \,|\, I \text{ is a tight annihilating ideal for } L, \operatorname{reg}(L) = r\}$$

be the family of Artinian Gorenstein ideals L containing I as tight annihilating ideal, of regularity $\operatorname{reg}(L) = r$*. Then, the parameter space of* $\mathcal{G}(I, r)$ *is an open subset of an affine space of dimension* $\deg(R/I)$.

Proof. The statement in the case of codimension 3 is part of the more general Theorem 5.31 in [6]. But the proof holds verbatim in our hypotheses. □

Now, we can state our main result.

THEOREM 8. *Let $I \subseteq R$ be a dimension* 1*, locally Gorenstein, Cohen-Macaulay ideal. Let $d \geq s_I + 1$ be an integer. Then $\mathcal{G}(I, s_I + d) = \mathcal{BDL}(I, s_I + d)$.*

Proof. If $d \geq s_I + 1$, then $h_{R/L}(j) = h_{R/I}(j)\ \forall j \leq s_I$. Moreover, by using arguments as in the proof of Proposition 13, we can prove that a minimal free resolution of L is

$$0 \to R(-t) \to \begin{array}{c} F_{n-1} \\ \oplus \\ F_1^*(-t) \end{array} \to \cdots \to \begin{array}{c} F_1 \\ \oplus \\ F_{n-1}^*(-t) \end{array} \to L \to 0,$$

where $t = s_I + n + d - 1$, and so I is a tight annihilating ideal for every $L \in \mathcal{BDL}(I, s_I + d)$. Hence, $\mathcal{BDL}(I, s_I + d) \subseteq \mathcal{G}(I, s_I + d)$. Moreover, both $\mathcal{BDL}(I, s_I + d)$ and $\mathcal{G}(I, s_I + d)$ are parametrized by open subsets of affine spaces of dimension $\deg(R/I) = h_{R/I}(d)$. Now, to prove that $\mathcal{BDL}(I, s_I + d) = \mathcal{G}(I, s_I + d)$, it is enough to prove that $\mathcal{BDL}(I, s_I + d)$ is closed in $\mathcal{G}(I, s_I + d)$.

If $L \in \mathcal{G}(I, s_I + d)$, and it is the flat limit of a 1-parameter flat family of ideals in $\mathcal{BDL}(I, s_I + d)$, then there exists a 1-parameter family of polynomials $f_t \in [R]_d$ such that $L_t = I + f_t I_1 \to L$ for $t \to 0$.

It is evident that $I + f_t I_1 \to I + f_0 I_1 = L$, for $t \to 0$, with $f_0 \in [R]_d$.

If f_0 is not regular for I, then there exists a minimal homogeneous prime ideal $\mathcal{P} \in R/I$ that contains $f_0 R + I$, and so $\dim \frac{R}{I + f_0 I_1} \neq 0$. But this is a contradiction because L is Artinian, and so f_0 is regular for I.

Hence, L is a basic double G-link of G on I, and so we get the claim. □

The same argument as above proves also the following

PROPOSITION 16. *Let I, J be as above, and let $1 \leq d \leq s_I$ be an integer. Then $\mathcal{BDL}(I, s_I + d)$ is a quasi projective subscheme of $\mathcal{G}(I, s_I + d)$ of codimension* $\deg(R/I) - h_{R/I}(d)$.

In $R = K[x, y, z]$, it is known that the first half of the Hilbert function of the Artinian Gorenstein ring R/L is admissible as Hilbert function of a dimension 1 Cohen-Macaulay ring R/I. Moreover, if $h_{R/L}(j) = s$ for at least 3 consecutive integers, then there exists a dimension 1 Cohen-Macaulay ring R/I of degree $\deg(R/I) = s$ that is a tight annihilating ring for L. In higher codimension, we could not find an analogous numerical condition to guarantee the existence of a tight annihilating ring for L.

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