Rend. Sem. Mat. Univ. Pol. Torino - Vol. 65, 4 (2007)

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ON ASSOCIATIVITY OF GRADED ALGEBRAS

Abstract. Necessary and sufficient conditions are given to guarantee associativity of algebras and co-algebras. As main application, it is discussed the associativity of the tensor algebra $T(V)$ of a free module V over any commutative ring R.

1. Review of basic definitions

Recall that a vector space A, or more generally a (free) module over a commutative ring R is an *R-algebra* if a multiplication map $m_A : A \otimes A \to A$ and a unit map $\eta_A : R \to A$ are defined with the following properties:

(1) $$m_A, \eta_A \text{ are } R\text{-linear;}$$

(2) $$m_A(\mathrm{id}_A \otimes m_A) = m_A(m_A \otimes \mathrm{id}_A) \qquad \text{(associativity);}$$

(3) $$m_A(\mathrm{id}_A \otimes \eta_A) = m_A(\eta_A \otimes \mathrm{id}_A) \cong \mathrm{id}_A \qquad \text{(unit).}$$

(2) $$\begin{array}{ccc} A\otimes A\otimes A & \xrightarrow{m_A\otimes id} & A\otimes A \\ \Big\downarrow{\scriptstyle \mathrm{id}_A\otimes m_A} & & \Big\downarrow{\scriptstyle m_A} \\ A\otimes A & \xrightarrow{m_A} & A \end{array}$$

(3) $$\begin{array}{ccccc} R\otimes A & \xrightarrow{\eta_A\otimes \mathrm{id}_A} & A\otimes A & \xleftarrow{\mathrm{id}_A\otimes\eta_A} & A\otimes R \\ & \lambda\searrow & \Big\downarrow{\scriptstyle m_A} & \swarrow\rho & \\ & & A & & \end{array}$$

where we denote by $\lambda : R\otimes A \xrightarrow{\sim} A$ and by $\rho : A\otimes R \xrightarrow{\sim} A$ the natural isomorphisms called left unit constraintand respectively right unit constraint. Throughout this paper $\otimes$ means $\otimes_R$. Note that (2) expresses the associativity of m:

$$\begin{aligned} &(a\cdot b)\cdot c = a\cdot(b\cdot c) \iff m_A(m_A(a\otimes b)\otimes c) = m_A(a\otimes m_A(b\otimes c)) \\ &\iff m_A\circ(m_A\otimes \mathrm{id}_A)((a\otimes b)\otimes c) = \\ &\qquad\qquad = m_A\circ(\mathrm{id}_A\otimes m_A)(a\otimes(b\otimes c)) \quad \forall a,b,c\in A \\ &\iff m_A(m_A\otimes\mathrm{id}_A) = m_A(\mathrm{id}_A\otimes m_A). \end{aligned}$$

(3) says that A has a unit element $1_A = \eta_A(1) =$identity element:

$$\begin{aligned} a\cdot 1_A &= a\cdot\eta_A(1) = m_A(\mathrm{id}_A\otimes\eta_A)(a\otimes 1) = 1_A\otimes a = \\ &= m_A(\eta_A\otimes\mathrm{id}_A)(1\otimes a) \stackrel{\lambda}{=} a \stackrel{\rho}{=} a \qquad \forall a\in A. \end{aligned}$$

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*Partially supported by MURST.

We write $A=(A,m_A,\eta_A)$ to denote an associative algebra over a commutative ring R. Also recall that dually a module over a commutative ring R is an *R-co-algebra* if a co-multiplication map $\Delta_A : A \to A\otimes A$ and a co-unit map $\varepsilon_A : A \to R$ are defined with the following properties:

(4) Δ_A, ε_A are R-linear;

(5) $(\mathrm{id}_A \otimes \Delta_A)\Delta_A = (\Delta_A \otimes \mathrm{id}_A)\Delta_A$ (co-associativity);

(6) $(\mathrm{id}_A \otimes \varepsilon_A)\Delta_A = (\varepsilon_A \otimes \mathrm{id}_A)\Delta_A$ (co-unit).

We write $A=(A,\Delta_A,\varepsilon_A)$ to denote a co-associative algebra A over a commutative ring R.Let (A,m_A,η_A), (B,m_B,η_B) be two R-algebras;then the tensor product $A\otimes B$ over R of the R-algebras A and B is defined in the customary manner by taking the underlying R-module to be the R-module $A\otimes B$ and setting $m_{A\otimes B}$ to be the composition:

(7) $$A\otimes B\otimes A\otimes B \stackrel{\mathrm{id}_A\otimes\tau_{B,A}\otimes \mathrm{id}_B}{\longrightarrow} A\otimes A\otimes B\otimes B \stackrel{m_A\otimes m_B}{\longrightarrow} A\otimes B$$

where $\tau_{B,A} : B\otimes A \to A\otimes B$ is the R-map defined by $\tau_{B,A}(b\otimes a) = a\otimes b$ for $a\in A$, $b\in B$ (τ is the *twisting morphism*). The unit map $\eta_{A\otimes B}$ is given by:

(8) $$\eta_A\otimes\eta_B : R \stackrel{\sim}{\longrightarrow} R\otimes R \to A\otimes B.$$

In particular $1_A\otimes 1_B = 1_{A\otimes B}$ is the identity element for the algebra $A\otimes B$.Similarly if $(A,\Delta_A,\varepsilon_A)$, $(B,\Delta_B,\varepsilon_B)$ are two R-co-algebras, then the tensor product over R of the R-co-algebras A and B is again a co-algebra with co-multiplication $\Delta_{A\otimes B}$ given by the composition:

(9) $$A\otimes B \stackrel{\Delta_A\otimes\Delta_B}{\longrightarrow} A\otimes A\otimes B\otimes B \stackrel{1_A\otimes\tau_{A,B}\otimes 1_B}{\longrightarrow} A\otimes B\otimes A\otimes B$$

and the co-unit map $\varepsilon_{A\otimes B}$ is given by:

(10) $$\eta_A\otimes\eta_B : A\otimes B \to R\otimes R \stackrel{\sim}{\longrightarrow} R.$$

In what follows we will identify any R-algebra A with the tensor products $A\otimes R \cong R\otimes A$ via ρ and λ respectively. We will also use id_A for the identity map of A. From the customary definitions (7), (9) it is clear that in order to ensure the R-module $A\otimes B$ with an algebra (resp. co-algebra) structure we need the morphism τ which makes possible to define tensor products of algebras (resp. co-algebras). In what follows we give a criterion which allows any R-linear map β to play the same role as τ. We start with some known classical notation and identities for braiding (see [6, 7]) which here instead are used as definitions.

2. The associativity and co-associativity conditions on β

DÉFINITION 2. *Let X,Y be R-algebras and let $\beta\in \mathrm{Hom}_R(X\otimes Y, Y\otimes X)$ be any R-linear map. Let by definition*

$$\beta_{X\otimes X,Y} = (\beta\otimes \mathrm{id}_X)(\mathrm{id}_X\otimes\beta) : X\otimes X\otimes Y \to Y\otimes X\otimes X,$$
$$\beta_{X,Y\otimes Y} = (\mathrm{id}_Y\otimes\beta)(\beta\otimes \mathrm{id}_Y) : X\otimes Y\otimes Y \to Y\otimes Y\otimes X.$$

LEMME 4. *The following relations are satisfied:*

(4.1) $(\mathrm{id}_Y \otimes \beta_{X\otimes X,Y})(\beta \otimes \mathrm{id}_{X\otimes Y}) = (\beta_{X,Y\otimes Y} \otimes \mathrm{id}_X)(\mathrm{id}_{X\otimes Y} \otimes \beta)$,

(4.2) $(m_Y \otimes m_X)(\eta_Y \otimes \beta \otimes \eta_X) = \beta$.

Proof. These are direct applications of the definitions. □

PROPOSITION 1. *Let* (A, m_A, η_A) *and* (B, m_B, η_B) *two associative R-algebras with identity elements* 1_A *and* 1_B *respectively. Let* $\beta \in \mathrm{Hom}_R(B \otimes A, A \otimes B)$ *be any R-linear map such that:*

(1.1) $(\mathrm{id}_A \otimes m_B)\beta_{B\otimes B,A} = \beta(m_B \otimes \mathrm{id}_A)$,

(1.2) $(m_A \otimes \mathrm{id}_B)\beta_{B,A\otimes A} = \beta(\mathrm{id}_B \otimes m_A)$,

then the morphism $\tilde{m}_{A\otimes B} = (m_A \otimes m_B)(\mathrm{id}_A \otimes \beta \otimes \mathrm{id}_B)$ *determines* [*] *an* associative *R-algebra structure on* $A \otimes B$*. Conversely assume that* $\tilde{m}_{A\otimes B}$ *defines an associative multiplication on* $A \otimes B$*. If* β *satisfies the following additional conditions* [†]*:*

(1.3) $\beta(\mathrm{id}_B \otimes \eta_A) = \tau(\mathrm{id}_B \otimes \eta_A)$,

(1.4) $\beta(\eta_B \otimes \mathrm{id}_A) = \tau(\eta_B \otimes \mathrm{id}_A)$,

then β *satisfies conditions (1.1) and (1.2).*

Proof. We need to prove first that

$$(1.1), (1.2) \implies \tilde{m}_{A\otimes B}(\tilde{m}_{A\otimes B} \otimes \mathrm{id}_{A\otimes B}) = \tilde{m}_{A\otimes B}(\mathrm{id}_{A\otimes B} \otimes \tilde{m}_{A\otimes B}).$$

Calculation of $\tilde{m}_{A\otimes B}(\tilde{m}_{A\otimes B} \otimes \mathrm{id}_{A\otimes B})$:

$$(\bullet)\quad \tilde{m}_{A\otimes B}(\tilde{m}_{A\otimes B} \otimes \mathrm{id}_{A\otimes B}) = (m_A \otimes m_B)(\mathrm{id}_A \otimes \beta \otimes \mathrm{id}_B)$$
$$[(m_A \otimes m_B)(\mathrm{id}_A \otimes \beta \otimes \mathrm{id}_B) \otimes \mathrm{id}_{A\otimes B}] =$$
$$= (m_A \otimes m_B)(\mathrm{id}_A \otimes \beta \otimes \mathrm{id}_B)\,[(m_A \otimes m_B \otimes \mathrm{id}_{A\otimes B})$$
$$(\mathrm{id}_A \otimes \beta \otimes \mathrm{id}_B \otimes \mathrm{id}_{A\otimes B})] =$$
$$= (m_A \otimes m_B)\,[(\mathrm{id}_A \otimes \beta \otimes \mathrm{id}_B)\,[m_A \otimes (m_B \otimes \mathrm{id}_A) \otimes \mathrm{id}_B]]$$
$$(\mathrm{id}_A \otimes \beta \otimes \mathrm{id}_B \otimes \mathrm{id}_{A\otimes B}) =$$
$$= (m_A \otimes m_B)\,[\mathrm{id}_A \circ m_A \otimes \beta(m_B \otimes \mathrm{id}_A) \otimes \mathrm{id}_B \circ \mathrm{id}_B]$$
$$(\mathrm{id}_A \otimes \beta \otimes \mathrm{id}_{B\otimes A\otimes B}) =$$
$$= (m_A \otimes m_B)\,[m_A \otimes \beta(m_B \otimes \mathrm{id}_A) \otimes \mathrm{id}_B]\,(\mathrm{id}_A \otimes \beta \otimes \mathrm{id}_{B\otimes A\otimes B}).$$

[*]Just before submitting this article the author discovered that part of Proposition 1 was stated without proof in[4].

[†]In a letter to the authorJ. A. Green offered helpful remarks concerning these conditions.

Calculation of $\tilde{m}_{A\otimes B}(\mathrm{id}_{A\otimes B}\otimes\tilde{m}_{A\otimes B})$:

$$(\bullet\bullet)\quad \tilde{m}_{A\otimes B}(\mathrm{id}_{A\otimes B}\otimes\tilde{m}_{A\otimes B}) = (m_A\otimes m_B)(\mathrm{id}_A\otimes\beta\otimes\mathrm{id}_B)$$
$$[\mathrm{id}_{A\otimes B}\otimes(m_A\otimes m_B)(\mathrm{id}_A\otimes\beta\otimes\mathrm{id}_B)] =$$
$$= (m_A\otimes m_B)(\mathrm{id}_A\otimes\beta\otimes\mathrm{id}_B)\,[(\mathrm{id}_{A\otimes B}\otimes m_A\otimes m_B)]\,.$$
$$[(\mathrm{id}_{A\otimes B\otimes A}\otimes\beta\otimes\mathrm{id}_B)] =$$
$$= (m_A\otimes m_B)(\mathrm{id}_A\otimes\beta\otimes\mathrm{id}_B)\,[\mathrm{id}_A\otimes(\mathrm{id}_B\otimes m_A)\otimes m_B]$$
$$(\mathrm{id}_{A\otimes B\otimes A}\otimes\beta\otimes\mathrm{id}_B) =$$
$$= (m_A\otimes m_B)\,[\mathrm{id}_A\otimes\beta(\mathrm{id}_B\otimes m_A)\otimes m_B]\,(\mathrm{id}_{A\otimes B\otimes A}\otimes\beta\otimes\mathrm{id}_B).$$

Assume now that m_A, m_B are associative and (1.1),(1.2) hold. Then:

$$(\bullet) = (m_A\otimes m_B)\,[m_A\otimes\beta(m_B\otimes\mathrm{id}_A)\otimes\mathrm{id}_B]\,(\mathrm{id}_A\otimes\beta\otimes\mathrm{id}_{B\otimes A\otimes B}) =$$
$$= (m_A\otimes m_B)\,[m_A\otimes(\mathrm{id}_A\otimes m_B)\beta_{B\otimes B,A}\otimes\mathrm{id}_B]$$
$$(\mathrm{id}_A\otimes\beta\otimes\mathrm{id}_{B\otimes A\otimes B}) =$$
$$= (m_A\otimes m_B)\,[m_A\circ(\mathrm{id}_A\otimes\mathrm{id}_A)\otimes(\mathrm{id}_A\otimes m_B)\beta_{B\otimes B,A}\otimes\mathrm{id}_B\circ\mathrm{id}_B]$$
$$(\mathrm{id}_A\otimes\beta\otimes\mathrm{id}_{B\otimes A\otimes B}) =$$
$$= (m_A\otimes m_B)\,[[m_A\otimes(\mathrm{id}_A\otimes m_B)\otimes\mathrm{id}_B]\circ(\mathrm{id}_{A\otimes A}\otimes\beta_{B\otimes B,A}\otimes\mathrm{id}_B)$$
$$(\mathrm{id}_A\otimes\beta\otimes\mathrm{id}_{B\otimes A\otimes B})] =$$
$$= (m_A\otimes m_B)\,[(m_A\otimes\mathrm{id}_A)(m_B\otimes\mathrm{id}_B)]\,[\mathrm{id}_A\otimes(\mathrm{id}_A\otimes\beta_{B\otimes B,A})\otimes\mathrm{id}_B)]$$
$$[\mathrm{id}_A\otimes(\beta\otimes\mathrm{id}_{B\otimes A})\otimes\mathrm{id}_B] =$$
$$= [m_A(m_A\otimes\mathrm{id}_A)\otimes m_B(m_B\otimes\mathrm{id}_B)]\,[\mathrm{id}_A\otimes(\mathrm{id}_A\otimes\beta_{B\otimes B,A})$$
$$(\beta\otimes\mathrm{id}_{B\otimes A})\otimes\mathrm{id}_B)]\,.$$

Also

$$(\bullet\bullet) = (m_A\otimes m_B)\,[\mathrm{id}_A\otimes\beta(\mathrm{id}_B\otimes m_A)\otimes m_B]\,(\mathrm{id}_{A\otimes B\otimes A}\otimes\beta\otimes\mathrm{id}_B) =$$
$$= (m_A\otimes m_B)\,[\mathrm{id}_A\otimes(m_A\otimes\mathrm{id}_B)\beta_{B,A\otimes A}\otimes m_B]$$
$$(\mathrm{id}_{A\otimes B\otimes A}\otimes\beta\otimes\mathrm{id}_B) =$$
$$= (m_A\otimes m_B)\,[\mathrm{id}_A\circ\mathrm{id}_A\otimes(m_A\otimes\mathrm{id}_B)\beta_{B,A\otimes A}\otimes m_B\circ(\mathrm{id}_B\otimes\mathrm{id}_B)]$$
$$(\mathrm{id}_{A\otimes B\otimes A}\otimes\beta\otimes\mathrm{id}_B) =$$
$$= (m_A\otimes m_B)\,[\mathrm{id}_A\otimes(m_A\otimes\mathrm{id}_B)\otimes m_B]\circ(\mathrm{id}_A\otimes\beta_{B,A\otimes A}\otimes\mathrm{id}_{B\otimes B})$$
$$(\mathrm{id}_{A\otimes B\otimes A}\otimes\beta\otimes\mathrm{id}_B) =$$
$$= (m_A\otimes m_B)\,[(\mathrm{id}_A\otimes m_A)(\mathrm{id}_B\otimes m_B)]\circ[id_A\otimes(\beta_{B,A\otimes A}\otimes\mathrm{id}_B)$$
$$(\mathrm{id}_{B\otimes A}\otimes\beta)\otimes\mathrm{id}_B] =$$
$$= [m_A(\mathrm{id}_A\otimes m_A)\otimes m_B(\mathrm{id}_B\otimes m_B)]\,[\mathrm{id}_A\otimes(\beta_{B,A\otimes A}\otimes\mathrm{id}_B)$$
$$(\mathrm{id}_{B\otimes A}\otimes\beta)\otimes\mathrm{id}_B)]\,.$$

Now associativity of m_A and m_B gives:

$$m_A(\mathrm{id}_A\otimes m_A) = m_A(m_A\otimes\mathrm{id}_A),\quad m_B(\mathrm{id}_B\otimes m_B) = m_B(m_B\otimes\mathrm{id}_B).$$

Also (4.1) gives:

$$(\mathrm{id}_A \otimes \beta_{B\otimes B,A})(\beta \otimes \mathrm{id}_{B\otimes A}) = (\beta_{B,A\otimes A} \otimes \mathrm{id}_B)(\mathrm{id}_{B\otimes A} \otimes \beta).$$

This proves that:

"(1.1), (1.2), associativity of m_A and of m_B

$\Longrightarrow$ associativity of $\tilde{m}_{A\otimes B}$."

Now assume that $\tilde{m}_{A\otimes B}$[‡] is associative and consider ($\bullet\bullet$) above restricted to the sub-R-algebra $R\otimes B\otimes R\otimes B\otimes A\otimes R \cong B\otimes B\otimes A$. Then we have

$$\begin{aligned}
&\tilde{m}(\mathrm{id}_{A\otimes B}\otimes\tilde{m})(\eta_A\otimes\mathrm{id}_B\otimes\eta_A\otimes\mathrm{id}_B\otimes\mathrm{id}_A\otimes\eta_B) = \\
&= \tilde{m}\,[\mathrm{id}_{A\otimes B}\otimes(m_A\otimes m_B)(\mathrm{id}_A\otimes\beta\otimes\mathrm{id}_B)] \\
&\qquad\qquad (\eta_A\otimes\mathrm{id}_B\otimes\eta_A\otimes\mathrm{id}_B\otimes\mathrm{id}_A\otimes\eta_B) = \\
&= \tilde{m}\,[(\mathrm{id}_{A\otimes B}\otimes m_A\otimes m_B)(\mathrm{id}_{A\otimes B\otimes A}\otimes\beta\otimes\mathrm{id}_B)] \\
&\qquad\qquad [\eta_A\otimes\mathrm{id}_B\otimes\eta_A\otimes(\mathrm{id}_B\otimes\mathrm{id}_A)\otimes\eta_B] = \\
&= \tilde{m}(\mathrm{id}_{A\otimes B}\otimes m_A\otimes m_B)\,[\eta_A\otimes\mathrm{id}_B\otimes\eta_A\otimes\beta(\mathrm{id}_B\otimes\mathrm{id}_A)\otimes\eta_B] = \\
&= \tilde{m}\,[\mathrm{id}_{A\otimes B}\otimes(m_A\otimes m_B)]\,[\eta_A\otimes\mathrm{id}_B\otimes(\eta_A\otimes\beta\otimes\eta_B)] = \\
&= \tilde{m}\,[\eta_A\otimes\mathrm{id}_B\otimes(m_A\otimes m_B)(\eta_A\otimes\beta\otimes\eta_B)] \overset{(4.2)}{=} \tilde{m}(\eta_A\otimes\mathrm{id}_B\otimes\beta) = \\
&= (m_A\otimes m_B)\,[\mathrm{id}_A\otimes(\beta\otimes\mathrm{id}_B)]\,[\eta_A\otimes(\mathrm{id}_B\otimes\beta)] = \\
&= (m_A\otimes m_B)\,[\eta_A\otimes(\beta\otimes\mathrm{id}_B)(\mathrm{id}_B\otimes\beta)] = \\
&\overset{\mathrm{def}}{=} (m_A\otimes m_B)(\eta_A\otimes\beta_{B\otimes B,A}) = (\mathrm{id}_A\otimes m_B)(\beta_{B\otimes B,A}).
\end{aligned}$$

Similarly consider ($\bullet$) restricted to $R\otimes B\otimes R\otimes B\otimes A\otimes R \cong B\otimes B\otimes A$. Then we have:

$$\begin{aligned}
&\tilde{m}(\tilde{m}\otimes\mathrm{id}_{A\otimes B})(\eta_A\otimes\mathrm{id}_B\otimes\eta_A\otimes\mathrm{id}_B\otimes\mathrm{id}_A\otimes\eta_B) = \tilde{m}\,[(m_A\otimes m_B) \\
&\qquad (\mathrm{id}_A\otimes\beta\otimes\mathrm{id}_B)\otimes\mathrm{id}_{A\otimes B}]\,(\eta_A\otimes\mathrm{id}_B\otimes\eta_A\otimes\mathrm{id}_B\otimes\mathrm{id}_A\otimes\eta_B) = \\
&= \tilde{m}(m_A\otimes m_B\otimes\mathrm{id}_{A\otimes B})(\mathrm{id}_A\otimes\beta\otimes\mathrm{id}_{B\otimes A\otimes B}) \\
&\qquad\qquad (\eta_A\otimes\mathrm{id}_B\otimes\eta_A\otimes\mathrm{id}_B\otimes\mathrm{id}_A\otimes\eta_B) = \\
&= \tilde{m}(m_A\otimes m_B\otimes\mathrm{id}_{A\otimes B})\,[\eta_A\otimes\beta(\mathrm{id}_B\otimes\eta_A)\otimes\mathrm{id}_{B\otimes A}\otimes\eta_B)] = \\
&\overset{(1.3)}{=} \tilde{m}(m_A\otimes m_B\otimes\mathrm{id}_{A\otimes B})\,[\eta_A\otimes\beta(\mathrm{id}_B\otimes\eta_A)\otimes\mathrm{id}_{B\otimes A}\otimes\eta_B] = \\
&= \tilde{m}(m_A\otimes m_B\otimes\mathrm{id}_{A\otimes B})\,[\eta_A\otimes\tau(\mathrm{id}_B\otimes\eta_A)\otimes\mathrm{id}_{B\otimes A}\otimes\eta_B] = \\
&= \tilde{m}(\eta_A\otimes m_B\otimes\mathrm{id}_A\otimes\eta_B) =
\end{aligned}$$

$$\begin{aligned}
&= (m_A\otimes m_B)(\mathrm{id}_A\otimes\beta\otimes\mathrm{id}_B)(\eta_A\otimes m_B\otimes\mathrm{id}_A\otimes\eta_B) = \\
&= (m_A\otimes m_B)[\eta_A\otimes\beta(m_B\otimes\mathrm{id}_A)\otimes\eta_B] = \beta(m_B\otimes\mathrm{id}_A).
\end{aligned}$$

[‡]For short we will often write $\tilde{m}$ if no confusion occurs.

This proves that:

$$\text{“Associativity of } \tilde{m} \text{ and (1.3)} \implies \text{(1.1).”}$$

In a similar fashion one proves that:

$$\text{“Associativity of } \tilde{m} \text{ and (1.4)} \implies \text{(1.2).”}$$

□

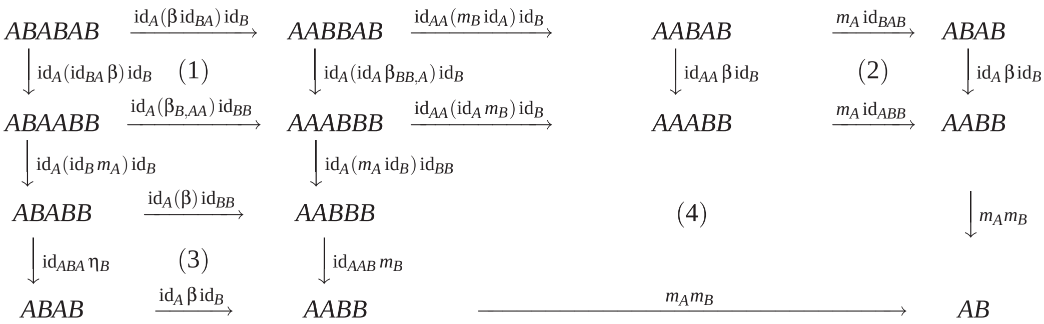

Figure 1: The commutative diagram expressing the associativity of $\tilde{m}$. The regions (1), (2), (3) and (4) in the diagram represent always commutative sub-diagrams. (For short we omit the $\otimes$ symbols in the above diagram.)

COROLLAIRE 1. *Let A, B be R-algebras and let* $\beta \in \mathrm{Hom}_R(B \otimes A, A \otimes B)$ *such that for all* $a \in A$ *and all* $b \in B$ *the following relations:*

$$\beta(1_B \otimes a) = a \otimes 1_B, \qquad \beta(b \otimes 1_A) = 1_A \otimes b$$

are satisfied. Then:

“$\tilde{m}_{A\otimes B}$ *is associative if and only if (1.1), (1.2) hold.*”

Dual to Proposition 1 we have the following:

PROPOSITION 2. *Let* $(A, \Delta_A, \varepsilon_A)$, $(B, \Delta_B, \varepsilon_B)$ *be two R-co-algebras and let* $\beta \in \mathrm{Hom}_R(A \otimes B, B \otimes A)$ *be any R-linear map such that:*

(2.1) $\beta_{A,B\otimes B}(\mathrm{id}_A \otimes \Delta_B) = (\Delta_B \otimes \mathrm{id}_A)\beta,$

(2.2) $\beta_{A\otimes A,B}(\Delta_A \otimes \mathrm{id}_B) = (\mathrm{id}_B \otimes \Delta_A)\beta,$

then the morphism $\tilde{\Delta}_{A\otimes B} = (\mathrm{id}_A \otimes \beta \otimes \mathrm{id}_B)(\Delta_A \otimes \Delta_B)$ *determines a co-associative co-multiplication on* $A \otimes B$*. Conversely assume that* $\Delta_{A\otimes B}$ *defines a co-associative co-multiplication on* $A \otimes B$*. If* β *satisfies the following additional conditions:*

(2.3) $(\mathrm{id}_B \otimes \varepsilon_A)\beta = (\mathrm{id}_B \otimes \varepsilon_A)\tau$,

(2.4) $(\varepsilon_B \otimes \mathrm{id}_A)\beta = (\varepsilon_B \otimes \mathrm{id}_A)\tau$,

then β *satisfies conditions (2.1) and (2.2).*

Proof. The proof is dual to 1. □

3. The main example

Recall that an R-algebra $A = (A, m_A, \eta_B)$ is a *graded R-algebra* if the R-module A is a direct sum of submodules A_i ($i \geq 0$) such that:

(11) $$m_A(A_i \otimes A_j) \subseteq A_{i+j} \quad \text{and} \quad \eta_A(R) \cong A_0.$$

Dually, an R-co-algebra $A = (A, \Delta_A, \varepsilon_A)$ is a *graded R-co-algebra* if $A = \sum_{i \geq 0} A_i$ as R-modules such that

(12) $$\Delta_A(A_k) \subseteq \sum_{i+j=k} A_i \otimes A_j \quad \text{and} \quad \varepsilon_A(A_0) \cong R.$$

In this section we will discuss the following situation. $A = \sum_{i \geq 0} A_i$ be a graded R-algebra and let $F : A \otimes A \to A \otimes A$ be any R-linear map such that:

(13) $$F(A_i \otimes A_j) \subseteq A_j \otimes A_i,$$

(14) $$F(1_R \otimes a) = a \otimes 1_R, F(a \otimes 1_R) = 1_R \otimes a \quad \text{for all } a \in A.$$

Statement of the problem:

PROBLEM 15. Study the graded R-algebras (resp. R-co-algebras) for which the map $\tilde{m}_{A \otimes A}$ (resp. $\tilde{\Delta}_{A \otimes A}$) defined in proposition 1 is associative (resp. co-associative).

DESCRIPTION OF THE MAIN EXAMPLE 16. Let V be a free R-module and let $A = T(V) = \sum_{r \geq 0} T_r(V)$be the tensor algebra on V. A is a graded R-algebra with $A_i = T_i(V) = V^{\otimes i}$, with multiplication $m_A : A_i \otimes A_j \to A_{i+j}$ given by $m_A(x \otimes y) = x \otimes y$ and with unit $\eta_A : R \to A$ given by $\eta_A(r) = r \in A_0 = T_0(V) = R$ for all $r \in R$.

Consider the following standard definition of β_i, see [5].

DEFINITIONS 17. Let $\beta \in \mathrm{End}_R(V \otimes V)$ and β_i given by

(18) $$\beta_i = \mathrm{id}_V^{\otimes i-1} \otimes \beta \otimes \mathrm{id}_V^{\otimes n-i-1} \in \mathrm{End}_K(V^{\otimes n}).$$

For any i, j with $i, j > 0$, we can define now $\beta_{i,j} \in \mathrm{Hom}_R(A_i \otimes A_j, A_j \otimes A_i)$ as follows:

(19) $$\beta_{i,j} = (\beta_j \beta_{j+1} \cdots \beta_{i+j-1})(\beta_{j-1} \beta_j \cdots \beta_{i+j-2}) \cdots (\beta_1 \beta_2 \cdots \beta_i).$$

For example:

$$\beta_{3,5} = (\beta_5\beta_6\beta_7)(\beta_4\beta_5\beta_6)(\beta_3\beta_4\beta_5)(\beta_2\beta_3\beta_4)(\beta_1\beta_2\beta_3),$$
$$\beta_{5,3} = (\beta_3\beta_4\beta_5\beta_6\beta_7)(\beta_2\beta_3\beta_4\beta_5\beta_6)(\beta_1\beta_2\beta_3\beta_4\beta_5).$$

We have:

PROPOSITION 3. *If β_i, β_j are as in* (18)*, then the following relations hold:*

(i) $|i-j| \geq 2 \Longrightarrow \beta_i\beta_j = \beta_j\beta_i$*;*

(ii) if $i = i_1 + i_2$*;* $i_1, i_2 \geq 1 \Longrightarrow \beta_{i,j} = (\beta_{i_1,j} \otimes \mathrm{id}_{A_{i_2}})(\mathrm{id}_{A_{i_1}} \otimes \beta_{i_2,j})$*;*

(iii) if $j = j_1 + j_2$*;* $j_1, j_2 \geq 1 \Longrightarrow \beta_{i,j} = (\mathrm{id}_{A_{j_1}} \otimes \beta_{i,j_2})(\beta_{i,j_1} \otimes \mathrm{id}_{A_{j_2}})$.

Proof. (i) is trivial. In fact assume for example $i < j$ then

$$\begin{aligned}\beta_i\beta_j &= (\mathrm{id}_V^{\otimes i-1} \otimes \beta \otimes \mathrm{id}_V^{\otimes n-i-1})(\mathrm{id}_V^{\otimes j-1} \otimes \beta \otimes \mathrm{id}_V^{\otimes n-j-1})\\ &= \mathrm{id}_V^{\otimes i-1} \otimes \beta \otimes \mathrm{id}_V^{\otimes j-1} \otimes \beta \otimes \mathrm{id}_V^{\otimes n-j-1} = \beta_j\beta_i.\end{aligned}$$

Similarly for $i > j$.(ii) By definition:

$$\begin{aligned}(\beta_{i_1,j} \otimes \mathrm{id}_{A_{i_2}}) &= [(\beta_j\beta_{j+1}\cdots\beta_{j+i_1-1})(\beta_{j-1}\beta_j\cdots\beta_{j+i_1-2})\\ &\qquad \cdots(\beta_1\beta_2\cdots\beta_{i_1})] \otimes \mathrm{id}_{A_{i_2}}\\ (\mathrm{id}_{A_{i_1}} \otimes \beta_{i_2,j}) &= \mathrm{id}_{A_{i_1}} \otimes [(\beta_j\beta_{j+1}\cdots\beta_{j+i_2-1})(\beta_{j-1}\beta_j\cdots\beta_{j+i_2-2})\\ &\qquad \cdots(\beta_1\beta_2\cdots\beta_{i_2})] =\\ &= (\beta_{j+i_1}\beta_{j+i_1+1}\cdots\beta_{j+i_1+i_2-1})(\beta_{j+i_1-1}\beta_{j+i_1}\cdots\beta_{j+i_1+i_2-2})\\ &\qquad \cdots(\beta_{i_1+1}\beta_{i_1+2}\cdots\beta_{i_1+i_2})\end{aligned}$$

For $k = 1, 2, \ldots, j$ let

$$b_k = \beta_{j-k+1}\beta_{j-k+2}\cdots\beta_{j-k+i_1}, \quad b'_k = \beta_{j-k+i_1+1}\beta_{j-k+i_1+2}\cdots\beta_{j-k+i_1+i_2}$$

then

$$(\beta_{i_1,j} \otimes \mathrm{id}_{A_{i_2}})(\mathrm{id}_{A_{i_1}} \otimes \beta_{i_2,j}) = b_1b_2\cdots b_jb'_1b'_2\cdots b'_j.$$

Notice that the "words" b_k have length $i_1 \geq 1$,while the "words" b'_k have length $i_2 \geq 1$. Notice also that by definition:

$$\beta_{i,j} = b_1b'_1b_2b'_2b_3b'_3\cdots b_jb'_j.$$

So in order to prove (ii) we need to prove that the relation:

$$(b_1b_2\cdots b_j)(b'_1b'_2\cdots b'_j) = b_1b'_1b_2b'_2\cdots b_jb'_j \tag{20}$$

is true for all $j \geq 1$.Now for $j = 1$ (20) is trivial since it reduces just to

$$b_1b'_1 = \beta_{i,1} = (\beta_1\beta_2\cdots\beta_{i_1})(\beta_{i_1+1}\cdots\beta_{i_1+i_2})$$

which holds by definition of $\beta_{i,1}$. Assume now that $j \geq 2$, then the first factor of b'_1 which is β_{j+i_1} and the last factor of b_j which is β_{i_1} commute since $|j+i_1-i_1| = j \geq 2$. It follows that:

$$(b_1b_2\cdots b_j)(b'_1b'_2\cdots b'_j) = (b_1b_2\cdots b_{j-1}b'_1b_j)(b'_2b'_3\cdots b'_j).$$

If $j=2$, we are done. If $j \geq 3$ we can bring b'_1 up to the position of b_{j-1} and so on up to the position of b_2. More precisely $|j+i_1-(j+i_1-2)| = 2 \Longrightarrow$

$$(b_1b_2)b_3\cdots b_j(b'_1)b'_2\cdots b'_j = (b_1b'_1b_2)b_3\cdots b_jb'_2\cdots b'_j,$$

$|(j+i_1-1)-(j+i_1-3)| = 2 \Longrightarrow$

$$(b_1b_2b_3)b_4\cdots b_jb'_1(b'_2)b'_3\cdots b'_j = (b_1b'_1b_2b'_2)b_3b_4\cdots b_jb'_3\cdots b'_j,$$

etc.In general the first factor β_{j+i_1-k+1} of b'_k and the last factor β_{j+i_1-k+2} of b_{k+1} are such that $|j+i_1-k-(j+i_1-k+2)| = 2$. This implies that:

$$b_{k+1}(b_{k+2}\cdots b_j)b'_k = b'_kb_{k+1}(b_{k+2}\cdots b_j),$$

hence

$$(b_1b_2\cdots b_j)(b'_1b'_2\cdots b'_j) = b_1b'_1b_2b'_2\cdots b_jb'_j = \beta_{i,j}.$$

We illustrate the above argument with the following example. Let $j=4$, $i=5$, $i_1=2$, $i_2=3$. Then

$$\begin{aligned}\beta_{5,4} &\stackrel{\text{def}}{=} (\beta_4\beta_5\beta_6\beta_7\beta_8)(\beta_3\beta_4\beta_5\beta_6\beta_7)(\beta_2\beta_3\beta_4\beta_5\beta_6)(\beta_1\beta_2\beta_3\beta_4\beta_5) = \\ &= [\underbrace{(\beta_4\beta_5)}_{b_1}\underbrace{(\beta_6\beta_7\beta_8)}_{b'_1}][\underbrace{(\beta_3\beta_4)}_{b_2}\underbrace{(\beta_5\beta_6\beta_7)}_{b'_2}][\underbrace{(\beta_2\beta_3)}_{b_3}\underbrace{(\beta_4\beta_5\beta_6)}_{b'_3}][\underbrace{(\beta_1\beta_2)}_{b_4}\underbrace{(\beta_3\beta_4\beta_5)}_{b'_4}]\end{aligned}$$

i.e. we associate each factor with i-term in the above expression,into sub-factors with i_1 and i_2 terms. In our case $i_1=2$ and $i_2=3$. From the proof given before we have that:

$$\begin{aligned}&b'_1 \text{ commutes with } b_2,b_3,b_4 \Rightarrow b_1\underline{b'_1b_2}b'_2b_3b'_3b_4b'_4 = b_1b_2b'_1b'_2b_3b'_3b_4b'_4\\ &b'_2 \text{ commutes with } b_3,b_4 \Rightarrow b_1b_2b'_1\underline{b'_2b_3}b'_3b_4b'_4 = b_1b_2b'_1b_3b'_2b'_3b_4b'_4\\ &b'_3 \text{ commutes with } b_4 \Rightarrow b_1b_2b'_1b_3b'_2\underline{b'_3b_4}b'_4 = b_1b_2b'_1b_3b'_2b_4b'_3b'_4\end{aligned}$$

$$\begin{aligned}b'_1b_3 = b_3b'_1 &\Rightarrow b_1b_2\underline{b'_1b_3}b'_2b_4b'_3b'_4 = b_1b_2b_3b'_1b'_2b_4b'_3b'_4\\ b'_2b_4 = b_4b'_2 &\Rightarrow b_1b_2b_3b'_1\underline{b'_2b_4}b'_3b'_4 = b_1b_2b_3b'_1b_4b'_2b'_3b'_4\\ b'_1b_4 = b_4b'_1 &\Rightarrow b_1b_2b_3\underline{b'_1b_4}b'_2b'_3b'_4 = b_1b_2b_3b_4b'_1b'_2b'_3b'_4.\end{aligned}$$

(iii) Let $j = j_1 + j_2$, then we want to prove that

$$\beta_{i,j} = (\mathrm{id}_{A_{j_1}} \otimes \beta_{i,j_2})(\beta_{i,j_1} \otimes \mathrm{id}_{A_{j_2}}).$$

But

$$(\mathrm{id}_{A_{j_1}} \otimes \beta_{i,j_2}) = \mathrm{id}_{A_{j_1}} \otimes [(\beta_{j_2}\beta_{j_2+1}\cdots\beta_{j_2+i-1})(\beta_{j_2-1}\beta_{j_2}\cdots\beta_{j_2+i_1-2}) \cdots(\beta_1\beta_2\cdots\beta_i)] =$$
$$= (\beta_j\beta_{j+1}\cdots\beta_{j+i-1})(\beta_{j-1}\beta_j\cdots\beta_{j+i-2})\cdots(\beta_{j_1+1}\beta_{j_1+2}\cdots\beta_{j_1+i})$$
$$(\beta_{i,j_1} \otimes \mathrm{id}_{A_{j_2}}) = (\beta_{j_1}\beta_{j_1+1}\cdots\beta_{j_1+i-1})(\beta_{j_1-1}\beta_{j_1}\cdots\beta_{j_1+i-2}) \cdots(\beta_1\beta_2\cdots\beta_i).$$

Then

$$(\mathrm{id}_{A_{j_1}} \otimes\beta_{i,j_2})(\beta_{i,j_1} \otimes \mathrm{id}_{A_{j_2}}) = [(\beta_j\cdots\beta_{j+i-1})\cdots(\beta_{j_1+1}\cdots\beta_{j_1+i})] [(\beta_{j_1}\cdots\beta_{j_1+i-1})\cdots(\beta_1\beta_2\cdots\beta_i)] = \beta_{i,j}$$

□

REMARQUE 1. Note that $\beta_{2,2} = \beta_{A_1\otimes A_2,A_1} = \beta_{A_1,A_1\otimes A_2}$.

Now we are ready to describe the case $A = T(V)$.

THEOREM 1. *Let $\beta \in \mathrm{End}_R(V \otimes V)$ be any R-linear map. Consider the graded R-algebra $A = T(V)$ and let $F : A\otimes A \to A\otimes A$ be defined on each $A_i \otimes A_j$ by*

$$\beta_{i_j} : A_i \otimes A_j \to A_j \otimes A_i$$

if $i, j > 0$. For the remaining cases define:

$$\beta_{0,j} : A_0 \otimes A_j \to A_j \otimes A_0 \quad \textit{and} \quad \beta_{i,0} : A_i \otimes A_0 \to A_0 \otimes A_i$$

respectively by

$$\beta_{0,j}(1\otimes a) = a\otimes 1 \quad \textit{and} \quad \beta_{i,0}(a\otimes 1) = 1\otimes a.$$

Then, for $F = \sum_{i,j\geq 0}\beta_{i,j}$ the conditions (13) *and* (14) *hold. Moreover $\tilde{m}_{A\otimes A} = (m_A \otimes m_A)(\mathrm{id}_A \otimes F \otimes \mathrm{id}_A)$ defines a new associative algebra structure on $A\otimes A$.*

Proof. From 1 we know that $\tilde{m}_{A\otimes A}$ is associative if and only if the following conditions hold:

(21) $$(\mathrm{id}_A \otimes m_A)F_{A\otimes A,A} = F(m_A \otimes \mathrm{id}_A)$$

(22) $$(m_A \otimes \mathrm{id}_A)F_{A,A\otimes A} = F(\mathrm{id}_A \otimes m_A)$$

where

$$F_{A\otimes A,A} = (F \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes F), \qquad F_{A,A\otimes A} = (\mathrm{id}_A \otimes F)(F \otimes \mathrm{id}_A).$$

But by definition $F_{A\otimes A,A}$ on $(A_i \otimes A_j) \otimes A_k$ is the following composition:

$$(A_i \otimes A_j) \otimes A_k \xrightarrow{\mathrm{id}_{A_i} \otimes\beta_{j,k}} A_i \otimes A_j \otimes A_k \xrightarrow{\beta_{i,k}\otimes \mathrm{id}_{A_j}} A_k \otimes (A_i \otimes A_j).$$

Similarly $F_{A,A\otimes A}$ on $A_i \otimes (A_j \otimes A_k)$ is the following composition:

$$A_i \otimes (A_j \otimes A_k) \xrightarrow{\beta_{i,j}\otimes \mathrm{id}_{A_k}} A_i \otimes A_j \otimes A_k \xrightarrow{\mathrm{id}_{A_j}\otimes \beta_{i,k}} (A_j \otimes A_k) \otimes A_i.$$

Hence for each $i, j, k \geq 1$, (21) and (22) become

$$(\mathrm{id}_{A_k} \otimes m_A)(\beta_{i,k} \otimes \mathrm{id}_{A,j})(\mathrm{id}_{A_i} \otimes \beta_{j,k}) = \beta_{i+j,k}(m_A \otimes \mathrm{id}_{A_k}) \tag{23}$$

$$(m_A \otimes \mathrm{id}_{A_i})(\mathrm{id}_{A,j} \otimes \beta_{i,k})(\beta_{i,j} \otimes \mathrm{id}_{A_k} = \beta_{i,j+k}(\mathrm{id}_{A_i} \otimes m_A). \tag{24}$$

But, by definition $m_A : A_r \otimes A_s \to A_{r+s}$ is "essentially"the identity map. So (23) and (24) become

$$(\beta_{i,k} \otimes \mathrm{id}_{A_j})(\mathrm{id}_{A_i} \otimes \beta_{j,k}) = \beta_{i+j,k}$$
$$(\mathrm{id}_{A_j} \otimes \beta_{i,k})(\beta_{i,j} \otimes \mathrm{id}_{A_k}) = \beta_{i,j+k}$$

which hold for 3 (ii), (iii). □

In what follows we want to *announce* that the results developed above, expecially in 3 and 1,lead to the consideration of the group $\tilde{\mathcal{B}}_k$ generated by elements $\{\tilde{b}_i \,|\, 1 \leq i < k\}$ with fundamental relations just $\tilde{b}_i\tilde{b}_j = \tilde{b}_j\tilde{b}_i(|i-j| \geq 2)$. Clearly there is an action $\tilde{\rho}_k : \tilde{\mathcal{B}}_k \to \mathrm{End}_R(A_k)$ given by:

$$\tilde{b}_i \to \beta_i \in \mathrm{End}_R(A_k).$$

If $\rho_k : \mathcal{B}_k \underline{\to} \mathrm{End}_R(A_k)$ denotes the usual action of the braid group $\mathcal{B}_k$ generated by elements $\{\overline{b}_i \,|\, 1 \leq i < k\}$, also given by

$$\overline{b}_i \to \beta_i \in \mathrm{End}_R(A_k),$$

then $\tilde{\rho}_k$ factors through the representation ρ_ki.e. $\rho_k \circ \pi_k = \tilde{\rho}_k$, where $\pi_k : \tilde{\mathcal{B}}_k \twoheadrightarrow \mathcal{B}_k$ is the canonical projection. We remark that it is well known, see [3], that there is a well defined map

$$\psi_k : S_k \to \mathcal{B}_k,$$

where S_k denotes the symmetric group of degree k. This results is due to Iwahori in the Coxeter group case and ψ_k is defined as a "lift" of $\sigma \in S_k$ in $\mathcal{B}_k$.

Acknowledgments. The author wants to thank D. Flores and J. A. Green for having brought this problem to her attention.

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AMS Subject Classification: 81R50, 16W50, 16W35.

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Lavoro pervenuto in redazione il 26.12.06.