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GLOBAL ANALYTIC AND GEVREY REGULARITY FOR NON-LINEAR OPERATORS ON THE TORUS

Abstract. We prove a result of global Gevrey and analytic regularity on the torus for non-linear operators constructed from rigid vector fields, with coefficients depending on the solution u and on its first derivatives. We use the method of majorant series.

1. Introduction and main notation

Let $\mathbb{T}^N$ be the N-dimensional torus and split $\mathbb{T}^N_z \simeq \mathbb{T}^m_t \times \mathbb{T}^n_x$. Let us then consider, for $u \in C^\infty(\mathbb{T}^N)$ and for some integer $n' \geq n$, the operator

$$\begin{aligned} P = P_u = P(t,x,u,D) = &\sum_{i,j=1}^{n'} a_{ij}(t,x,u,X_1u,\dots,X_{n'}u)X_iX_j \\ &+\sum_{j=1}^{n'} b_j(t,x,u,X_1u,\dots,X_{n'}u)X_j + X_o + c(t,x,u,X_1u,\dots,X_{n'}u) \end{aligned} \tag{1}$$

defined for $z=(t,x) \in \mathbb{T}^m \times \mathbb{T}^n$, where the real analytic coefficients a_{ij}, b_j and c are complex valued, but the real analytic rigid vector fields

$$X_j = \sum_{k=1}^{n} d_{jk}(x)\frac{\partial}{\partial x_k} + \sum_{k=1}^{m} e_{jk}(x)\frac{\partial}{\partial t_k}, \qquad j=0,\dots,n' \tag{2}$$

are real valued (*rigid* means that the coefficients d_{jk}, e_{jk} do not depend on t).

We shall also assume that, for every $x \in \mathbb{T}^n$, the fields

$$X'_j = \sum_{k=1}^{n} d_{jk}(x)\frac{\partial}{\partial x_k}, \qquad j=1,\dots,n'$$

span the tangent space $T_x(\mathbb{T}^n)$.

We shall prove a result of Gevrey and analytic regularity for the operator (1), under a suitable a-priori estimate.

We already proved in [3] a result of analytic regularity for the operator

$$\begin{aligned} P = P_u = &\sum_{i,j=1}^{n'} a_{ij}(t,x,u(t,x))X_iX_j + \sum_{j=1}^{n'} b_j(t,x,u(t,x))X_j \\ &+X_0 + c(t,x,u(t,x)). \end{aligned} \tag{3}$$

This operator is in particular of the form (1). Moreover, if for a fixed $u \in C^\infty(\mathbb{T}^N)$ we define the transposed operator ${}^tP_u$ of P_u by the relation

$$\langle {}^tP_uv, w\rangle = \langle v, P_uw\rangle \qquad \forall v,w \in C^\infty(\mathbb{T}^N),$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(\mathbb{T}^N)$, then we can easily compute that also the transposed operator ${}^tP_u$ of P_u is of the form (1). We thus give here a generalization of the results of [3].

In §4 we apply our main result (Theorem 1) to the operator

$$P_u = \partial_x^2 + \partial_y^2 + \sin^2 x(1 + a^2(u(t,x)))\partial_t^2,$$

for a real analytic function $a(u)$, and to its transposed operator ${}^tP_u$, showing its different behaviour in $\mathbf{R}^3$ and on the torus $\mathbb{T}^3$. This enlightens the interest of considering such operators on the torus.

Let us remark that, for P_u defined by (1), we can obtain a result of C^∞-hypoellipticity, following [10] by the use of para-differential operators. We shall therefore assume, in the following, that $u \in C^\infty(\mathbb{T}^N)$ is a fixed solution of the equation $P_u u = f$, for $f \in G^s(\mathbb{T}^N)$. We shall also assume that the following a-priori estimate is satisfied for some $0 < \delta \le \delta'$ and for all $v \in C^\infty(\mathbb{T}^N)$:

$$\begin{aligned} |||v|||_\mu := \sum_{i,j=1}^{n'} \|X_i X_j v\|_\mu + \sum_{j=1}^{n'} \|X_j v\|_{\mu+\delta} + \|v\|_{\mu+\delta'} \\ \le C_u(\|P_u v\|_\mu + \|v\|_\mu), \end{aligned} \tag{4}$$

where $C_u = C_u(u, X_1 u, \dots, X_{n'} u) \le C$ is a positive bounded function and μ is a fixed integer with $\mu > N/2$, so that the Sobolev space $H^\mu(\mathbb{T}^N)$ is an algebra and

$$\|fg\|_\mu \le \Lambda \|f\|_\mu \cdot \|g\|_\mu \qquad \forall f, g \in H^\mu(\mathbb{T}^N), \tag{5}$$

for some $\Lambda > 0$.

Under these assumptions we shall obtain a global result of reguality on the torus for P_u of the form (1) in the Gevrey classes $G^s(\mathbb{T}^N)$ for $s \ge 1$, identifying $G^1(\mathbb{T}^N)$ with the real analytic class $\mathcal{A}(\mathbb{T}^N)$. We shall use the method of majorant series, as we did in [3] for proving the hypoellipticity of P of the form (3) in the analytic case ($s = 1$). To this aim we need to modify here the majorant formal power series imployed in the analytic case.

2. Gevrey formal power series

Let $s \ge 1$ and choose $c > 0$ such that if we define, for every multiindex $\alpha \in \mathbf{N}^N$, the sequence $m_{|\alpha|} = c|\alpha|!^s/(|\alpha|+1)^2$, then

$$\sum_{0 \le \beta \le \alpha} \binom{\alpha}{\beta} m_{|\beta|} m_{|\alpha-\beta|} \le m_{|\alpha|} \qquad \forall \alpha \in \mathbf{N}^N. \tag{6}$$

Let us then define, for $\varepsilon > 0$ and $q \ge 1$, $M_q = \varepsilon^{1-q} m_q$. From (6) it follows that

$$\sum_{0 < \beta < \alpha} \binom{\alpha}{\beta} M_{|\beta|} M_{|\alpha-\beta|} \le \varepsilon M_{|\alpha|}, \tag{7}$$

and hence, if we consider the formal power series

$$\theta(Y) = \sum_{\alpha > 0} \frac{M_{|\alpha|}}{\alpha!} Y^{\alpha},$$

for $Y = (t, x) \in \mathbf{R}^N$, we obtain that

$$\theta^q(Y) \ll \varepsilon^{q-1}\theta(Y) \qquad \forall q \geq 1,\ Y \in \mathbf{R}^N, \tag{8}$$

meaning that each coefficient of the formal power series on the left is less than or equal to the corresponding coefficient of the formal power series on the right hand-side.

Let us now consider $A, R > 0$ such that the coefficients of P_u defined by (1), which are real analytic functions of the variables $t, x, u_0, u_1, \ldots, u_{n'} \in \mathbf{R}^{n'+3}$, satisfy

$$\begin{aligned} &\sum_{i,j=1}^{n'} \|\partial_t^{q_t}\partial_{u_0}^{q_0}\cdots\partial_{u_{n'}}^{q_{n'}} a_{ij}\|_{H^\mu(\mathbb{T}^N, u(\mathbb{T}^N), X_1u(\mathbb{T}^N), \ldots, X_{n'}u(\mathbb{T}^N))} + \\ &+ \sum_{j=1}^{n'} \|\partial_t^{q_t}\partial_{u_0}^{q_0}\cdots\partial_{u_{n'}}^{q_{n'}} b_j\|_{H^\mu(\mathbb{T}^N, u(\mathbb{T}^N), X_1u(\mathbb{T}^N), \ldots, X_{n'}u(\mathbb{T}^N))} + \\ &+ \|\partial_t^{q_t}\partial_{u_0}^{q_0}\cdots\partial_{u_{n'}}^{q_{n'}} c\|_{H^\mu(\mathbb{T}^N, u(\mathbb{T}^N), X_1u(\mathbb{T}^N), \ldots, X_{n'}u(\mathbb{T}^N))} \leq AR^q q! \end{aligned}$$

for all $q, q_t, q_0, \ldots, q_{n'} \geq 0$ with $q_t + q_0 + \ldots + q_{n'} = q$. Define then, as in [3], the formal power series

$$\phi(w) = \sum_{q=1}^{+\infty} AR^q w^q, \qquad \text{for } w \in \mathbf{R}. \tag{9}$$

Then for every $\rho > 0$:

$$\phi(\rho\theta(Y)) \ll \frac{A}{\varepsilon}\theta(Y)\sum_{q=1}^{+\infty}(\rho R\varepsilon)^q = \frac{AR\rho}{1 - \varepsilon\rho R}\theta(Y) \tag{10}$$

for all $\varepsilon > 0$ such that $\varepsilon\rho R < 1$.

Let us now fix an index $k \in \{1, \ldots, m\}$ and denote, for simplicity, $t = t_k$. In order to compute the $T^p = \partial_t^p = \partial_{t_k}^p$-derivative of $\phi(\rho\theta(Y))$, for some integer $p \geq 1$, we recall the following formula (which can be easily proved by induction on p) for the derivative of a composite function:

$$T^p(\phi \circ w) = \sum_{\substack{r_i \in \mathbf{N}\setminus\{0\} \\ r_1 + \ldots + r_q = p}} C_{q,r}\phi^{(q)}(w)\partial_t^{r_1}w\cdots\partial_t^{r_q}w$$

for some $C_{q,r} > 0$ (these numbers are explicitly given in the formula of Faà Di Bruno). Then

$$\begin{aligned} T^p(\phi(\rho\theta)) &= \sum_{\substack{r_1 + \ldots + r_q = p \\ r_i > 0}} C_{q,r}\phi^{(q)}(\rho\theta)\rho^q\partial_t^{r_1}\theta\cdots\partial_t^{r_q}\theta \\ &= \phi'(\rho\theta)\rho\partial_t^p\theta + \sum_{\substack{r_1 + \ldots + r_q = p \\ 0 < r_i < p}} C_{q,r}\phi^{(q)}(\rho\theta)\rho^q\partial_t^{r_1}\theta\cdots\partial_t^{r_q}\theta, \end{aligned}$$

and since $\partial_t^r \theta(0) = M_r$ and $\phi^{(q)}(0) = AR^q q!$:

$$T^p(\phi(\rho\theta(Y)))|_{Y=0} = \sum_{\substack{r_1+\ldots+r_q=p \\ r_i>0}} C_{q,r} AR^q q! \rho^q \partial_t^{r_1}\theta(0)\cdots\partial_t^{r_q}\theta(0)$$

$$= AR\rho M_p + \sum_{\substack{r_1+\ldots+r_q=p \\ 0<r_i<p}} C_{q,r} AR^q q! \rho^q \partial_t^{r_1}\theta(0)\cdots\partial_t^{r_q}\theta(0). \tag{11}$$

Moreover, from (10) we have that, for $\varepsilon\rho R < 1$,

$$T^p\phi(\rho\theta(Y)) \ll \frac{AR\rho}{1-\varepsilon\rho R} T^p\theta(Y)$$

and hence

$$T^p\phi(\rho\theta(Y))|_{Y=0} \leq \frac{AR\rho}{1-\varepsilon\rho R} M_p. \tag{12}$$

3. Gevrey and analytic hypoellipticity

We have now all the instruments to prove some hypoellipticity results. Let us first consider the case of coefficients not depending on t and x:

THEOREM 1. *Let P be an operator of the form*

$$P = \sum_{i,j=1}^{n'} a_{ij}(u, X_1u, \ldots, X_{n'}u) X_i X_j + \sum_{j=1}^{n'} b_j(u, X_1u, \ldots, X_{n'}u) X_j$$
$$+ X_o + c(u, X_1u, \ldots, X_{n'}u),$$

defined for $z = (t,x) \in \mathbb{T}^m \times \mathbb{T}^n$, where all the coefficients a_{ij}, b_j and c are real analytic and the rigid vector fields $\{X_j\}_{j=0,\ldots,n'}$ are defined by (2). Assume that, for every fixed $x \in \mathbb{T}^n$, the $\{X_j'\}_{j=1,\ldots,n'}$ span $T_x(\mathbb{T}^n)$.

Assume moreover that $u \in C^\infty(\mathbb{T}^N)$ is a solution of the equation $Pu = f$, for some $f \in G^s(\mathbb{T}^N)$, with $s \geq 1$, and that the a-priori estimate (4) is satisfied. Then also $u \in G^s(\mathbb{T}^N)$.

Proof. From the given assumptions on the vector fields X_j, it is sufficient to prove the Gevrey estimate for $\|\partial_{t_k}^b u\|_\mu$ for $k = 1, \ldots, m$ and for every $b \geq 1$.

We fix k and denote, for simplicity, $t = t_k$ and $T = \partial_t = \partial_{t_k}$. Then we define

$$[u]_{t,r} = \sup_{0<q\leq r} \frac{|||T^q u|||_\mu}{M_q},$$

and prove by induction on $r \geq 1$ that there exist $\varepsilon, M > 0$ such that for all $r \geq 1$:

$$[u]_{t,r} \leq M. \tag{13}$$

Since $M_1 = c/4$ and $M_2, M_3 \geq M_1$ for ε small enough, we can take

$$M = \max\{1, \frac{4}{c} \max_{1\leq q\leq 3} |||\, T^q u \,|||_\mu\}, \tag{14}$$

and find $0 < \varepsilon' < 1$ such that for $0 < \varepsilon < \varepsilon'$ the required estimate (13) is satisfied for $r = 1,2,3$.

Let us now assume that (13) is satisfied for all integers $r < b$, with $b \geq 3$, and let us prove it for $r = b$ (the above request $b \geq 3$ will be understood in the following).

We follow the same ideas of [3] to estimate $|||\, T^b u \,|||_\mu$. By the a-priori estimate (4) we have that

$$\begin{aligned} |||\, T^b u \,|||_\mu &= \sum_{j,k=1}^{n'} \|X_j X_k T^b u\|_\mu + \sum_{j=1}^{n'} \|X_j T^b u\|_{\mu+\delta} + \|T^b u\|_{\mu+\delta'} \\ &\leq C_u(\|PT^b u\|_\mu + \|T^b u\|_\mu). \end{aligned} \tag{15}$$

For every $\varepsilon_1 > 0$ we can find a positive constant $C_{\varepsilon_1} > 0$ such that:

$$\|T^b(X)u\|_\mu \leq \varepsilon_1 \|T^b(X)u\|_{\mu+\delta} + C_{\varepsilon_1}\|T^b(X)u\|_{\mu-1} \tag{16}$$

$$\|PT^b u\|_\mu \leq \|[P, T^b]u\|_\mu + \|T^b P u\|_\mu, \tag{17}$$

where (X) denotes the generic term of the form 1 or X_j. In the following, the term $\varepsilon_1\|T^b(X)u\|_{\mu+\delta}$ will be absorbed in the left hand-side, while $\|T^b P u\|_\mu$ will not give any problems because $Pu = f \in G^s(\mathbb{T}^N)$. Moreover

$$\begin{aligned} \|(X)T^b u\|_{\mu-1} &\leq \|(X)T^{b-1}u\|_{\mu+\delta} \leq |||\, T^{b-1}u \,|||_\mu \\ &\leq [u]_{t,b-1} M_{b-1} \leq M M_{b-1} \leq \varepsilon M M_b, \end{aligned} \tag{18}$$

since $M_{b-1} \leq \varepsilon M_b$ for $b \geq 3$ (here we need $b \geq 3$).

Let us now estimate $\|[P, T^b]u\|_\mu$. To this aim we compute

$$\begin{aligned} [T^b, P] = T^b P - P T^b &= \sum_{i,j=1}^{n'} \sum_{b'=1}^{b} \binom{b}{b'} T^{b'} a_{ij}(u, X_1 u, \ldots, X_{n'} u) X_i X_j T^{b-b'} \\ &+ \sum_{j=1}^{n'} \sum_{b'=1}^{b} \binom{b}{b'} T^{b'} b_j(u, X_1 u, \ldots, X_{n'} u) X_j T^{b-b'} \\ &+ \sum_{b'=1}^{b} \binom{b}{b'} T^{b'} c(u, X_1 u, \ldots, X_{n'} u) T^{b-b'} \end{aligned}$$

since the X_j's do not depend on the variable t.

Denoting by $a \circ (X)u$ the generic coefficient $a_{ij}(u, X_1 u, \ldots, X_{n'} u)$ or $b_j(u, X_1 u,$

$\dots, X_{n'}u)$ or $c(u, X_1u, \dots, X_{n'}u)$, we thus have to estimate terms of the form

$$\left\| \sum_{b'=1}^{b} \binom{b}{b'} T^{b'}(a \circ (X)u)(X^2)T^{b-b'}u \right\|_\mu \leq \Lambda \|T^b(a \circ (X)u)\|_\mu \cdot \|(X^2)u\|_\mu$$

$$+\Lambda \sum_{b'=1}^{b-1} \binom{b}{b'} \|T^{b'}(a \circ (X)u)\|_\mu \cdot \|(X^2)T^{b-b'}u\|_\mu, \tag{19}$$

where (X^2) denotes the generic term of the form 1, or X_j or X_jX_k.

To this aim we first write the following formula for $T^p(a \circ (X)u)$, which can be easily proved by induction on $p \geq 1$:

$$\begin{aligned} T^p(a \circ (X)u) &= \partial_t^p a(u, X_1u, \dots, X_{n'}u) \\ &= \sum_{\substack{r_1+\dots r_q=p \\ q_0+\dots+q_{n'}=q \\ r_i>0}} C_{q,r}(\partial_{u_0}^{q_0} \cdots \partial_{u_{n'}}^{q_{n'}} a)(u, X_1u, \dots, X_{n'}u)\partial_t^{r_1}(X)u \cdots \partial_t^{r_q}(X)u. \end{aligned}$$

From the choice of A, R made in Section 2, denoting by $a^{(p)}((X)u)$ the generic derivative $(\partial_{u_0}^{q_0} \cdots \partial_{u_{n'}}^{q_{n'}} a)(u, X_1u, \dots, X_{n'}u)$ with $q_0 + \dots + q_{n'} = p$, we thus have:

$$\begin{aligned} \|T^b(a \circ (X)u)\|_\mu &\leq \Lambda \|a'((X)u)\|_\mu \cdot \|T^b(X)u\|_\mu \\ &\quad + \sum_{\substack{r_1+\dots r_q=b \\ 0<r_i<b}} C_{q,r}\|a^{(q)}((X)u)\|_\mu \Lambda^q \|\partial_t^{r_1}(X)u\|_\mu \cdots \|\partial_t^{r_q}(X)u\|_\mu \\ &\leq \Lambda AR\|T^b(X)u\|_\mu + \sum_{\substack{r_1+\dots r_q=b \\ 0<r_i<b}} C_{q,r}A(R\Lambda[u]_{t,b-1})^q q! \partial_t^{r_1}\theta(0) \cdots \partial_t^{r_q}\theta(0) \\ &= \Lambda AR\|T^b(X)u\|_\mu + T^b\phi(\Lambda[u]_{t,b-1}\theta(Y))|_{Y=0} - AR\Lambda[u]_{t,b-1}M_b \\ &\leq \Lambda AR\|T^b(X)u\|_\mu + \left(\frac{AR\Lambda[u]_{t,b-1}}{1-\varepsilon R\Lambda[u]_{t,b-1}} - AR\Lambda[u]_{t,b-1} \right) M_b \\ &\leq \Lambda AR(\varepsilon_1\|T^b(X)u\|_{\mu+\delta} + C_{\varepsilon_1}\|T^b(X)u\|_{\mu-1}) \\ &\qquad + AR\Lambda[u]_{t,b-1}\left(\frac{1}{1-\varepsilon_o} - 1 \right) M_b \\ &\leq \varepsilon_1 \Lambda AR\|T^b(X)u\|_{\mu+\delta} + \varepsilon\Lambda ARC_{\varepsilon_1}MM_b + \frac{\varepsilon_o}{1-\varepsilon_o}AR\Lambda MM_b \end{aligned} \tag{20}$$

for $\varepsilon_1 > 0$ and $\varepsilon = \varepsilon_o/(MR\Lambda)$ with $0 < \varepsilon_o < 1$ to be chosen in the following, because of (11), (12), the inductive assumption on $[u]_{t,b-1}$, (16) and (18).

Analogously, for $1 \leq b' \leq b-1$:

$$\begin{aligned} \|T^{b'}(a \circ (X)u)\|_\mu &\leq \sum_{\substack{r_1+\dots+r_q=b'\leq b-1 \\ r_i>0}} C_{q,r}AR^q q!(\Lambda[u]_{t,b-1})^q \partial_t^{r_1}\theta(0) \cdots \partial_t^{r_q}\theta(0) \\ &= T^{b'}\left(\phi(\Lambda[u]_{t,b-1}\theta(Y))\right)\Big|_{Y=0} \\ &\leq \frac{\Lambda AR}{1-\varepsilon_o}[u]_{t,b-1}M_{b'} \leq \frac{\Lambda AR}{1-\varepsilon_o}MM_{b'} \end{aligned} \tag{21}$$

and

$$\|(X^2)T^{b-b'}u\|_\mu \le |\!|\!|\, T^{b-b'}u\, |\!|\!|_\mu \le [u]_{t,b-1}M_{b-b'} \le MM_{b-b'}. \tag{22}$$

Substituting (20), (21) and (22) in (19), and setting $m = |\!|\!|\, u\, |\!|\!|_\mu$, we finally have:

$$\begin{aligned}
&\left\| \sum_{b'=1}^{b} \binom{b}{b'} T^{b'}(a \circ (X)u)(X^2)T^{b-b'}u \right\|_\mu \\
&\le \Lambda m \left(\varepsilon_1 \Lambda AR \|T^b(X)u\|_{\mu+\delta} + \varepsilon \Lambda ARC_{\varepsilon_1} MM_b + \frac{\varepsilon_o}{1-\varepsilon_o} AR\Lambda MM_b \right) \\
&\qquad + \Lambda \sum_{b'=1}^{b-1} \binom{b}{b'} \frac{\Lambda AR}{1-\varepsilon_o} MM_{b'}MM_{b-b'} \\
&\le \varepsilon_1 \Lambda^2 ARm \|T^b(X)u\|_{\mu+\delta} + \frac{\varepsilon_o}{1-\varepsilon_o} AR\Lambda^2 mMM_b \\
&\qquad + \varepsilon \Lambda^2 ARM \left(mC_{\varepsilon_1} + \frac{M}{1-\varepsilon_o} \right) M_b \\
&\le \varepsilon_1 \Lambda^2 ARm \|T^b(X)u\|_{\mu+\delta} + \varepsilon_o \Lambda A \left(\frac{\Lambda RMm}{1-\varepsilon_o} + mC_{\varepsilon_1} + \frac{M}{1-\varepsilon_o} \right) M_b
\end{aligned}$$

Therefore, substituting in (17) and (15):

$$\begin{aligned}
&|\!|\!|\, T^b u\, |\!|\!|_\mu \le C(u)(n'^2 + n' + 1) \Big[\varepsilon_1 \Lambda^2 ARm \|T^b(X)u\|_{\mu+\delta} \\
&+ \varepsilon_o \Lambda A \left(\frac{\Lambda RMm}{1-\varepsilon_o} + mC_{\varepsilon_1} + \frac{M}{1-\varepsilon_o} \right) M_b \Big] \\
&+ C(u) \left(B_f^{b+1} \frac{b!^s}{(b+1)^2} + \varepsilon_1 \|T^b u\|_{\mu+\delta} + C_{\varepsilon_1} \varepsilon MM_b \right) \\
&\le \varepsilon_1 C[\Lambda^2 ARm(n'^2 + n' + 1) + 1] \|(X)T^b u\|_{\mu+\delta} \\
&+ \varepsilon_o CM \left[\Lambda A(n'^2 + n' + 1) \left(\frac{\Lambda Rm}{1-\varepsilon_o} + \frac{mC_{\varepsilon_1}}{M} + \frac{1}{1-\varepsilon_o} \right) + \frac{C_{\varepsilon_1}}{MR\Lambda} \right] M_b \\
&+ CB_f^{b+1} \frac{b!^s}{(b+1)^2} M.
\end{aligned}$$

where $B_f > 0$ is given by the G^s-estimate of $f = Pu$.

We now choose $0 < \varepsilon_1 < 1$ sufficiently small so that

$$A_{\varepsilon_1} = \varepsilon_1 C[\Lambda^2 ARm(n'^2 + n' + 1) + 1] < 1.$$

Then we take $0 < \varepsilon_0 < 1$ sufficiently small so that, for $b \ge 3$,

$$\begin{aligned}
&\frac{\varepsilon_o C}{1 - A_{\varepsilon_1}} \left[\Lambda A(n'^2 + n' + 1) \left(\frac{\Lambda Rm}{1-\varepsilon_o} + \frac{mC_{\varepsilon_1}}{M} + \frac{1}{1-\varepsilon_o} \right) + \frac{C_{\varepsilon_1}}{MR\Lambda} \right] \le \frac{1}{2} \\
&C \frac{B_f^{b+1}}{1 - A_{\varepsilon_1}} \frac{b!^s}{(b+1)^2} \le \frac{1}{2} M_b = \frac{c}{2} \frac{\varepsilon_o^{1-b}}{(MR\Lambda)^{1-b}} \frac{b!^s}{(b+1)^2}.
\end{aligned}$$

With such choices we finally have that $|||\,T^b u\,|||_\mu \leq MM_b$ for all $b \geq 3$, and hence, by the inductive assumption:

$$[u]_{t,b} = \sup_{0<q\leq b} \frac{|||\,T^q u\,|||_\mu}{M_q} = \max\left\{ \sup_{0<q<b} \frac{|||\,T^q u\,|||_\mu}{M_q}, \frac{|||\,T^b u\,|||_\mu}{M_b} \right\} \leq M.$$

The theorem is therefore proved. ☐

Let us now consider the generic case:

THEOREM 2. *Let P be an operator of the form (1), with all the coefficients a_{ij}, b_j and c real analytic. Assume that the vector fields $\{X_j\}_{j=0,\dots,n'}$ are rigid and that for every fixed $x \in \mathbb{T}^n$ the $\{X'_j\}_{j=1,\dots,n'}$ span $T_x(\mathbb{T}^n)$.*

Assume moreover that $u \in C^\infty(\mathbb{T}^N)$ is a solution of the equation $Pu = f$, for some $f \in G^s(\mathbb{T}^N)$, with $s \geq 1$, and that the a-priori estimate (4) is satisfied. Then also $u \in G^s(\mathbb{T}^N)$.

Proof. As in [3], the proof of hypoellipticity in the case of coefficients depending also on t and x is a slight modification of that in the case of coefficients which do not depend on t and x. Therefore, we give here only a sketch of it.

The main point is to estimate the H^μ-norm of a term of the form

$$\begin{aligned}
&\partial_t^p a(t,x,u,X_1u,\dots,X_{n'}u) \\
&= \sum_{s=1}^{p} \binom{p}{s} \sum_{\substack{r_1+\dots r_q=s \\ q_o+\dots+q_{n'}=q \\ r_i>0}} C_{q,r}(\partial_t^{p-s}\partial_{u_o}^{q_o}\cdots\partial_{u_{n'}}^{q_{n'}}a)(t,x,u,X_1u,\dots,X_{n'}u)\cdot \\
&\qquad\qquad \cdot\, \partial_t^{r_1}(X)u\cdots\partial_t^{r_q}(X)u
\end{aligned}$$

for $1 \leq p \leq b$. For $p = b \geq 3$ we can write

$$\begin{aligned}
\partial_t^b a(t,x,u,X_1u,\dots,X_{n'}u) &= (\partial_t^b a)(t,x,u,X_1u,\dots,X_{n'}u) \\
&\quad + \partial_t^b a(\tau,x,u,X_1u,\dots,X_{n'}u)|_{\tau=t} \\
+ \sum_{s=1}^{b-1} \binom{b}{s} &\sum_{\substack{r_1+\dots r_q=s \\ q_o+\dots+q_{n'}=q \\ r_i>0}} C_{q,r}(\partial_t^{b-s}\partial_{u_o}^{q_o}\cdots\partial_{u_{n'}}^{q_{n'}}a)(t,x,u,X_1u,\dots,X_{n'}u)\cdot \\
&\qquad \cdot\, \partial_t^{r_1}(X)u\cdots\partial_t^{r_q}(X)u.
\end{aligned} \tag{23}$$

The H^μ-norm of the first term on the right hand-side of (23) can be esality estimated, if we take $M = \max\{1, \frac{4}{c}\max_{1\leq q\leq 3} |||\,T^q u\,|||_\mu, 2A\}$ instead of (14), by

$$\|\partial_t^b a \circ (X)u\|_\mu \leq AR^b b! \leq \frac{M}{2} M_b$$

for ε small enough, because of the analyticity of the coefficients of the operator and because of

$$R^p p! \leq (2R)^p p! \leq M_p \qquad \forall p \geq 2. \tag{24}$$

The H^μ-norm of the second term on the right hand-side of (23) has already been estimated in the proof of Theorem 1 by (20).

Finally, recalling that $(k+j)! \leq 2^{k+j} k!\, j!$, and defining the formal power series $\phi(w)$ as in (9) with $\bar{R} = 2R$ instead of R, we can give also the estimate of the H^μ-norm of the third term on the right hand-side of (23):

$$\begin{aligned}
&\Bigg\| \sum_{s=1}^{b-1} \binom{b}{s} \sum_{\substack{r_1+\dots r_q=s \\ q_o+\dots q_{n'}=q \\ r_i>0}} C_{q,r} (\partial_t^{b-s} \partial_{u_o}^{q_o} \cdots \partial_{u_{n'}}^{q_{n'}} a)(t,x,uX_1u,\dots,X_{n'}u) \cdot \\
&\qquad\qquad \cdot \partial_t^{r_1}(X)u \cdots \partial_t^{r_q}(X)u \Bigg\|_\mu \\
&\leq \sum_{s=1}^{b-1} \binom{b}{s} \bar{R}^{b-s}(b-s)! \sum_{\substack{r_1+\dots r_q=s \\ r_i>0}} C_{q,r} A \bar{R}^q q! (\Lambda[u]_{t,b-1})^q \partial_t^{r_1}\theta(0) \cdots \partial_t^{r_q}\theta(0) \\
&\leq \bar{R} b T^{b-1} \phi(\Lambda[u]_{t,b-1}\theta(Y))|_{Y=0} + \sum_{s=1}^{b-2} \binom{b}{s} M_{b-s} T^s \phi(\Lambda[u]_{t,b-1}\theta(Y))|_{Y=0} \\
&\leq \frac{A\bar{R}\Lambda[u]_{t,b-1}}{1-\varepsilon R \Lambda[u]_{t,b-1}} \left[\bar{R} b M_{b-1} + \sum_{s=1}^{b-2} \binom{b}{s} M_{b-s} M_s \right] \\
&\leq \varepsilon \left(\frac{3A\bar{R}(\bar{R}+1)\Lambda M}{1-\varepsilon M \Lambda R} \right) M_b.
\end{aligned}$$

because of (24), (12), (7) and $bM_{b-1} \leq 2\varepsilon M_b$ for $b \geq 3$.

The same arguments hold to estimate $\|T^{b'}(a \circ (X)u)\|_\mu$ for $1 \leq b' \leq b-1$. We can thus conclude analogously as in the proof of Theorem 1. ☐

4. An example

of Hörmander (cf. [7]) shows that the operator of Grušin type $P = \partial_x^2 + x^2 \partial_t^2$ is C^∞-hypoelliptic locally in $\mathbf{R}^2$. Moreover, it is also G^s-hypoelliptic, for all $s \geq 1$, locally in $\mathbf{R}^2$ (cf. [6]).

The same arguments hold for

$$P = \partial_x^2 + \sin^2 x \partial_t^2. \tag{25}$$

These results imply that (25) is globally hypoelliptic on $\mathbb{T}^2$ in the C^∞, analytic and Gevrey classes.

We proved in [3] that the non-linear operator

$$P = \partial_x^2 + \sin^2 x(1 + a^2(u(t,x)))\partial_t^2,$$

for a real analytic function $a(u)$, is globally analytic hypoelliptic on the torus $\mathbb{T}^2$.

Things radically change if we introduce a third variable and consider the operator

$$P = \partial_x^2 + \partial_y^2 + \sin^2 x\partial_t^2 \tag{26}$$

in $\mathbf{R}^3$. It is C^∞ hypoelliptic locally in $\mathbf{R}^3$ (cf. [7]), but it is not analytic hypoelliptic locally in $\mathbf{R}^3$ (cf. [2]). Moreover, Bove and Tartakoff obtained in [4] a sharp result of non-isotropic Gevrey hypoellipticity for the operator (26), proving that it is $G^{3/2,1,2}$-hypoelliptic locally in $\mathbf{R}_x \times \mathbf{R}_y \times \mathbf{R}_t$. Finally, (26) is analytic hypoelliptic globally on $\mathbb{T}^3$ (cf. [5], [9]).

Let us now consider the non-linear operator

$$P = \partial_x^2 + \partial_y^2 + \sin^2 x(1 + a^2(u(t,x)))\partial_t^2 \tag{27}$$

for a real analytic function $a(u)$.

We want to apply Theorem 1 and prove that, for P_u given by (27), both P_u and ${}^tP_u$ are G^s-hypoelliptic globally on $\mathbb{T}^3$, for all $s \geq 1$.

To this aim, let us first remark that (27) is of the form (1) with

$$\begin{aligned}
&X_0 = 0,\ X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = \sin x\partial_t \\
&a_{11} = 1,\ a_{22} = 1,\ a_{33} = 1 + a^2(u) \\
&a_{ij} = 0 \quad \text{for } i, j = 1,2,3 \text{ with } i \neq j \\
&b_j = 0 \quad \text{for all } j = 1,2,3 \text{ and } c = 0.
\end{aligned} \tag{28}$$

Let us now compute, for a fixed $u \in C^\infty(\mathbb{T}^3)$, the transposed operator ${}^tP_u$ of P_u. For all

$v, w \in G^s(\mathbb{T}^3)$:

$$\begin{aligned}
\langle w, {}^tP_u v\rangle &= \langle P_u w, v\rangle = \langle \partial_x^2 w, v\rangle + \langle \partial_y^2 w, v\rangle + \langle \sin^2 x(1+a^2(u))\partial_t^2 w, v\rangle\\
&= \langle w, \partial_x^2 v\rangle + \langle w, \partial_y^2 v\rangle - \int_{\mathbb{T}^3} \partial_t w\cdot \partial_t[\sin^2 x(1+a^2(u))v]\,dtdxdy\\
&= \langle w, (\partial_x^2+\partial_y^2)v\rangle\\
&\quad - \int_{\mathbb{T}^3} \partial_t w\cdot \sin^2 x[2a(u)a'(u)\partial_t u\cdot v + (1+a^2(u))\partial_t v]\,dtdxdy\\
&= \langle w, (\partial_x^2+\partial_y^2)v\rangle\\
&\quad + \int_{\mathbb{T}^3} w\sin^2 x\,\partial_t[2a(u)a'(u)\partial_t u\cdot v + (1+a^2(u))\partial_t v]\,dtdxdy\\
&= \langle w, (\partial_x^2+\partial_y^2)v\rangle\\
&\quad + \int_{\mathbb{T}^3} w\sin^2 x[2a'(u)^2(\partial_t u)^2 v + 2a(u)a''(u)(\partial_t u)^2 v]\,dtdxdy\\
&\quad + \int_{\mathbb{T}^3} w\sin^2 x[2a(u)a'(u)\partial_t^2 u\cdot v + 2a(u)a'(u)\partial_t u\cdot\partial_t v]\,dtdxdy\\
&\quad + \int_{\mathbb{T}^3} w\sin^2 x[2a(u)a'(u)\partial_t u\cdot\partial_t v + (1+a^2(u))\partial_t^2 v]\,dtdxdy\\
&= \langle w, [\partial_x^2+\partial_y^2+\sin^2 x(1+a^2(u))\partial_t^2]v\rangle + \langle w, 4\sin^2 x\, a(u)a'(u)\partial_t u\cdot\partial_t v\rangle\\
&\quad + \langle w, 2\sin^2 x[a'(u)^2(\partial_t^2 u)^2 + a(u)a''(u)(\partial_t u)^2 + a(u)a'(u)\partial_t^2 u]v\rangle.
\end{aligned}$$

Therefore

$$\begin{aligned}
{}^tP_u u &= \partial_x^2 u + \partial_y^2 u + \sin^2 x(1+a^2(u))\partial_t^2 u + 4\sin^2 x\, a(u)a'(u)(\partial_t u)(\partial_t u)\\
&\qquad + 2\sin^2 x[a'(u)^2(\partial_t u)^2 u + a(u)a''(u)(\partial_t u)^2 u + a(u)a'(u)\partial_t^2 u\cdot u]\\
&= \partial_x^2 u + \partial_y^2 u + [1 + a^2(u) + 2a(u)a'(u)u]\sin^2 x\,\partial_t^2 u\\
&\qquad + [4\sin x\, a(u)a'(u)(\partial_t u) + 2\sin x\, a'(u)^2(\partial_t u)u\\
&\qquad + 2\sin x\, a(u)a''(u)(\partial_t u)u]\sin x\,\partial_t u
\end{aligned}$$

is of the form $P_u u$ with P_u given by (1), for X_0, X_1, X_2, X_3 as in (28) and

$$\begin{aligned}
&a_{11} = 1,\ a_{22} = 1\\
&a_{33} = 1 + a^2(u) + 2a(u)a'(u)u\\
&a_{ij} = 0 \text{ for } i, j = 1,2,3 \text{ with } i \neq j\\
&b_1 = b_2 = 0,\ b_3 = 4\sin x\, a(u)a'(u)\partial_t u\\
&c = 2\sin^2 x[a'(u)^2(\partial_t u)^2 + a(u)a''(u)(\partial_t u)^2].
\end{aligned}$$

We want to prove that both P_u and ${}^tP_u$ satisfy the a-priori estimate (4). From [8] it follows that

$$\|v\|^2_{\mu+\frac{1}{2}} \leq c\left(\sum_{j=1}^{3}\|X_j v\|^2_\mu + \|v\|^2_\mu\right) \qquad \forall v \in C^\infty(\mathbb{T}^3)$$

for some $c > 0$.

This implies, by standard arguments, the following a-priori estimate for the operator $\tilde{P} = \partial_x^2 + \partial_y^2 + \sin^2 x \partial_t^2$:

$$\sum_{i,j=1}^{3} \|X_i X_j v\|_\mu^2 + \sum_{j=1}^{3} \|X_j v\|_{\mu+\frac{1}{2}}^2 + \|v\|_{\mu+1} \le c' |\langle \tilde{P}v, v\rangle_\mu| + \|v\|_\mu^2 \tag{29}$$

for some $c' > 0$ and for all $v \in C^\infty(\mathbb{T}^3)$.

Since P_u defined by (27) can be written as $P_u = \tilde{P} + a^2(u) X_3^2$, we have that

$$\begin{aligned} |\langle \tilde{P}v, v\rangle_\mu| &\le |\langle P_u v, v\rangle_\mu| + |\langle a^2(u) X_3^2 v, v\rangle_\mu| \\ &\le \frac{1}{2}\|Pv\|_\mu^2 + \frac{1}{2}\|v\|_\mu^2 + \varepsilon \|a^2(u) X_3^2 v\|_\mu^2 + \frac{1}{4\varepsilon}\|v\|_\mu^2 \\ &\le \frac{1}{2}\|Pv\|_\mu^2 + \frac{2\varepsilon+1}{4\varepsilon}\|v\|_\mu^2 + \varepsilon K \|X_3^2 v\|_\mu^2 \end{aligned}$$

for some constant $K = K(D^\mu u) > 0$ and for all $\varepsilon > 0$. Substituting in (29) we obtain, for ε sufficiently small, the desired a-priori estimate (4) for P_u.

Analogously,

$$\begin{aligned} {}^tP_u = \tilde{P} &+ (a^2(u) + 2a(u)a'(u)u) X_3^2 + 4a(u)a'(u)(X_3 u) X_3 \\ &+ 2(a'(u)^2 + a(u)a''(u))(X_3 u)^2 \end{aligned}$$

and hence

$$\begin{aligned} |\langle \tilde{P}v, v\rangle_\mu| &\le |\langle {}^tP_u v, v\rangle_\mu| + |\langle (a^2(u) + 2a(u)a'(u)u) X_3^2 v, v\rangle_\mu| \\ &\quad + 4|\langle a(u)a'(u)(X_3 u) X_3 v, v\rangle_\mu| + 2|\langle (a'(u)^2 + a(u)a''(u))(X_3 u)^2 v, v\rangle_\mu| \\ &\le \frac{1}{2}\|{}^tP_u v\|_\mu^2 + \frac{1}{2}\|v\|_\mu^2 + \varepsilon \|(a^2(u) + 2a(u)a'(u)u) X_3^2 v\|_\mu^2 \\ &\quad + \frac{1}{4\varepsilon}\|v\|_\mu^2 + 4\varepsilon \|a(u)a'(u)(X_3 u) X_3 v\|_\mu^2 + \frac{1}{4\varepsilon}\|v\|_\mu^2 \\ &\quad + \|(a'(u)^2 + a(u)a''(u)))(X_3 u)^2 v\|_\mu^2 + \|v\|_\mu^2 \\ &\le \frac{1}{2}\|{}^tP_u v\|_\mu^2 + \left(\frac{3}{2} + \frac{1}{2\varepsilon}\right)\|v\|_\mu^2 + \varepsilon K \|X_3^2 v\|_\mu^2 + \varepsilon K \|X_3 v\|_\mu^2 + K\|v\|_\mu^2 \end{aligned}$$

for some constant $K = K(D^{\mu+1}u) > 0$ and for all $\varepsilon > 0$.

Substituting this estimate in (29) we obtain, for ε sufficiently small, the desired a-priori estimate (4) for ${}^tP_u$.

Thus both P_u and ${}^tP_u$ satisfy all the hypothesis of Theorem 1 and hence if $u \in C^\infty(\mathbb{T}^N)$ is a solution of $P_u u = f$ or of ${}^tP_u u = f$ for $f \in G^s(\mathbb{T}^N)$, with $s \ge 1$, then also $u \in G^s(\mathbb{T}^N)$.

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