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MONOTONIC MODAL LOGICS RELATED TO THE VON WRIGHT'S LOGIC OF PLACE

Abstract. In this paper we introduce the monotonic modal logics $\mathbf{M4}_{wn}$, $\mathbf{M4}_n$ and $\mathbf{MB}_n$ obtained from the basic strong monotonic modal logic **MON** by adding some formulas considered by R. Jansana in [8]. For each logic defined we prove completeness with respect to their characteristic classes of monotonic frames. The canonicity of these logics is proved using the representation theory for monotonic algebras developed in [4]. We also introduce the logics **MS4** and **MS5** as a monotonic counterpart of the normal logics **S4** and **S5**, respectively. Finally, we prove that there exists a translation of the logic **MS4** in $\mathbf{M4}_{wn}$, and a translation of the logic **MS5** in $\mathbf{M4}_{wn}+\mathbf{MB}_n$.

1. Introduction

In [8] R. Jansana introduces some normal modal logics related to the logic of place presented by Von Wright in [10] and studied semantically by Segerberg in [9]. In the Von Wright's logics the modal operator $\Box$ is interpreted intuitively as "everywhere else" and a sentence $\Box\varphi$ is valid in a place x if the sentence φ is valid in every other place that can be reached from x. In [8] R. Jansana introduces a weakening of Von Wright's logic of place. The main idea of Jansana is to study the logic of "in every other place that can be reached in fewer than $n+1$ steps". In a Kripke frame $\langle X,R\rangle$ the steps are represented by the accessibility relation $R\subseteq X\times X$ in the following way: each indice is a place, and from one place $x\in X$ a place $y\in X$ can be reached directly when xRy, and from a place x a place y can be reached in j steps when xR^jy.

In this paper we are interested in other weakening of the Von Wright's logic of place. We can give the interpretation saying that a sentence $\Box\varphi$ is valid in a place x if the sentence φ is valid in every *set* of places that can be reached from x. With this interpretation we have a non-normal modal logic, i.e., a modal logic where the formulas $\Box(\varphi\wedge\psi)\rightarrow\Box\varphi\wedge\Box\psi$ and $\Box\top$ are valid but the formula $\Box\varphi\wedge\Box\psi\rightarrow\Box(\varphi\wedge\psi)$ is not valid. These classes of modal logics are called strong monotonic modal logics [2], or fused modal logic [6]. Clearly the Kripke frames do not constitute an adequate semantics for the monotonic modal logics. Instead, strong monotonic modal logics are interpreted over monotonic frames (or neighbourhood frames in the terminology of Chellas [2], or fused Kripke frames in the terminology of J. Jaspars [7]), i.e. structures of the type $\langle X,R\rangle$ where X is a set and R is a relation between elements of X and non-empty subsets of X, such that $R(x)=\{Z\subseteq X:(x,Z)\in R\}$ is closed under supersets, for each $x\in X$.

The purpose of this paper is to study extensions of the minimal strong monotonic modal logic with the axioms introduced by R. Jansana in [8]. In Section 2 we

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give the basic definition of strong monotonic modal logics, and we recall the definitions of monotonic frames, monotonic modal algebras and the relation between these semantics. In Section 3 we prove some new and general results on strong monotonic frames that we will used in the paper. In Section 4 we introduce the monotonic logics $\mathbf{M4}_{\mathrm{wn}}$, $\mathbf{M4}_{\mathrm{n}}$ and $\mathbf{MB}_{\mathrm{n}}$ obtained from the basic strong monotonic modal logic **MON** by adding some formulas considered by R. Jansana in [8]. We prove that these logics are canonical by showing that the variety of normal monotonic algebras associated with each logic is closed under canonical extensions. In Section 5 we introduce the monotonic modal logics obtained from **MON** by adding some or all of the traditional modal axioms **4**, **T**, and **B**. We prove that these logics are complete with respect to they characteristic classes of monotonic frames. Finally, we prove that there exists a translation of the logic $\mathbf{MS4} = \mathbf{MON} + \{\mathbf{4}, \mathbf{T}\}$ in $\mathbf{M4}_{\mathrm{wn}}$, and a translation of the logic $\mathbf{MS5} = \mathbf{MON} + \{\mathbf{4}, \mathbf{T}, \mathbf{B}\}$ in $\mathbf{M4}_{\mathrm{wn}} + \mathbf{MB}_{\mathrm{n}}$.

2. Preliminaries

Let us consider a propositional language $\mathcal{L}$ defined by using a denumerable set of propositional variables *Var*, the connectives $\vee$ and $\wedge$, the negation $\neg$ and the propositional constant $\top$. The modal language $\mathcal{L}_{\Box}$ is obtained extending $\mathcal{L}$ by means of the a unary modal operator $\Box$. We shall denote by $\Diamond$ the operator defined by $\Diamond p = \neg\Box\neg p$, for $p \in Var$. The set of all well formed formulas as well as the formula algebra in the language $\mathcal{L}_{\Box}$ will be denoted by Fm.

A *strong monotonic modal logic* is a set of formulas $\mathbf{\Lambda}$ in the language $\mathcal{L}_{\Box}$, which contains the Classical Propositional Calculus **CP**, is closed under substitutions, $\Box\top \in \mathbf{\Lambda}$, and is closed under the following inference rules:

R1. If $\varphi, \varphi \to \psi \in \mathbf{\Lambda}$, then $\psi \in \mathbf{\Lambda}$ (Modus Pones).

R2. If $\varphi \to \psi \in \mathbf{\Lambda}$, then $\Box\varphi \to \Box\psi \in \mathbf{\Lambda}$.

The strong monotonic modal logic generated by a finite set of formulas Γ will be denoted by $\mathbf{\Lambda} + \{\Gamma\}$. For more details on monotonic modal logic see [2], [4], and [6]. The smallest *strong monotonic modal logic* will be denoted by **MON**. We note that the logic **MON** is the modal logic **RB** studied by J. Jaspars in [6].

Relational semantic

Let X be a non-empty set. We denote by $\mathcal{P}(X)$ the power set algebra. Let $\mathcal{P}_0(X) = \mathcal{P}(X) - \{\emptyset\}$. The complement of a subset $Y \subseteq X$ we denote by $Y - X$ or Y^c.

DEFINITION 1. [6] *A monotonic* frame*, or m-frame for short, is a structure* $\mathcal{F} = \langle X, R\rangle$ *such that* $X \neq \emptyset$, $R \subseteq X \times \mathcal{P}_0(X)$, *and for any* $x \in X$ *and for any* $Y, Y' \in \mathcal{P}_0(X)$, *if* $Y' \subseteq Y$ *and* $Y' \in R(x)$, *then* $Y \in R(x)$, *where* $R(x) = \{Z \in \mathcal{P}_0(X) : (x, Z) \in R\}$.

Let $\mathcal{F} = \langle X, R\rangle$ be an m-frame. For each $U \in \mathcal{P}(X)$, we define the sets

$$L_U = \{Y \in \mathcal{P}_0(X) \mid Y \cap U \neq \emptyset\}$$

and

$$L_U^c = \{Y \in \mathcal{P}_0(X) \mid Y \cap U = \emptyset\}.$$

LEMMA 1. *Let $\mathcal{F} = \langle X, R\rangle$ be an m-frame. Then:*

1. $L_{U\cap V} \subseteq L_U \cap L_V$ *and* $L_{U\cup V} = L_U \cup L_V$, *for every* $U, V \in \mathcal{P}(X)$.
2. $L_X = \mathcal{P}_0(X)$ *and* $L_\emptyset = \emptyset$.

Proof. It is easy and left to the reader. ☐

Let $\mathcal{F} = \langle X, R\rangle$ be an m-frame. We define a unary operation $\Box_R$ on $\mathcal{P}(X)$ as follows:

$$\begin{aligned}\Box_R(U) &= \{x \in X \mid \forall Y \in R(x)\ (Y \cap U \neq \emptyset)\}\\ &= \{x \in X \mid R(x) \subseteq L_U\},\end{aligned}$$

for each $U \in \mathcal{P}(X)$. We note that $\Box_R(X) = X$, and $\Box_R(U \cap V) \subseteq \Box_R(U) \cap \Box_R(V)$, for all $U, V \in \mathcal{P}(X)$. The dual operator $\Diamond_R$ is defined by

$$\begin{aligned}\Diamond_R(U) &= \{x \in X \mid \exists Y \in R(x) : Y \subseteq U\}\\ &= \{x \in X \mid R(x) \cap L^c_{U^c} \neq \emptyset\},\end{aligned}$$

for each $U \in \mathcal{P}(X)$.

A *valuation* V on an m-frame $\mathcal{F} = \langle X, R\rangle$ is a function $V : Var \to \mathcal{P}(X)$. A valuation can be extended recursively to the set of all formulas by means of the following clauses:

1. $V(\top) = X$,
2. $V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi)$, $\quad V(\varphi \vee \psi) = V(\varphi) \cup V(\psi)$,
3. $V(\neg\varphi) = V(\varphi)^c$,
4. $V(\Box\varphi) = \left\{x \in X \mid R(x) \subseteq L_{V(\varphi)}\right\} = \Box_R(V(\varphi))$.

An *m-model* is a pair $\mathcal{M} = \langle \mathcal{F}, V\rangle$ where $\mathcal{F}$ is an m-frame and V is a valuation on $\mathcal{F}$. The notions of truth at a point, validity in a model and validity in an m-frame for formulas are defined in the usual way. A formula φ is *valid at point x* in a model $\mathcal{M}$, in symbols $\mathcal{M} \vDash_x \varphi$ if $x \in V(\varphi)$. The formula φ is *valid in a model* $\mathcal{M}$, in symbols $\mathcal{M} \vDash \varphi$, if $V(\varphi) = X$. Finally, the formula φ is *valid in an* m-frame $\mathcal{F}$, in symbols $\mathcal{F} \vDash \varphi$, if $V(\varphi) = X$ for all valuations V defined on $\mathcal{F}$.

The monotonic modal logic of a class of monotonic frames $\mathbf{K}$ is $\mathrm{Th}(\mathbf{K}) = \{\varphi \in Fm : \mathcal{F} \vDash \varphi \text{ for all } \mathcal{F} \in \mathbf{K}\}$. Let $\mathbf{\Lambda}$ be a monotonic modal logic. The class of all m-frames $\mathcal{F}$ such that $\mathbf{\Lambda} \subseteq \mathrm{Th}(\{\mathcal{F}\}) = \mathrm{Th}(\mathcal{F})$ is called the *characteristic class* of $\mathbf{\Lambda}$, and it is denoted by $\mathrm{Fr}(\mathbf{\Lambda})$. A monotonic logic $\mathbf{\Lambda}$ is *frame complete with respect to a class of* m-frames $\mathbf{K}$ if $\mathbf{\Lambda} = \mathrm{Th}(\mathbf{K})$. The logic **MON** is frame complete with respect to the class of all m-frames (see [2], [4], and [5]).

Algebraic semantic

The algebraic semantic of strong monotonic modal logics is given by means of Boolean algebras with a monotonic modal operator. Let us recall that a *strong monotonic modal* algebra, or m-algebra, is an pair $\boldsymbol{A} = \langle A, \Box \rangle$, where A is a Boolean algebra and $\Box$ is a unary operator defined on A such that

M1. $\Box(a \wedge b) \leq \Box a \wedge \Box b$,

M2. $\Box 1 = 1$.

The dual operator $\Diamond$ is defined by $\Diamond a = \neg \Box \neg a$. It is clear that the class of m-algebras is a variety that will be denoted by **MA**.

Given a monotonic frame $\mathcal{F} = \langle X, R \rangle$, the algebra

$$\langle \mathcal{P}(X), \cup, \cap, -, \Box_R, \emptyset, X \rangle$$

is a monotonic modal algebra called the (full) *complex algebra* of $\mathcal{F}$. The complex algebra of $\mathcal{F}$ we will also denote by $\langle \mathcal{P}(X), \Box_R \rangle$. A *complex algebra* is a subalgebra of a full complex algebra $\langle \mathcal{P}(X), \Box_R \rangle$ for some m-frame $\mathcal{F}$.

REMARK 1. The standard semantic tool used to interpret strong monotonic modal logics is the neighbourhood semantics (see [2], [4] or [5]). A *monotonic neighbourhood model* is a pair $\langle \mathcal{F}, V \rangle$ where $\mathcal{F}$ is an m-frame and V is a valuation on $\mathcal{F}$. The notion of a formula being true is inductively defined for boolean connectives the same way as for m-models, and for formulas of type $\Box \varphi$ is defined by

$$V(\Box p) = \{x \in X \mid V(\varphi) \in R(x)\}.$$

In accordance with this interpretation we can define in $\mathcal{P}(X)$ a monotonic operator $m_R : \mathcal{P}(X) \to \mathcal{P}(X)$ as:

$$m_R(U) = \{x \in X \mid U \in R(x)\}, \tag{1}$$

for each $U \in \mathcal{P}(X)$. Clearly the pair $\langle \mathcal{P}(X), m_R \rangle$ is a monotonic modal algebra, called the neighbourhood complex algebra of $\mathcal{F}$. We note that if $\langle \mathcal{F}, V \rangle$ is a neighbourhood model, then $V(\Box p) = m_R(V(p))$, for each $p \in Var$. In the next result we establish the relation between monotonic neighbourhood models and m-models by proving that any complex algebra induces an equivalent neighbourhood complex algebra, and reciprocally any neighbourhood complex algebra induces an equivalent complex algebra.

LEMMA 2. *1. Let $\langle \mathcal{P}(X), \Box_R \rangle$ be a complex algebra of an m-frame $\mathcal{F} = \langle X, R \rangle$. Then there exists a neighbourhood complex algebra $\langle \mathcal{P}(X), m_J \rangle$ of an m-frame $\mathcal{F}_J = \langle X, J \rangle$ such that $\Box_R(U) = m_J(U)$, for all $U \in \mathcal{P}(X)$.*

2. Let $\langle \mathcal{P}(X), m_J \rangle$ be a neighbourhood complex algebra of an m-frame $\mathcal{F} = \langle X, J \rangle$. Then there exists a complex algebra $\langle \mathcal{P}(X), \Box_R \rangle$ of an m-frame $\mathcal{F}_R = \langle X, R \rangle$ such that $m_J(U) = \Box_R(U)$, for all $U \in \mathcal{P}(X)$.

Proof. 1. Let $\mathcal{F} = \langle X, R\rangle$ be an m-frame. We define a relation $J \subseteq X \times \mathcal{P}_0(X)$ by:

$$(x,U) \in J \text{ if and only if } \forall Z \in \mathcal{P}_0(X)\,((x,Z) \in R \text{ implies that } Z \cap U \neq \emptyset).$$

It is clear that $\langle X, J\rangle$ is an m-frame. We prove that $\Box_R(U) = m_J(U)$, for all $U \in \mathcal{P}(X)$. Let $x \in \Box_R(U)$ and we suppose that $x \notin m_J(U)$. Then, $(x,U) \notin J$. By the definition of J, we have that there exists $Z \in \mathcal{P}_0(X)$ such that $(x,Z) \in R$ and $Z \cap U = \emptyset$. Since $x \in \Box_R(U)$ and $(x,Z) \in R$, $Z \cap U \neq \emptyset$, which is a contradiction. It follows that $x \in m_J(U)$.

Suppose that $x \in m_J(U)$. Then $(x,U) \in J$. Let $Z \in \mathcal{P}_0(X)$ such that $(x,Z) \in R$. As $(x,U) \in J$, $Z \cap U \neq \emptyset$. Thus, $x \in \Box_R(U)$.

2. Let $\mathcal{F} = \langle X, J\rangle$ be an m-frame. Let us define the relation $R \subseteq X \times \mathcal{P}_0(X)$ as follows:

$$(x,Y) \in R \text{ if and only if } \forall Z \in \mathcal{P}_0(X)\,(Z \in J(x) \text{ implies that } Y \cap Z \neq \emptyset).$$

It is clear that for all $Y, K \in \mathcal{P}_0(X)$, if $Y \subseteq K$ and $(x,Y) \in R$, then $(x,K) \in R$. So, $\mathcal{F}_R = \langle X, R\rangle$ is also an m-frame. We prove that $m_J(U) = \Box_R(U)$, for all $U \in \mathcal{P}(X)$. If $x \in m_J(U)$, $U \in J(x)$. Let $(x,Y) \in R$. By the definition of R, since $U \in J(x)$, we get $Y \cap U \neq \emptyset$. Then, $x \in \Box_R(U)$.

Assume that $x \in \Box_R(U)$. Suppose that $U \notin J(x)$. Since $\mathcal{F} = \langle X, J\rangle$ is monotonic, $Z \nsubseteq U$ for all $Z \in J(x)$, i.e, $Z \cap U^c \neq \emptyset$ for all $Z \in J(x)$. By the definition of the relation R, $(x, U^c) \in R$. As $x \in \Box_R(U)$, $U \cap U^c \neq \emptyset$, which is a contradiction. Thus, $U \in J(x)$. ☐

Let $\boldsymbol{A}$ be an m-algebra. We denote the set of all ultrafilters of $\boldsymbol{A}$ by $\mathrm{Ul}(\boldsymbol{A})$ and the set of all proper filters of $\boldsymbol{A}$ by $\mathrm{Fi}(\boldsymbol{A})$. For each $a \in A$ we consider the set $\beta_{\boldsymbol{A}}(a) = \{P \in \mathrm{Ul}(\boldsymbol{A}) : a \in P\}$. For each proper filter F of $\boldsymbol{A}$ consider the set

$$\hat{F} = \{P \in \mathrm{Ul}(\boldsymbol{A}) : F \subseteq P\}.$$

We note that for each proper filter F of $\boldsymbol{A}$,

$$\hat{F} = \bigcap \{\beta_{\boldsymbol{A}}(a) : a \in F\}.$$

Now we define a relation between ultrafilters and subsets of ultrafilters of a m-algebra. We have only to consider particular sets of ultrafilters. More precisely, consider the set

$$\mathcal{C}_0(\mathrm{Ul}(\boldsymbol{A})) = \left\{Y \subseteq \mathrm{Ul}(\boldsymbol{A}) : Y = \hat{F} \text{ for some proper filter } F \text{ of } \boldsymbol{A}\right\}.$$

We define a relation $R_{\boldsymbol{A}} \subseteq \mathrm{Ul}(\boldsymbol{A}) \times \mathcal{C}_0(\mathrm{Ul}(\boldsymbol{A}))$ as follows:

$$(P, \hat{F}) \in R_{\boldsymbol{A}} \quad \Leftrightarrow \quad \forall \Box a \in P \;\left(\hat{F} \cap \beta_{\boldsymbol{A}}(a) \neq \emptyset\right). \tag{2}$$

The *ultrafilter* m-frame of $\boldsymbol{A}$, is the m-frame $\mathcal{F}(\boldsymbol{A}) = \langle \mathrm{Ul}(\boldsymbol{A}), R_{\boldsymbol{A}}\rangle$. We note that the relation $R_{\boldsymbol{A}}$ can also be defined as a subset of $\mathrm{Ul}(\boldsymbol{A}) \times \mathrm{Fi}(\boldsymbol{A})$ as follows:

$$\begin{aligned}(P,F) \in R_{\boldsymbol{A}} \quad &\Leftrightarrow \quad \forall \Box a \in P \;\left(\hat{F} \cap \beta_{\boldsymbol{A}}(a) \neq \emptyset\right)\\ &\Leftrightarrow \quad F \subseteq \Diamond^{-1}(P).\end{aligned}$$

Any of these definitions we will used in the rest of this work. The following theorem follows from the results given by H. H. Hansen [4] (see also [5] and [6]). We give a proof for completeness.

THEOREM 1. *Let* $\boldsymbol{A}$ *be an m-algebra. Let* $a \in A$*, and let* $P \in \mathrm{Ul}(\boldsymbol{A})$.

1. $\Diamond a \in P$ *if and only if there exists* $F \in \mathrm{Fi}(\boldsymbol{A})$ *such that* $(P, \hat{F}) \in R_{\boldsymbol{A}}$ *and* $a \in F$.
2. $\Box a \in P$ *if and only if for all* $F \in \mathrm{Fi}(\boldsymbol{A})$ *such that* $(P, \hat{F}) \in R_{\boldsymbol{A}}$*, implies that* $\hat{F} \cap \beta_{\boldsymbol{A}}(a) \neq \emptyset$.

Proof. We prove 1. Let $\Diamond a \in P$. Let us consider the filter $F = F(a)$ generated by a. Then it is easy to see that $F(a) \subseteq \Diamond^{-1}(P)$. So, $(P, \hat{F}) \in R_{\boldsymbol{A}}$ and $a \in F$.

Assume that there exists $F \in \mathrm{Fi}(\boldsymbol{A})$ such that $(P, \hat{F}) \in R_{\boldsymbol{A}}$ and $a \in F$. From $a \in F \subseteq \Diamond^{-1}(P)$, we get $\Diamond a \in P$. □

Let $\boldsymbol{A}$ be a monotonic algebra. The complex algebra

$$A(\mathcal{F}(\boldsymbol{A})) = \left\langle \mathcal{P}(\mathrm{Ul}(\boldsymbol{A})), \cup, \cap, ^{c}, \Box_{R_{\boldsymbol{A}}}, \emptyset, \mathrm{Ul}(\boldsymbol{A}) \right\rangle$$

of $\mathcal{F}(\boldsymbol{A})$ is called the *canonical extension* of $\boldsymbol{A}$.

THEOREM 2. [4] *Every m-algebra* $\boldsymbol{A}$ *is isomorphic to the subalgebra of the m-algebra* $A(\mathcal{F}(\boldsymbol{A}))$ *by means of the mapping* $\beta_{\boldsymbol{A}} : \boldsymbol{A} \to \mathcal{P}(\mathrm{Ul}(\boldsymbol{A}))$ *defined by* $\beta_{\boldsymbol{A}}(a) = \{P \in \mathrm{Ul}(\boldsymbol{A}) : a \in P\}$.

Proof. It is clear that $\beta_{\boldsymbol{A}}$ is an injective Boolean homomorphism. From Theorem 1 we have that $\beta_{\boldsymbol{A}}(\Box a) = \Box_{R_A}\beta_{\boldsymbol{A}}(a)$, for any $a \in A$. Thus, $\beta_{\boldsymbol{A}}$ is an injective homomorphism of monotonic modal algebras. □

Let $\mathcal{F} = \langle X, R \rangle$ be an m-frame. As the elements of $\langle \mathcal{P}(X), \Box_R \rangle$ are subsets of the universe of $\mathcal{F}$, a valuation in $\langle \mathcal{P}(X), \Box_R \rangle$ is nothing but a valuation on $\mathcal{F}$. In other words, for any formula φ, $\mathcal{F} \vDash \varphi$ iff the equation $\varphi \approx 1$ is valid in $\langle \mathcal{P}(X), \Box_R \rangle$. If $\mathbf{K}$ is a class of m-frames, then we denote the class of all full complex algebras of m-frames in $\mathbf{K}$ by Cm$\mathbf{K}$. We note that for any formula φ, $\varphi \in \mathrm{Th}(\mathbf{K})$ iff the equation $\varphi \approx 1$ is valid in the class Cm$\mathbf{K}$, and that for any formulas φ and ψ, the equation $\varphi \approx \psi$ is valid in the class Cm$\mathbf{K}$ iff the formula $(\varphi \to \psi) \wedge (\psi \to \varphi) \in \mathrm{Th}(\mathbf{K})$. Thus, we get that the monotonic modal logic $\mathrm{Th}(\mathbf{K})$ of a class of monotonic frames $\mathbf{K}$ can be identified with the equational theory of the class of complex algebras Cm$\mathbf{K}$, that is, the variety $\mathcal{V}(\mathrm{Cm}\mathbf{K}) = \mathrm{HSP}(\mathrm{Cm}\mathbf{K})$.

If $\mathbf{\Lambda}$ is a monotonic modal logic, then the class of monotonic modal algebras $\mathcal{V}(\mathbf{\Lambda}) = \{\boldsymbol{A} \in \mathbf{MA} : \boldsymbol{A} \vDash \varphi$, for all $\varphi \in \mathbf{\Lambda}\}$ is a variety defined by the equations $\varphi \approx 1$, for all $\varphi \in \mathbf{\Lambda}$. It is easy to check that for any logic $\mathbf{\Lambda}$, $\varphi \in \mathbf{\Lambda}$ iff φ is valid in every algebra of $\mathcal{V}(\mathbf{\Lambda})$. Thus, we get an algebraic completeness results for each logic $\mathbf{\Lambda}$. Moreover, $\mathbf{MA} = \mathcal{V}(\mathbf{MON})$ (see Chapter 7 of [4] or [5]).

Completely analogous to the case of normal modal logic (see [1] Chapter 5), a variety $\mathcal{V}$ of monotonic modal algebras is said to be *complete* if there exists a class of m-frames $\mathbf{K}$ which generates $\mathcal{V}$, i.e. $\mathcal{V} = \mathcal{V}(\mathrm{Cm}\mathbf{K})$. Then we have that a logic $\mathbf{\Lambda}$ is frame complete with respect to a class of m-frames $\mathbf{K}$ iff the variety $\mathcal{V}(\mathbf{\Lambda})$ is a complete variety. In other words,

$$\mathbf{\Lambda} = \mathrm{Th}(\mathbf{K}) \text{ iff } \mathcal{V}(\mathbf{\Lambda}) = \mathcal{V}(\mathrm{Cm}\mathbf{K}).$$

On the other hand, a class of monotonic modal algebras $\mathbf{M}$ is *canonical* if $\mathbf{M}$ is closed under canonical extensions, i.e. $A(\mathcal{F}(\boldsymbol{A})) \in \mathbf{M}$ whenever $\boldsymbol{A} \in \mathbf{M}$. As in the case of normal modal logic (see [1] Proposition 5.45, or [4]), we can prove that a logic $\mathbf{\Lambda}$ is canonical if the variety $\mathcal{V}(\mathbf{\Lambda})$ is canonical. In order obtain this characterization it is sufficient to show that for any algebra $\boldsymbol{A}$ in the variety $\mathcal{V}(\mathbf{\Lambda})$, the ultrafilter frame $\mathcal{F}(\boldsymbol{A})$ of $\boldsymbol{A}$ is a frame of the logic $\mathbf{\Lambda}$, i.e., $\mathcal{F}(\boldsymbol{A}) \in \mathrm{Fr}(\mathbf{\Lambda})$, or equivalently, that $A(\mathcal{F}(\boldsymbol{A}))$ belongs to $\mathcal{V}(\mathbf{\Lambda})$.

THEOREM 3. *The logic* **MON** *is canonical and complete with respect to the class of all m-frames.*

3. Some useful properties

In this section we prove some results which will be used in the next sections.

Let $\varphi \in Fm$. For each $n \geq 0$ we define inductively the formula $\Box^n\varphi$ as $\Box^0\varphi = \varphi$ and $\Box^{n+1}\varphi = \Box\Box^n\varphi$, and the formula $t_n(\varphi) = \varphi \wedge \Box\varphi \wedge \ldots \wedge \Box^n\varphi$. Similarly we define the formulas $\Diamond^n\varphi$ and $d_n(\varphi) = \varphi \vee \Diamond\varphi \vee \ldots \vee \Diamond^n\varphi$.

Let $\mathcal{F} = \langle X, R\rangle$ be an m-frame. We define the binary relation $\overline{R} \subseteq \mathcal{P}_0(X) \times \mathcal{P}_0(X)$ as follows:

$$(Z,Y) \in \overline{R} \Leftrightarrow \forall x \in Z: \ (x,Y) \in R.$$

Define inductively the *n-composition* R^n of R as follows:

$$(x,Y) \in R^0 \quad \Leftrightarrow \quad x \in Y.$$

$$(x,Y) \in R^{n+1} \quad \Leftrightarrow \quad \exists Z_1,\ldots,Z_n \in \mathcal{P}_0(X) \text{ such that } (x,Z_1) \in R, \ (Z_i,Z_{i+1}) \in \overline{R}, \text{ for } 1 \leq i \leq n-1 \text{ and } (Z_n,Y) \in \overline{R}.$$

Finally, we define the relation $\overline{R^n} \subseteq \mathcal{P}_0(X) \times \mathcal{P}_0(X)$ by:

$$(Y,Z) \in \overline{R^n} \Leftrightarrow \forall y \in Y: (y,Z) \in R^n.$$

We note that $(Y,Z) \in \overline{R^0}$ if and only if $Y \subseteq Z$. With this notation, we have that

$$(x,Y) \in R^{n+1} \quad \Leftrightarrow \quad \exists Z \in \mathcal{P}_0(X): (x,Z) \in R \text{ and } (Z,Y) \in \overline{R^n} \quad \Leftrightarrow \quad \exists Z \in \mathcal{P}_0(X): (x,Z) \in R^n \text{ and } (Z,Y) \in \overline{R}. \tag{3}$$

LEMMA 3. *Let $\mathcal{F} = \langle X, R\rangle$ be an m-frame. Then for every $n \geq 0$, the relation R^n is increasing, i.e., for every $Y, Z \in \mathcal{P}_0(X)$, if $Y \subseteq Z$ and $Y \in R^n(x)$, then $Z \in R^n(x)$.*

Proof. The proof is by induction on n. Let $n = 0$. Let $Y \subseteq Z$ and $(x,Y) \in R^0$. Then $x \in Y \subseteq Z$. So, $(x,Z) \in R^0$. Let $Y,Z \in \mathcal{P}_0(X)$ such that $Y \subseteq Z$ and $(x,Y) \in R^{n+1}$. Then there exist $B_1,...,B_n \in \mathcal{P}_0(X)$ such that

$$(x,B_1) \in R, (B_i,B_{i+1}) \in \overline{R}, \text{ with } 1 \leq i \leq n-1, \text{ and } (B_n,Y) \in \overline{R}.$$

So, for every $b \in B_n$, $(b,Y) \in R$. As $Y \subseteq Z$ and R is increasing, for every $b \in B_n$, $(b,Z) \in R$. It follows that $(B_n,Z) \in \overline{R}$. Thus, $(x,Z) \in R^{n+1}$. □

Let $\mathcal{F} = \langle X,R \rangle$ be an m-frame. For any $U \in \mathcal{P}(X)$ we define inductively the operator $\Box_R^n(U)$ by:

$$\Box_R^0(U) = U$$

$$\Box_R^{n+1}(U) = \Box_R^n(\Box_R(U)), \quad \text{for } n > 0.$$

LEMMA 4. *Let $\mathcal{F} = \langle X,R \rangle$ be an m-frame. Then, $\Box_R^n(U) = \Box_{R^n}(U)$ for every $n \geq 0$ and for every $U \in \mathcal{P}(X)$.*

Proof. The proof is by induction on n. Let $n = 0$. Since $\Box_R^0(U) = U$, we prove that $U = \Box_{R^0}(U)$. Let $x \in \Box_{R^0}(U)$. So, for every $Y \in R^0(x)$, $Y \cap U \neq \emptyset$, in particular as $(x,\{x\}) \in R^0$, $x \in U$. Let $x \in U$. Then for every Y such that $x \in Y$, $Y \cap U \neq \emptyset$. Thus, $R^0(x) \subseteq L_U$, i.e., $x \in \Box_{R^0}(U)$. Suppose that the result holds for n. Let $U \in \mathcal{P}(X)$ and let $x \in X$. Suppose that $x \in \Box_R^{n+1}(U) = \Box_R^n(\Box_R(U))$. We prove that $R^{n+1}(x) \subseteq L_U$. Let $Y \in \mathcal{P}_0(X)$ such that $(x,Y) \in R^{n+1}$. Then there exists $V \in \mathcal{P}_0(X)$ such that $(x,V) \in R^n$ and $(V,Y) \in \overline{R}$, i.e.,

$$(x,V) \in R^n \text{ and } (v,Y) \in R, \text{ for every } v \in V.$$

By assumption and inductive hypothesis we have that $x \in \Box_R^n(\Box_R(U)) = \Box_{R^n}(\Box_R(U))$. So, $R^n(x) \subseteq L_{\Box_R(U)}$. Since, $(x,V) \in R^n$, $V \cap \Box_R(U) \neq \emptyset$. Thus, there exists $v \in V$ such that $R(v) \subseteq L_U$. Since $(v,Y) \in R$, $Y \cap U \neq \emptyset$, i.e., $Y \in L_U$. Suppose now that $x \notin \Box_R^{n+1}(U)$. By inductive hypothesis we have

$$\Box_R^{n+1}(U) = \Box_R^n(\Box_R(U)) = \Box_{R^n}(\Box_R(U)).$$

Then there exists $Y \in \mathcal{P}_0(X)$ such that $(x,Y) \in R^n$ and $Y \cap \Box_R(U) = \emptyset$. Since for every $y \in Y$, $R(y) \subsetneq L_U$, we have that for each $y \in Y$ there exists $V_y \in \mathcal{P}_0(X)$ such that $(y,V_y) \in R$ and $V_y \cap U = \emptyset$. Consider the set

$$V = \bigcup \{V_y : y \in Y\}.$$

Since $\mathcal{F}$ is an m-frame, and $V_y \subseteq V$, $(y,V) \in R$. Thus, for every $y \in Y$ we get $(y,V) \in R$, i.e., $(Y,V) \in \overline{R}$. As $(x,Y) \in R^n$ and $(Y,V) \in \overline{R}$, we get $(x,V) \in R^{n+1}$, and taking into account that $V \cap U = \emptyset$, we have $R^{n+1}(x) \subsetneq L_U$. Therefore, $x \notin \Box_{R^{n+1}}(U)$. □

COROLLARY 1. *Let $\langle X,R,V \rangle$ be an m-model. For every formula φ, for all $n \neq 0$, and for every $x \in X$:*

$$x \in V(\Box^n \varphi) \Leftrightarrow R^n(x) \subseteq L_{V(\varphi)}.$$

Proof. The results follows from the previous Lemma and by the fact that $V(\Box^n\varphi) = \Box_R^n(V(\varphi))$, for every formula φ. $\square$

THEOREM 4. *Let $\boldsymbol{A}$ be an m-algebra. Let $P \in \mathrm{Ul}(\boldsymbol{A})$ and $F \in \mathrm{Fi}(\boldsymbol{A})$. Then for all $n \geq 0$*

$$(P,F) \in R_{\boldsymbol{A}}^n \Leftrightarrow F \subseteq \{a \in A : \Diamond^n a \in P\}.$$

Proof. The proof is by induction on n. The case $n = 0$ follows by the following equivalences:

$$\begin{aligned}(P,F) \in R_{\boldsymbol{A}}^0 \quad &\Leftrightarrow \quad P \in \hat{F} \\ &\Leftrightarrow \quad F \subseteq \left\{a \in A : \Diamond^0 a = a \in P\right\}.\end{aligned}$$

We note that the case $n = 1$ follows from Theorem 1. Suppose that the result holds for n. If $(P,F) \in R_A^{n+1}$, then it is easy to see that $F \subseteq \left\{a \in A : \Diamond^{n+1} a \in P\right\}$. Suppose that

$$F \subseteq \left\{a \in A : \Diamond^{n+1} a \in P\right\}.$$

Consider the set $X = \{\Diamond^n a \in A : a \in F\}$ and let H be the filter generated by X. We prove that

$$H \subseteq \Diamond^{-1}(P). \tag{4}$$

Let $x \in H$. Then there exist $a_1, ..., a_k \in F$ such that $\Diamond^n a_1 \wedge \ldots \wedge \Diamond^n a_k \leq x$. It follows that

$$\Diamond(\Diamond^n(a_1 \wedge \ldots \wedge a_k)) \leq \Diamond^n a_1 \wedge \ldots \wedge \Diamond^n a_k \leq \Diamond x.$$

Since, $a_1 \wedge \ldots \wedge a_k \in F$,

$$\Diamond(\Diamond^n(a_1 \wedge \ldots \wedge a_k)) = \Diamond^{n+1}(a_1 \wedge \ldots \wedge a_k) \leq \Diamond x \in P.$$

Thus, $H \subseteq \Diamond^{-1}(P)$. Now we prove that $(H,F) \in \overline{R}_{\boldsymbol{A}}^n$. Let $Q \in \mathrm{Ul}(\boldsymbol{A})$ such that $H \subseteq Q$. By inductive hypothesis, the condition $(Q,F) \in R_{\boldsymbol{A}}^n$ is equivalent to

$$F \subseteq \{a \in A : \Diamond^n a \in Q\}.$$

Let $a \in F$. Then $\Diamond^n a \in X \subseteq H \subseteq Q$. So, $\Diamond^n a \in Q$. Thus, $(Q,F) \in R_{\boldsymbol{A}}^n$, for all $Q \in \hat{H}$, and hence we conclude $(P,F) \in R_{\boldsymbol{A}}^{n+1}$ because $(P,H) \in R_{\boldsymbol{A}}$. $\square$

4. The logics $\mathbf{M4}_{\mathrm{wn}}$, $\mathbf{M4}_{\mathrm{n}}$ and $\mathbf{MB}_{\mathrm{n}}$

Let us consider the following formulas:

$$\begin{array}{ll} \mathbf{4}_{\mathrm{n}} & \Box^n\varphi \to \Box^{n+1}\varphi. \\ \mathbf{4}_{\mathrm{wn}} & t_n(\varphi) \to \Box^{n+1}\varphi. \\ \mathbf{B}_{\mathrm{n}} & \varphi \to t_n(d_n(\varphi)). \end{array} \tag{5}$$

Now we investigate the characteristic class of frames of extensions of the logic **MON** obtained by adding formulas from the set $\{\mathbf{4}_{\mathrm{n}}, \mathbf{4}_{\mathrm{wn}}, \mathbf{B}_{\mathrm{n}}\}$.

DEFINITION 2. *Let* $\mathcal{F} = \langle X, R \rangle$ *be an m-frame. The relation* R *is* weakly n-transitive *if and only if* $\forall x \in X\ \forall Y \in \mathcal{P}_0(X)$, *if* $(x, Y) \in R^{n+1}$, *then* $x \in Y$ *or there exists* $1 \leq j \leq n$ *such that* $(x, Y) \in R^j$.

THEOREM 5. *Let* $\mathcal{F} = \langle X, R \rangle$ *be an m-frame. Then* $\mathcal{F} \models \mathbf{4}_{\mathrm{wn}}$ *if and only if* R *is weakly* n*-transitive.*

Proof. Let $\mathcal{F} = \langle X, R \rangle$ be an m-frame such that $\mathcal{F} \models \mathbf{4}_{wn}$. Let $Z \in \mathcal{P}_0(X)$ such that $(x, Z) \in R^{n+1}$. Suppose $x \notin Z$ and $(x, Z) \notin R^j$ for every $1 \leq j \leq n$ and consider the valuation V defined by

$$V(p) = X - Z = Z^c.$$

So, $x \in V(p)$. Moreover, if $(x, Y) \in R^j$ for some $1 \leq j \leq n$, then $Y \cap V(p) \neq \emptyset$, because in the opposite case $Y \subseteq Z$ and as R^j is increasing for every j, $(x, Z) \in R^j$, which is a contradiction. Thus, $x \in V\left(\Box^j p\right)$ for every $1 \leq j \leq n$. It follows that $x \in V\left(\Box^{n+1} p\right)$, and since $(x, Z) \in R^{n+1}$, $Z \cap V(p) \neq \emptyset$, which is a contradiction. Therefore, $x \in Z$ or there exists $1 \leq j \leq n$ such that $(x, Z) \in R^j$.

Assume now that R is weakly n-transitive. Let $x \in X$ and let $x \in V(t_n(p))$. Let $Y \in \mathcal{P}_0(X)$ such that $(x, Y) \in R^{n+1}$. If $x \in Y$, then $Y \cap V(p) \neq \emptyset$. So, $x \in V\left(\Box^{n+1} p\right)$. If $x \notin Y$, there exists $1 \leq j \leq n$ such that $(x, Y) \in R^j$, and as $x \in V\left(\Box^j p\right)$, $Y \cap V(p) \neq \emptyset$. Thus, $x \in V\left(\Box^{n+1} p\right)$. □

DEFINITION 3. *Let* $\mathcal{F} = \langle X, R \rangle$ *be an m-frame. The relation* R is n-transitive *if and only if* $\forall x \in X\ \forall Y \in \mathcal{P}_0(X)$, if $(x, Y) \in R^{n+1}$, *then* $(x, Y) \in R^n$.

THEOREM 6. *Let* $\mathcal{F} = \langle X, R \rangle$ *be an m-frame. Then* $\mathcal{F} \models \mathbf{4}_{\mathrm{n}}$ *if and only if* R *is* n*-transitive.*

Proof. Let $\mathcal{F} = \langle X, R \rangle$ be an m-frame such that $\mathcal{F} \models \mathbf{4}_n$. Let $x \in X$ and let $Y \in \mathcal{P}_0(X)$ be such that $(x, Y) \in R^{n+1}$. Suppose that $(x, Y) \notin R^n$. As R^n is increasing, we get that for all $Z \in R^n(x)$, $Z \nsubseteq Y$. Consider the valuation V defined by

$$V(p) = X - Y = Y^c.$$

So, $R^n(x) \subseteq L_{V(p)}$, i.e., $x \in V(\Box^n p)$. Since $\mathcal{F} \models \mathbf{4}_n$, $x \in V\left(\Box^{n+1} p\right)$, i.e., $Y \cap Y^c \neq \emptyset$ which is a contradiction. Thus, $(x, Y) \in R^n$. The other direction it is easy and left to the reader. □

DEFINITION 4. *Let* $\mathcal{F} = \langle X, R \rangle$ *be an m-frame. We shall say that* R is n-symmetric *if and only if* $\forall x \in X\ \forall Y \in \mathcal{P}_0(X)$, *if* $(x, Y) \in R^j$ *for some* j, $0 \leq j \leq n$, *then* $x \in Y$ *or there is* $y \in Y$ *and* k *with* $0 \leq k \leq n$ *such that* $(y, \{x\}) \in R^k$.

THEOREM 7. *Let* $\mathcal{F} = \langle X, R \rangle$ *be an m-frame. Then* $\mathcal{F} \models \mathbf{B}_{\mathrm{n}}$ *if and only if* R *is* n*-symmetric.*

Proof. Suppose that $\mathcal{F} \models \mathbf{B}_n$. Let $x \in X$ and let $Y \in \mathcal{P}_0(X)$ such that $(x,Y) \in R^j$, for some $j \leq n$. Consider the valuation V defined by

$$V(p) = \{x\},$$

with $p \in Var$. Then, $x \in V(t_n(d_n(p)))$. Since $(x,Y) \in R^j$, for some $0 \leq j \leq n$, we get

$$Y \cap V(d_n(p)) = Y \cap (V(p) \cup V(\Diamond p) \cup ... \cup V(\Diamond^n p)) \neq \emptyset.$$

So, if $Y \cap V(p) \neq \emptyset$, then $x \in Y$. If $Y \cap V(p) = \emptyset$, then there exists $0 \leq k \leq n$ such that $Y \cap V\left(\Diamond^k p\right) \neq \emptyset$. It follows that there exists $y \in Y$ such that $y \in V\left(\Diamond^k p\right)$. Then $(y,K) \in R^k$ for some $K \in \mathcal{P}_0(X)$ such that $K \subseteq V(p) = \{x\}$. Thus, $K = \{x\}$ and $(y,\{x\}) \in R^k$.

Suppose that R is n-symmetric. Let $\varphi \in Fm$. Let V be a valuation over $\mathcal{F}$ and let $x \in X$ such that $x \in V(\varphi)$. Let $Y \in \mathcal{P}_0(X)$ such that $(x,Y) \in R^j$ for some $0 \leq j \leq n$. If $x \in Y$, $Y \cap V(\varphi) \neq \emptyset$, and as $V(\varphi) \subseteq V(d_n(\varphi))$, we get $Y \cap V(d_n(\varphi)) \neq \emptyset$. So, $x \in V(t_n(d_n(\varphi)))$..

If $x \notin Y$, then there exists $0 \leq k \leq n$ and there exists $y \in Y$ such that $(y,\{x\}) \in R^k$. As $\{x\} \subseteq V(\varphi)$, we have $y \in V\left(\Diamond^k \varphi\right) \subseteq V(d_n(\varphi))$. So, $Y \cap V(d_n(\varphi)) \neq \emptyset$. Therefore, $x \in V(t_n(d_n(\varphi)))$. □

Consider the monotonic logics

$$\begin{aligned}
\mathbf{M4}_{\mathrm{wn}} &= \mathbf{MON} + \{\mathbf{4}_{\mathrm{wn}}\}, \\
\mathbf{M4}_{\mathrm{n}} &= \mathbf{MON} + \{\mathbf{4}_{\mathrm{n}}\}, \\
\mathbf{MB}_{\mathrm{n}} &= \mathbf{MON} + \{\mathbf{B}_{\mathrm{n}}\}.
\end{aligned}$$

From Theorem 5, Theorem 6 and Theorem 7 we have

$$\begin{aligned}
\mathrm{Fr}(\mathbf{M4}_{\mathrm{wn}}) &= \{\mathcal{F} = \langle X,R\rangle \mid R \text{ is weakly } n\text{-transitive}\}, \\
\mathrm{Fr}(\mathbf{M4}_{\mathrm{n}}) &= \{\mathcal{F} = \langle X,R\rangle \mid R \text{ is } n\text{-transitive}\}, \\
\mathrm{Fr}(\mathbf{MB}_{\mathrm{n}}) &= \{\mathcal{F} = \langle X,R\rangle \mid R \text{ is } n\text{-symmetric}\}.
\end{aligned}$$

THEOREM 8. *The logic* $\mathbf{M4}_{\mathrm{wn}}$ *is canonical and complete with respect to the class* $\mathrm{Fr}(\mathbf{M4}_{wn})$.

Proof. It suffices to prove that the variety $\mathcal{V}(\mathbf{MON} + \{\mathbf{4}_{wn}\})$ is canonical, i.e., for each $\boldsymbol{A} \in \mathcal{V}(\mathbf{MON} + \{\mathbf{4}_{wn}\})$, $A(\mathcal{F}(\boldsymbol{A})) \in \mathcal{V}(\mathbf{MON} + \{\mathbf{4}_{wn}\})$. Let $\boldsymbol{A} \in \mathcal{V}(\mathbf{MON} + \{\mathbf{4}_{wn}\})$. Let $P \in \mathrm{Ul}(\boldsymbol{A})$ and let $F \in \mathrm{Fi}(\boldsymbol{A})$ such that $(P,F) \in R_{\boldsymbol{A}}^{n+1}$. If $F \subsetneq P$ and $(P,F) \notin R_{\boldsymbol{A}}^j$ for any $1 \leq j \leq n$, then there exists $a_0 \in F$ and $\neg a_0 \in P$, and by Theorem 4 there exist $a_j \in A$ for $1 \leq j \leq n$, such that

$$\Box^j a_j \in P \text{ and } \hat{F} \cap \beta_{\boldsymbol{A}}(a_j) = \emptyset.$$

Let $a = \neg a_0 \vee a_1 \vee \ldots \vee a_n$. Since

$$\neg a_0 \wedge \Box a_1 \wedge \ldots \wedge \Box^n a_n \leq \Box^j(a),$$

for every $1 \leq j \leq n$, we have

$$\neg a_0 \wedge \Box a_1 \wedge \ldots \wedge \Box^n a_n \leq \Box^{n+1}(a).$$

It follows that $\Box^{n+1}(a) \in P$. Since $(P,F) \in R_A^{n+1}$, $\hat{F} \cap \beta_{\boldsymbol{A}}(a) \neq \emptyset$, i.e., there exists $0 \leq j \leq n$ such that $\hat{F} \cap \beta_{\boldsymbol{A}}(a_j) \neq \emptyset$, which is a contradiction. Therefore $\mathcal{F}(\boldsymbol{A})$ is an m-frame of the logic $\mathbf{MON} + \{\mathbf{4}_{wn}\}$ and consequently it is canonical. □

THEOREM 9. *The logic* $\mathbf{M4}_{\mathrm{n}}$ *is canonical and thus complete with respect to the class* $\mathrm{Fr}(\mathbf{M4}_{\mathrm{n}})$.

Proof. It is very similar to the proof of Theorem 8. □

Let us recall that for any set Boolean algebra $\boldsymbol{A}$ we can construct the dual Stone space of $\boldsymbol{A}$ as the the zero-dimensional, compact and Hausdorff topological space $\langle \mathrm{Ul}(\boldsymbol{A}), \mathcal{T}_{\boldsymbol{A}} \rangle$, where the topology $\mathcal{T}_{\boldsymbol{A}}$ is generated by the clopen basis consisting of the sets $\{\beta_{\boldsymbol{A}}(a) : a \in A\}$. For each filter F of $\boldsymbol{A}$, the set $\hat{F}$ is a closed subset, and as the space is compact, then $\hat{F}$ is compact. We used these facts in the the proof of the following theorem.

THEOREM 10. *The logic* $\mathbf{MB}_{\mathrm{n}}$ *is canonical and complete with respect to the class* $\mathrm{Fr}(\mathbf{MB}_{\mathrm{n}})$.

Proof. It suffices to prove that the variety $\mathcal{V}(\mathbf{MON} + \{\mathbf{S}_{\mathrm{n}}\})$ is canonical. Let $\boldsymbol{A} \in \mathcal{V}(\mathbf{MON} + \{\mathbf{S}_{\mathrm{n}}\})$. Let $P \in \mathrm{Ul}(\boldsymbol{A})$ and $F \in \mathrm{Fi}(\boldsymbol{A})$ such that $(P,F) \in R_{\boldsymbol{A}}^j$ for some $0 \leq j \leq n$. We prove that

$$F \subseteq P \text{ or there exists } Q \in \hat{F} \text{ such that } (Q,\{P\}) \in R_{\boldsymbol{A}}^k \text{ for some } 0 \leq k \leq n.$$

Suppose neither is the case. Then there exists $a_0 \in F$, such that $\neg a_0 \in P$ and for each $Q \in \hat{F}$ there exist $a_i \in P$, with $0 \leq i \leq n$, such that $\Diamond^i a_i \notin Q$. It follows that

$$a_Q = \neg a_0 \wedge a_1 \wedge \ldots \wedge a_n \in P \text{ and } \neg a_0 \vee \Diamond a_1 \vee \ldots \vee \Diamond^n a_n \notin Q.$$

By monotonicity $\Diamond^i a_Q \leq \Diamond^i a_i$ for all $0 \leq i \leq n$, and hence

$$a_Q \vee \Diamond a_Q \vee \ldots \Diamond^n a_Q = d_n(a_Q) \notin Q.$$

Since $Q \in \hat{F}$ is an arbitrary element of $\hat{F}$,

$$\hat{F} \subseteq \bigcup \left\{ \beta_{\boldsymbol{A}}(\neg d_n(a_Q)) : Q \in \hat{F} \right\}.$$

Since $\hat{F}$ is closed, then it is compact. So there exist $a_{Q_1}, \ldots, a_{Q_l} \in A$ such that

$$\begin{aligned} \hat{F} &\subseteq \beta_{\boldsymbol{A}}\left(\neg d_n(a_{Q_1}) \vee \neg d_n(a_{Q_2}) \vee \ldots \vee \neg d_n\left(a_{Q_l}\right)\right) \\ &= \beta_{\boldsymbol{A}}\left(\neg\left(d_n(a_{Q_1}) \wedge d_n(a_{Q_2}) \wedge \ldots \wedge d_n\left(a_{Q_l}\right)\right)\right). \end{aligned}$$

By monotonicity, we have

$$d_n(a_{Q_1} \wedge a_{Q_2} \wedge \ldots \wedge a_{Q_n}) \leq d_n(a_{Q_1}) \wedge d_n(a_{Q_2}) \wedge \ldots \wedge d_n\left(a_{Q_l}\right),$$

and since $\hat{F} \subseteq \beta_{\boldsymbol{A}}\left(\neg\left(d_n\left(a_{Q_1} \wedge a_{Q_2} \wedge \ldots \wedge a_{Q_l}\right)\right)\right)$, we get

$$\hat{F} \cap \beta_{\boldsymbol{A}}\left(\left(d_n\left(a_{Q_1} \wedge a_{Q_2} \wedge \ldots \wedge a_{Q_l}\right)\right)\right) = \emptyset. \tag{6}$$

It is clear that $a = a_{Q_1} \wedge a_{Q_2} \wedge \ldots \wedge a_{Q_l} \in P$, and as $a \leq t_n(d_n(a))$ and $(P,F) \in R^j$,

$$\hat{F} \cap \beta_{\boldsymbol{A}}(d_n(a)) \neq \emptyset,$$

contradicticts (6). Therefore, $F \subseteq P$, or there exists $Q \in \beta_{\boldsymbol{A}}(F)$ such that $(Q,\{P\}) \in R^k_{\boldsymbol{A}}$ for some $0 \leq k \leq n$. ☐

5. Relation with the logics MS4 and MS5

Let us consider the formulas

4	$\Box\varphi \to \Box^2\varphi.$
T	$\Box\varphi \to \varphi.$
B	$\varphi \to \Box\Diamond\varphi.$

We note that the formula **4** is the formula $\mathbf{4}_1$. From Theorems 6 and 9 it follows that the logic $\mathbf{MON}+\{\mathbf{4}\}$ is complete with respect to the class of frames $\mathcal{F} = \langle X,R\rangle$ where the relation R is 1-transitive, i.e., R satisfies the property that for all $x \in X$ and for all $Y \in \mathcal{P}_0(X)$, if $(x,Y) \in R^2$, then $(x,Y) \in R$.

THEOREM 11. *Let $\mathcal{F} = \langle X,R\rangle$ be an m-frame. Then*

1. *$\mathcal{F} \models \mathbf{T}$ if and only if $(x,\{x\}) \in R$, for all $x \in X$.*
2. *$\mathcal{F} \models \mathbf{B}$ if and only if $\forall x \in X\ \forall Y \in P_0(X)$, if $(x,Y) \in R$, then there is $y \in Y$ such that $(y,\{x\}) \in R$.*

Proof. 1. Suppose that $\mathcal{F} \models \mathbf{T}$ and let $x \in X$. Consider the valuation

$$V(p) = \{x\}^c = X - \{x\},$$

with $p \in Var$. Since $x \notin V(p)$, $x \notin V(\Box p)$. Then, there exists $Y \in \mathcal{P}_0(X)$ such that $(x,Y) \in R$ and $Y \cap V(p) = Y \cap \{x\}^c = \emptyset$. It follows that $Y \subseteq \{x\}$, and as $Y \neq \emptyset$, we get $Y = \{x\}$. Thus, $(x,\{x\}) \in R$. The other direction is easy.

The proof of 2 is similar to the proof of Theorem 7. ☐

REMARK 2. *Let us note that the above frame conditions are not exactly the same conditions given by Hansen in Proposition 5.1 of [4].*

DEFINITION 5. *Let $\mathcal{F} = \langle X, R\rangle$ be an m-frame. We shall say that the relation $R \subseteq X \times \mathcal{P}_0(X)$ is a generalized quasi-order if R is 1-transitive and $(x, \{x\}) \in R$, for all $x \in X$. We shall say that the relation R is a generalized equivalence if is a generalized quasi-order and it satisfies the property 2 of Theorem 11.*

Let us consider the monotonic logics

$$\begin{aligned}\mathbf{MS4} &= \mathbf{MON} + \{\mathbf{4}, \mathbf{T}\} \\ \mathbf{MS5} &= \mathbf{MON} + \{\mathbf{4}, \mathbf{T}, \mathbf{B}\}.\end{aligned}$$

By Theorem 11 we get that

$$\begin{aligned}\mathrm{Fr}(\mathbf{MS4}) &= \{\mathcal{F} = \langle X, R\rangle \mid R \text{ is a generalized quasi-order}\} \\ \mathrm{Fr}(\mathbf{MS5}) &= \{\mathcal{F} = \langle X, R\rangle \mid R \text{ is a generalized equivalence}\}.\end{aligned}$$

THEOREM 12. *Any extension of* **MON** *obtained by adding any subset of the set of formulas $\{\mathbf{4}, \mathbf{T}, \mathbf{B}\}$ is canonical and frame complete with respect to its characteristic class of frames.*

Proof. We consider only the case of $\mathbf{MON} + \{\mathbf{T}\}$ and we prove that the variety $\mathcal{V}(\mathbf{MON} + \{\mathbf{T}\})$ is canonical. Let $\boldsymbol{A} \in \mathcal{V}(\mathbf{MON} + \{\mathbf{T}\})$. Let $P \in \mathrm{Ul}(\boldsymbol{A})$. Since $\Box a \leq a$ for all $a \in A$, $(P, \{P\}) \in R_{\boldsymbol{A}}$.

Conversely, suppose that there exists $a \in A$ such that $\Box a \nleq a$. Then there exists $P \in \mathrm{Ul}(\boldsymbol{A})$ such that $\Box a \in P$ and $a \notin P$. Thus, $(P, \{P\}) \notin R_{\boldsymbol{A}}$. □

We now study the relationship between the logics **MS4** and $\mathbf{MON} + \{\mathbf{4}_{\mathrm{wn}}\}$, and between the logics **MS5** and $\mathbf{MON} + \{\mathbf{4}_{\mathrm{wn}}, \mathbf{B}_{\mathrm{n}}\}$. Consider the following translation (n) from modal formulas to defined by:

$$\begin{aligned}\top^{(n)} &\overset{def}{=} \top \\ p_j^{(n)} &\overset{def}{=} p_j \\ (\varphi \to \psi)^{(n)} &\overset{def}{=} \varphi^{(n)} \to \psi^{(n)} \\ (\Box\varphi)^{(n)} &\overset{def}{=} t_n\left(\varphi^{(n)}\right).\end{aligned}$$

Let $\mathcal{F} = \langle X, R\rangle$ be an m-frame. Define the relation Rn $(n \geq 0)$ by

$$Rn = R^0 \cup R \cup \ldots \cup R^n.$$

It is easy to see that the structure $\langle X, Rn\rangle$ is an m-frame.

LEMMA 5. *For every $n \geq 1$, every model $\langle X, R, V\rangle$, every $x \in X$, and every formula φ*

$$\langle X, R, V\rangle \vDash_x \varphi^{(n)} \text{ if and only if } \langle X, Rn, V\rangle \vDash_x \varphi.$$

Proof. The proof is by induction on the construction of the formula φ. We prove only the case of formulas $\Box\varphi$. Suppose that the result holds for φ and that $\langle X,R,V\rangle \vDash_x (\Box\varphi)^{(n)}$. Then $\langle X,R,V\rangle \vDash_x t_n\left(\varphi^{(n)}\right)$, i.e.,

$$x \in V\left(t_n\left(\varphi^{(n)}\right)\right) = V\left(\varphi^{(n)} \wedge \Box\varphi^{(n)} \wedge \ldots \wedge \Box^n\varphi^{(n)}\right).$$

We prove that

$$Rn(x) \subseteq L_{V(\varphi)}.$$

Let $(x,Y) \in Rn$. Then $x \in Y$ or there exists $1 \leq j \leq n$ such that $(x,Y) \in R^j$. If $x \in Y$, $Y \cap V\left(\varphi^{(n)}\right) \neq \emptyset$. Thus, $Rn(x) \subseteq L_{V(\varphi)}$. If there exists $1 \leq j \leq n$ such that $(x,Y) \in R^j$, then $Y \cap V\left(\varphi^{(n)}\right) \neq \emptyset$, because $x \in V\left(\Box^j\varphi^{(n)}\right)$. So there exists $y \in Y$ such that $y \in V\left(\varphi^{(n)}\right)$. By inductive hypothesis, $\langle X,Rn,V\rangle \vDash_y \varphi$. Thus, $Rn(x) \subseteq L_{V(\varphi)}$, i.e., $\langle X,Rn,V\rangle \vDash_x \Box\varphi$.

Conversely,assume that $\langle X,Rn,V\rangle \vDash_x \Box\varphi$ and that $\langle X,R,V\rangle \nvDash_x t_n\left(\varphi^{(n)}\right)$. Then there exists $j \leq n$ such that $x \notin V\left(\Box^j\varphi^{(n)}\right)$. So there exists $Y \in \mathcal{P}_0(X)$ such that $(x,Y) \in R^j$ and $Y \cap V\left(\varphi^{(n)}\right) = \emptyset$. It follows that for every $y \in Y$, $y \notin V\left(\varphi^{(n)}\right)$. Then by inductive hypothesis we get that, for every $y \in Y$, $\langle X,Rn,V\rangle \nvDash_y \varphi$. Since $(x,Y) \in R^j$, $(x,Y) \in Rn$. Then $Rn(x) \subsetneq L_{V(\varphi)}$. Thus, $\langle X,Rn,V\rangle \nvDash_x \Box\varphi$, which is a contradiction. ☐

REMARK 3. It is easy to see that in every m-frame $\mathcal{F} = \langle X,R\rangle$, the relation R is weakly n-transitive ($n > 0$) if and only if the relation Rn is a generalized quasi-order i.e., $\langle X,Rn\rangle \in \mathrm{Fr}(\mathbf{MS4})$.

Moreover, R is n-symmetric if and only if Rn satisfies the property that $\forall x \in X\ \forall Y \in \mathcal{P}_0(X)$, if $(x,Y) \in Rn$, then there is $y \in Y$ such that $(y,\{x\}) \in Rn$. Thus, R is weakly n-transitive and n-symmetric ($n > 0$) if and only if the relation Rn is a generalized equivalence i.e., $\langle X,Rn\rangle \in \mathrm{Fr}(\mathbf{MS5})$.

THEOREM 13. *1. For every* $n \geq 1$ *and every formula* φ,

$$\varphi \in \mathbf{MS4} \textit{ if and only if } \varphi^{(n)} \in \mathbf{MON} + \{\mathbf{4}_{\mathrm{wn}}\}.$$

2. For every $n \geq 1$ *and every formula* φ,

$$\varphi \in \mathbf{MS5} \textit{ if and only if } \varphi^{(n)} \in \mathbf{MON} + \{\mathbf{4}_{\mathrm{wn}}, \mathbf{B}_{\mathrm{n}}\}.$$

Proof. 1. $\Rightarrow$) Suppose that $\varphi^{(n)} \notin \mathbf{MON} + \{\mathbf{4}_{\mathrm{wn}}\}$. Then there exists an m-frame $\mathcal{F} = \langle X,R\rangle$ in which the relation R is weakly n-transitive and such that $\mathcal{F} \nvDash \varphi^{(n)}$. By Lemma 5 $\langle X,Rn\rangle \nvDash \varphi$. Since $\langle X,Rn\rangle \in \mathrm{Fr}(\mathbf{MON} + \{\mathbf{T},\mathbf{4}\})$, $\varphi \notin \mathbf{MON} + \{\mathbf{T},\mathbf{4}\}$. The direction $\Leftarrow$) is similar and left to the reader.

The proof of item 2 is similar to the previous one. ☐

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