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LAUDAL TYPE THEOREMS FOR ALGEBRAIC CURVES

Abstract. Laudal's Lemma states that if C is an integral curve in $\mathbb{P}^3$ of degree $d > s^2 + 1$ and Z is its general plane section, then C is contained in a surface of degree s provided that Z is contained in a curve of degree s. The aim of this paper is to extend Laudal's Lemma to possibly reducible curves proving that, under the unavoidable hypothesis that the Hilbert function of the generic plane section is of decreasing type, the bound $s^2 + s$ (not $s^2 + 1$) holds. Moreover we prove that the bound is sharp providing various examples of reducible curves in $\mathbb{P}^3$ of degree $d = s^2 + s$. Last we give an example where $d = s^2 + s - 1$, that is a d-degree curve satisfying an intermediate bound.

1. Introduction

Laudal's Lemma (see [7]) gives an answer to the *lifting problem* for curves in $\mathbb{P}^3$. Denoting by C a reduced and irreducible curve in $\mathbb{P}^3$ of degree d, the theorem in Laudal's original paper states that

> if the general plane section of C is contained in a curve of degree s, then C itself is contained in a surface of degree s, provided that $d > s^2 + s$.

In the last century, this theorem has been the object of different extensions, generalizations and improvements.

First of all in 1981 (see [5] for details), Gruson and Peskine improved the bound in Laudal's Lemma from $s^2 + s$ to $s^2 + 1$, by proving the following result:

> in projective 3-space, if C is an irreducible algebraic curve not contained in any surface of degree s and if the general plane section of C lies on a curve of degree s, then deg $C \leq s^2 + 1$.

Few years later, in papers [16], [17], [18], Strano slightly changed the point of view of Laudal's paper and introduced new techniques. More precisely, while Laudal's Lemma concerns the lifting of a single curve containing the general plane section Z of a curve C in $\mathbb{P}^3$, Strano's results concern the lifting of *all* curves containing Z. In reality, he studied the surjectivity of the restriction map between the cohomology groups $H^0\mathcal{J}_C(t)$ and $H^0\mathcal{J}_Z(t)$:

$$\varrho : H^0\mathcal{J}_C(t) \to H^0\mathcal{J}_Z(t). \tag{1}$$

In particular, Strano proved a theorem, from now on referred to as the Tor-Lemma, which is strictly related to Laudal type theorems. Such a theorem puts into close connection the surjectivity of the restriction map (1) and the presence, in suitable degrees, of syzygies for the homogeneous ideal corresponding to the plane section Z.

Laudal's Lemma has also been generalized to $\mathbb{P}^N$, $N \geq 4$, by different authors (see [9], [10], [11], [13], [14], [15]).

The fundamental principle behind the research on the *lifting theorems* is the study of the Hilbert function and the Castelnuovo function (the first difference of the Hilbert function), and the so-called *Uniform Position Property* (*UPP* for short) of points in $\mathbb{P}^2$, with its consequences on the Hilbert function.

In all the known literature, the study focuses on reduced and irreducible curves in $\mathbb{P}^3$ (or varieties in $\mathbb{P}^n$).

The aim of the present paper is to extend the discussion to possibly reducible curves. First of all, we give a short example in order to show that some assumptions are unavoidable. In particular, it is not possible to drop the hypothesis on the Hilbert function of being of decreasing type. When non-integral curves are considered, the plane section Z does not have the *Uniform Position Property* necessarily. Therefore Z is not forced to be contained in a complete intersection $CI(a_1, a_2)$. Nevertheless, by using the decreasing type property, we are able to prove that, if a_1 and a_2 are the degrees of the first two minimal generators of the homogeneous ideal of Z, then $d \leq a_1 \cdot a_2$, under the only assumption that the Hilbert function is of decreasing type. As a consequence, a bound on the degree of a reducible curve in terms of a non-lifting level is shown.

The paper proves that in this case (possibly reducible curve), under the hypothesis that the Hilbert function of the generic plane section is of decreasing type, the bound is the one proved by Laudal ($d \leq s^2 + s$) and it is sharp. Such a bound cannot be improved: examples of reducible curves with $d = s^2 + s$ are given. Moreover, we give examples of reducible curves of degree $d = s^2 + s - 1$.

2. Notation and preliminary results

From now on, we denote by:

1. $\mathbb{P}^3$ the projective space over an algebraically closed field k of characteristic 0;
2. C a curve in $\mathbb{P}^3$, i.e. a 1-dimensional locally C.M, equidimensional, non-degenerate closed subscheme;
3. $Z = C \cap H$ the general plane section of C;
4. $\alpha(I_Z)$ the initial degree of Z, i.e.:

$$\alpha(I_Z) = \min\{t \mid h^0\mathcal{J}_Z(t) \neq 0\}.$$

Now we list the results that represent essential tools in the following.

First of all, we want to mention the so-called Tor-Lemma (see [16], [17] for details) that allows to connect the vanishing of the Tor and the surjectivity of the restriction map in cohomology. It guarantees the lifting of all the hypersurfaces containing the general hyperplane section of a variety X in $\mathbb{P}^n$. The ideas and techniques are introduced in [16] and in [17] in the context of curves in $\mathbb{P}^3$.

REMARK 1. Consider a curve C in $\mathbb{P}^3$, denote by Z its general plane section and by I_Z the corresponding homogeneous ideal. In this particular case, the Tor-Lemma (see [16], [17]) implies that if the map:

$$H^0\mathcal{J}_C(s) \longrightarrow H^0\mathcal{J}_Z(s)$$

is not surjective, i.e. there exists a curve F of degree s in the plane containing Z that does not lift to a surface in $H^0\mathcal{J}_C(s)$, then there exists a syzygy of I_Z in degree less than or equal to $s+2$. This information will be useful in order to evaluate the number of generators of the homogeneous ideal I_Z in degree s and $s+1$, (see Corollary 2 on page 12 in [16]). The integer s is called a **non-lifting level**.

Another fundamental notion in the context of the *Lifting Problem* for curves in $\mathbb{P}^3$ is the so-called *Uniform Position Property* of points in $\mathbb{P}^2$ with its consequences on the behaviour of the Hilbert and Castelnuovo functions.

DEFINITION 1. *(see [6]) A set $Z \subset \mathbb{P}^2$ is said to have the **Uniform Position Property** if for any subset $Z' \subseteq Z$ consisting of n' points, the Hilbert function of Z' satisfies the following:*

$$h_{Z'}(i) = \min\{n', h_Z(i)\}.$$

Definition (1) is equivalent to the following:

DEFINITION 2. *A set of points is said to have the **Uniform Position Property** if all subsets of Z having the same cardinality have the same Hilbert functions.*

REMARK 2. Denoting by Δh_Z the first difference of the Hilbert function of the set of points Z, i.e. $\Delta h_Z(i) = h_Z(i) - h_Z(i-1)$, it holds:

$$\Delta h_Z(i) = i + 1 - \Delta h^0\mathcal{J}_Z(i).$$

When $H = \{h_i\}$ is the Hilbert function of a set of points $Z \subset \mathbb{P}^2$, it is $h_1 \leq 3$ and there exist integers $a_1 \leq a_2 \leq \tau$ such that:

$$\Delta h_Z(i) = \begin{cases} i+1 & \text{for } i = 0, \dots, a_1 - 1 \\ a_1 & \text{for } i = a_1, \dots, a_2 - 1 \\ < a_1 & \text{for } i = a_2 \\ \text{non-increasing} & \text{for } i = a_2, \dots, \tau \\ 0 & \text{for } i \geq \tau \end{cases}$$

where:

$$a_1 = \inf\left\{ i \in \mathbb{N} \mid h_i < \binom{i+2}{2} \right\} \tag{2}$$

$$a_2 = \begin{cases} a_1 & \text{if } h_{a_1} < \binom{a_1+2}{2} - 1 \\ \inf\{i > a_1 \mid \Delta h_Z(i) < a_1\} & \text{otherwise} \end{cases}$$

$$\tau = \inf\{i \in \mathbb{N} \mid h_i = h_{i+1}\} \tag{3}$$

DEFINITION 3. *We say that the Hilbert function h_Z is of **decreasing type** if $\Delta h_Z(i_1)\Delta h_Z(i_2)$ for $a_2 \leq i_1 < i_2 \leq \tau$.*

REMARK 3. In [8] it is proved that the Hilbert function of points with the *UPP* is of decreasing type. In general the converse is false. If the Castelnuovo function of a set Z of points is strictly decreasing, it is not always true that Z has the *UPP*. The only result in this direction is Theorem 4 in [8]: there exists an irreducible smooth ACM curve in $\mathbb{P}^3$ whose general plane section is a set of points Z' with the same Hilbert function as the one corresponding to Z. So the Castelnuovo function of Z' is strictly decreasing. By Harris' Uniform Position Lemma (see [6]), we can conclude that the new set of points Z' obtained as general plane section of this curve has the *UPP*.

PROPOSITION 1 (see [4] for a proof). *Let Z be a set of points in $\mathbb{P}^2$ with the* UPP. *Denote by $d_i = h^0\mathcal{J}_Z(i) - h^0\mathcal{J}_Z(i-1)$; then it holds:*

1. $d_i = 0$ *if* $i < \alpha(I_Z)$;
2. $d_i = i + 1 - \alpha(I_Z)$ *if* $\alpha(I_Z) < i < a_2$;
3. $d_{i+1} \geq d_i + 2$ *if* $a_2 \leq i \leq \tau$.

   The equality holds in (3) if and only if I_Z has no minimal generators in degree $i + 1$;
4. $d_i = i + 1$ *for* $i \geq \tau$;
5. $\sum_{i=0}^{\infty}(i + 1 - d_i) = \sum_{i=0}^{\tau-1}(i + 1 - d_i) = d$

3. A Laudal type theorem in the case of reducible curves

First of all, we prove a theorem that allows us to establish a bound on the degree of a curve C in $\mathbb{P}^3$ in terms of the degrees of the first two curves containing its general plane section.

THEOREM 1. *Let C be a locally C.M. and equidimensional curve in $\mathbb{P}^3$ of degree d, Z its general plane section. Suppose that the Hilbert function of Z is of decreasing type. Then:*

$$d \leq a_1 \cdot a_2,$$

where a_1 and a_2 are the degrees of the first two minimal generators of I_Z.

Proof. We give two different proofs: the first one is based on direct computation, the second one is based on a theorem by Maggioni and Ragusa.

Proof 1: Recall that (see section 2):

$$\Delta h_Z(i) = \begin{cases} i+1 & \text{for } i = 0, \dots, a_1 - 1 \\ a_1 & \text{for } i = a_1, \dots, a_2 - 1 \\ < a_1 & \text{for } i = a_2 \\ \text{strictly decreasing} & \text{for } i = a_2, \dots, \tau - 1 \\ 0 & \text{for } i \geq \tau \end{cases}$$

By simple computations:

$$\begin{aligned} d = \textstyle\sum_{i=0}^{\infty} \Delta h_Z(i) \leq \; & \underbrace{\sum_{i=0}^{a_1-1}(i+1)}_{=1+2+\cdots+a_1} + \underbrace{\sum_{i=a_1}^{a_2-1} a_1}_{\leq a_1(a_2-a_1)} + \underbrace{\sum_{i=a_2}^{\tau}(a_1-\epsilon_i)}_{\leq(a_1-1)+(a_1-2)+\cdots+1} \\ & \leq (1+\cdots+a_1) + a_1(a_2-a_1) + (a_1-1) + \cdots + 1 \\ & \leq \tfrac{a_1(a_1+1)}{2} + a_1(a_2-a_1) + \tfrac{(a_1-1)a_1}{2} \\ & \leq a_1 \cdot a_2 \end{aligned}$$

Observe that this computation works also when $a_1 = a_2$, in which case the decreasing starts at a_1.

Proof 2: The second proof is based on a theorem proved by Maggioni and Ragusa ([8], Lemma 3). Indeed, if $\Delta h_Z(i)$ is of decreasing type, then h_Z is also the Hilbert function of a set of points Z' with the *UPP*, because Z' is the plane section of an integral curve; therefore Z' is contained in a complete intersection $CI(a_1, a_2)$ and so $d \leq a_1 \cdot a_2$.

□

REMARK 4. Theorem 1 in fact does not concern curves in $\mathbb{P}^3$ but only sets of d points with Hilbert function of decreasing type.

REMARK 5. The theorem is not obvious as, in this case, the two curves of minimal degrees containing the generic plane section of the curve C may have a common factor.

COROLLARY 1. *If $d > a_1 \cdot a_2$, then h_Z is not of decreasing type.*

EXAMPLE 1. A set Z with $d > a_1 \cdot a_2$ and Δh_Z not of decreasing type. Consider a set Z of seven points lying on a conic consisting of two lines and on a cubic consisting of one of the two previous lines and a conic (see figure (1)). In this case $da_1 \cdot a_2$ and the Hilbert function cannot be of decreasing type.

By simple computation, the Hilbert function of Z is: $1, 3, 5, 6, 7 \;\; \forall\, n \geq 4$ and

$$\Delta h_Z : 1\ 2\ 2\ 1\ 1\ 0 \rightarrow$$

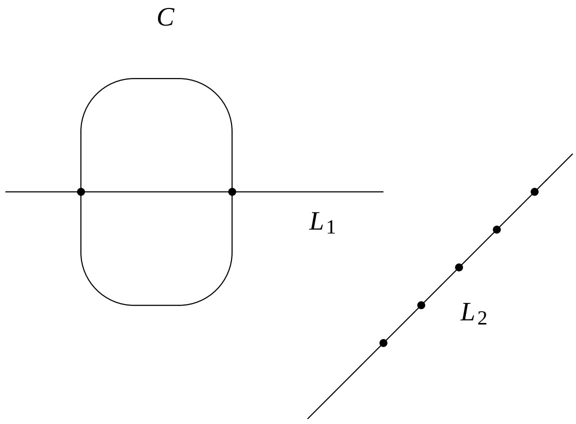

Figure 1: conic = $L_1 \cup L_2$, cubic = $L_2 \cup C$.

Observe that the set Z can be realized as general plane section of a curve Y, union of five lines on a plane and of two skew lines away from the plane.

We are now able to prove Laudal's Lemma for not necessarily irreducible curves. In this case we cannot improve the bound in the original version of Laudal's Lemma. Indeed it is possible to construct sets of points, general plane sections of reducible curves, with Hilbert function of decreasing type, in which the number of points itself is exactly $s^2 + s$. In the proof we use the Tor-Lemma (see [16] or [17]) or, better, its consequences on syzygies. The difference with the case of integral curves consists of the fact that the general plane section Z of an integral curve is made of points in uniform position while in our case we can only prove three facts: $d \leq a_1 \cdot a_2$, the syzygy property and the decreasing type of the Hilbert function.

THEOREM 2. *Let C be a locally C.M. and equidimensional curve in $\mathbb{P}^3$ and Z its general plane section. Assume that the Hilbert function of Z is of decreasing type.*

If s is a non-lifting level for C then:

1. *if $h^0\mathcal{J}_Z(s) = 1$, $d \leq s^2 + s$;*
2. *if $h^0\mathcal{J}_Z(s) \geq 2$, $d \leq s^2$.*

REMARK 6. We observe that $d \leq s^2$ when $h^0\mathcal{J}_Z(s) \geq 2$ was proved by Valenzano ([19]) only for integral curves.

Proof. It is not restrictive to suppose that $s = \zeta(C)$ is the minimal non-lifting level:

$$s = \zeta(C) = \min\{t \mid H^0\mathcal{J}_C(t) \to H^0\mathcal{J}_Z(t) \text{ is not surjective}\}.$$

In this case we have: $h^0\mathcal{J}_Z(s) \neq 0$ and $\alpha(Z) \leq s$. Let us analyze two possible cases:

1. $h^0\mathcal{J}_Z(s) = 1$.

   In this case we have: $a_1 = \alpha(Z) = s < \alpha(C)$. Moreover s is a non lifting level, hence the map:

   $$H^0\mathcal{J}_C(s) \to H^0\mathcal{J}_Z(s)$$

   is not surjective and there exists a syzygy in degree $\leq s+2$, by the Tor-Lemma. On the other hand, s is the minimal degree of a generator of the ideal I_Z and so it is not possible to have syzygies in degree $< s+1$. Consider the second generator of the homogeneous ideal defining Z and denote by a_2 its degree. Suppose $a_2 = s+2$. In this case the syzygy must have degree $s+2$ and is of the form $p_2F_s + G_{s+2} = 0$, where p_2 is a polynomial of degree 2. The second generator is multiple of the first one and so it is not a ”new” generator.

   In conclusion $a_2 = s+1$ (and the syzygy has degree $s+2$).

   By applying Theorem 1:

   $$d \leq a_1 \cdot a_2 \quad \text{i.e. } d \leq s(s+1).$$

2. $h^0\mathcal{J}_Z(s) \geq 2$.

   As before:

   $$a_1 = \alpha(Z) \leq s.$$

   We have three possibilities:

   (a) If $\alpha(Z) = s = a_1$, considering that $h^0\mathcal{J}_Z(s) \geq 2$, we have at least a second generator in degree s, so $a_2 = s$ and $d \leq s^2$.

   (b) If $a_1 < s$ and $a_2 = a_1$, we have $a_2 < s$ and $d < s^2$.

   (c) If $a_1 < s$ and $a_2 > s$, then every polynomial $F_s \in H^0\mathcal{J}_Z(s)$ must be a linear combination of generators of lower degrees:

   $$F_s = \alpha_1 G_1 + \cdots + \alpha_n G_n, \quad \deg G_i < s, \quad \deg \alpha_i = s - \deg G_i.$$

   But s is the minimal non-lifting level, hence $G_1, \ldots, G_n$ do lift and so does F_s, in contradiction with the fact that at least a non-liftable F_s must exist. Therefore $a_2 \leq s$. In this event $d \leq s^2$.

   Hence, it always holds:

   $$a_1 \leq a_2 \leq s$$

   and so $d \leq s^2$.

□

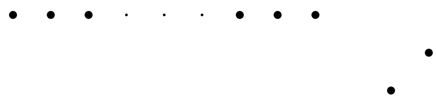

Figure 2: no bound.

EXAMPLE 2. No bound for d can exist when Δh_Z is not of decreasing type. Consider a curve $C \subset \mathbb{P}^3$ given by the union of a plane curve of degree δ and two skew lines apart. Denote by Z the general plane section of such a curve: a possible configuration is represented in figure (2).

We have:

$$s = \alpha(Z) = 2 < \alpha(C) = 3$$

but, on the other hand, $\deg(C) = \delta + 2$ and δ can be chosen as high as we want.
Therefore, it suffices to consider a curve of degree $\delta > 4$.
Let us observe that the Hilbert function of this set of points is not of decreasing type.
In fact, by simple computation, the Hilbert function of Z is: $1, 3, 5, 6, \ldots, = \delta + 2$, $\forall\ n \geq \delta - 1$. In conclusion the Castelnuovo function is:

t :	0	1	2	3	4	5	$\ldots$	$\delta - 1$	δ	$\to$
$\Delta h_Z(t)$:	1	2	2	1	1	1	$\ldots$	1	0	$\to$

and it is not of decreasing type.

REMARK 7. The fact that seems to be essential in order to obtain the bound $s^2 + 1$ on the degree of the curve is that the general plane section Z is a set of points with the *UPP*. In fact, this property yields two kinds of information:

1. first, on the set Z itself: it is contained in the complete intersection of the first two generators of the corresponding homogeneous ideal;
2. second, on the Castelnuovo function: it is strictly decreasing.

If Property 1 does not hold, then the *UPP* is missing and so $s^2 + s$ replaces $s^2 + 1$ in the bound.

4. Curves and sets of points with $\mathbf{d = s^2 + s}$

This section is dedicated to the construction of sets of points in $\mathbb{P}^2$, which we can think of as plane sections of reducible curves in $\mathbb{P}^3$, such that:

1. they have not the *UPP* and they are not contained in the complete intersection of the first two generators of the corresponding ideal;

2. the Hilbert function is of decreasing type;

3. the number of the points d, that is the degree of the curve they derive from, is exactly $s^2 + s$.

This shows that our bound is sharp.

Consider a curve X in $\mathbb{P}^3$ given by the union of

$$\begin{cases} 2s \text{ lines on a plane } \pi_1, \\ 2s-2 \text{ lines on a plane } \pi_2, \\ 2s-4 \text{ lines on a plane } \pi_3, \\ \dots \\ 4 \text{ lines on a plane } \pi_{s-1} \\ \text{two skew lines not contained in any of the preceding planes.} \end{cases}$$

Obviously X is not contained in any surface of degree s. The general plane section of such a curve is the following configuration of points:

$$\begin{cases} 2s \text{ points on a line } r_1, \\ 2s-2 \text{ points on a line } r_2, \\ 2s-4 \text{ points on a line } r_3, \\ \dots \\ 4 \text{ points on a line } r_{s-1} \\ \text{two points not contained in any of the preceding lines.} \end{cases}$$

Hence Z is contained in a curve of degree s, which is a non-lifting level.

In this case the Castelnuovo function is easily computed by using the method introduced in [1] (see Theorem 3.10), [2] (see Theorem 1.2) and [3] (see Lemma 1.3, Lemma 2.3, Theorem 3.3), described in the following. For each point we evaluate the minimal degree of a curve separating the point itself from all the preceding ones. We begin with the first point: the degree of the separator in this case is zero. Then, in order to separate the second point (on the first line) from the preceding one, we need a line, so the degree of the separator is 1. We go on until we reach the last point in the first line (if we have $2s$ points on the first line, the degree of the separator is exactly $2s-1$). Let us consider the second line and proceed as before: in order to separate the first point of the second line from all the preceding ones, we need a line so the degree of the separator is 1. We iterate this process until we reach the last point in the last line.

We prefer to introduce the following graphical visualization:

0	1	2	3	...	$s-1$	s	$s+1$	...	$2s-3$	$2s-2$	$2s-1$
	1	2	3	...	$s-1$	s	$s+1$	...	$2s-3$	$2s-2$	
		2	3	...	$s-1$	s	$s+1$	...	$2s-3$		
			3	...	...	...	...	...			
				...	...	...	...	...			
				...	...	...	...	...			
				$s-2$	$s-1$	s	$s+1$				
					$s-1$	s					
1	2	3	4	...	s	s	$s-1$	$s-2$	3	2	1

On each row, we have listed the minimal degree of a curve that separates each of the point from the preceding ones. The Castelnuovo function is computed counting how many separators appear in each degree.

Of course this method works because each line contains fewer points then the preceding one.

In conclusion, we have constructed a set of points satisfying:

- $d = s^2 + s$;

- they do not have the *UPP*, since the Hilbert function of a subset of Z consisting of three points depends on the choice of the three points;

- with Hilbert function of decreasing type.

REMARK 8. The above examples concern sets of points not contained in $CI(a_1, a_2)$.

EXAMPLE 3. There are sets of points not in uniform position, with Hilbert function of decreasing type and with $d = s^2 + s - 1$, i.e. satisfying an intermediate bound between $s^2 + 1$ and $s^2 + s$.
For example, consider:

$$\begin{cases} 2s-1 \text{ points on a line } r_1, \\ 2s-2 \text{ points on a line } r_2, \\ 2s-4 \text{ points on a line } r_3, \\ \dots \\ 4 \text{ points on a line } r_{s-1} \\ \text{two points on a line } r_s. \end{cases}$$

As before, we can compute the Castelnuovo function:

0	1	2	3	$\dots$	$s-1$	s	$s+1$	$\dots$	$2s-3$	$2s-2$
	1	2	3	$\dots$	$s-1$	s	$s+1$	$\dots$	$2s-3$	$2s-2$
		2	3	$\dots$	$s-1$	s	$s+1$	$\dots$	$2s-3$	
			3	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$		
				$\dots$	$\dots$	$\dots$	$\dots$	$\dots$		
				$\dots$	$\dots$	$\dots$	$\dots$	$\dots$		
				$s-2$	$s-1$	s	$s+1$			
					$s-1$	s				
1	2	3	4	$\dots$	s	s	$s-1$	$s-2\dots$	3	2

It is strictly decreasing.

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