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**ESSENTIAL SELF-ADJOINTNESS AND GLOBAL  
HYPOELLIPTICITY OF THE TWISTED LAPLACIAN**

**Abstract.** A scale of Sobolev spaces is introduced to measure the global hypoellipticity of the twisted Laplacian. The essential self-adjointness of the twisted Laplacian is established and the domain of the unique self-adjoint extension is determined in terms of the Sobolev spaces. The global hypoellipticity of the twisted Laplacian in the Gelfand–Shilov spaces is also proved.

**1. The twisted Laplacian**

Let  $\frac{\partial}{\partial z}$  and  $\frac{\partial}{\partial \bar{z}}$  be linear partial differential operators on  $\mathbb{R}^2$  given by

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i \frac{\partial}{\partial y}$$

and

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}.$$

Then we define the linear partial differential operator  $L$  on  $\mathbb{R}^2$  by

$$L = -\frac{1}{2}(Z\bar{Z} + \bar{Z}Z),$$

where

$$Z = \frac{\partial}{\partial z} + \frac{1}{2}\bar{z}, \quad \bar{z} = x - iy,$$

and

$$\bar{Z} = \frac{\partial}{\partial \bar{z}} - \frac{1}{2}z, \quad z = x + iy.$$

The vector fields  $Z$  and  $\bar{Z}$ , and the identity operator  $I$  form a basis for a Lie algebra in which the Lie bracket of two elements is their commutator. In fact,  $-\bar{Z}$  is the formal adjoint of  $Z$  and  $L$  is an elliptic partial differential operator on  $\mathbb{R}^2$  given by

$$L = -\Delta + \frac{1}{4}(x^2 + y^2) - i \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right),$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

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Thus,  $L$  is the ordinary Hermite operator  $-\Delta + \frac{1}{4}(x^2 + y^2)$  perturbed by the partial differential operator  $-iN$ , where

$$N = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$$

is the rotation operator. As such, we call  $L$  the twisted Laplacian. The vector fields  $Z$  and  $\bar{Z}$ , and the twisted Laplacian  $L$  are studied in the books [9, 10, 11]. The connection of the twisted Laplacian  $L$  with the sub-Laplacian on the Heisenberg group  $\mathbb{H}^1$  can be found in [10]. By way of comparison, the sub-Laplacian  $\Delta_{\mathbb{H}^1}$  on the Heisenberg group  $\mathbb{H}^1$  is given by

$$-\Delta_{\mathbb{H}^1} = -\Delta - \frac{1}{4}(x^2 + y^2) \frac{\partial^2}{\partial t^2} + N \frac{\partial}{\partial t}.$$

The twisted Laplacian  $L$  can be factored as

$$L = \left( D_x - \frac{1}{2}y \right)^2 + \left( D_y + \frac{1}{2}x \right)^2.$$

Its symbol  $\sigma$  is given by

$$\sigma(x, y; \xi, \eta) = \left( \xi - \frac{1}{2}y \right)^2 + \left( \eta + \frac{1}{2}x \right)^2$$

for all points  $(x, y)$  and  $(\xi, \eta)$  in  $\mathbb{R}^2$ . Thus,  $L$  is elliptic, but not globally elliptic. It is not globally elliptic even in the sense of [8]. In other words, we cannot find positive constants  $C$  and  $R$  such that

$$|\sigma(x, y; \xi, \eta)| \geq C(1 + |x|^2 + |y|^2 + |\xi|^2 + |\eta|^2)$$

for  $|x|^2 + |y|^2 + |\xi|^2 + |\eta|^2 \geq R$ . While the local solvability and local regularity of  $L$  are well understood, the corresponding global properties are not clear.

Besides these traditional questions, the twisted Laplacian  $L$  is closely connected with the mathematics of pseudo-differential operators in the framework of Weyl transforms and Wigner transforms. This perspective is most important about this operator.

The aim of this paper is to study the essential self-adjointness and the global hypoellipticity of the twisted Laplacian. See [7] and [2] for detailed expositions on, respectively, essential self-adjointness and global hypoellipticity.

Using the Weyl transforms and the Fourier–Wigner transforms of Hermite functions, which we recall in Sections 2 and 3, we can construct the heat kernel and Green function of the twisted Laplacian  $L$ . As is shown in [13], the Green function can be used to prove that the twisted Laplacian  $L$  is globally hypoelliptic in the sense that

$$u \in S'(\mathbb{R}^2), Lu \in S(\mathbb{R}^2) \Rightarrow u \in S(\mathbb{R}^2).$$

The heat kernel, the Green function and the global hypoellipticity of the twisted Laplacian  $L$  are described in Section 4. In Section 5 we show that the twisted Laplacian  $L$  is

essentially self-adjoint. A scale of Sobolev spaces tailored to the twisted Laplacian  $L$  is constructed in Section 6. The domain of the unique self-adjoint extension can then be computed explicitly. The global hypoellipticity of the twisted Laplacian  $L$  using these Sobolev spaces is given in Section 7. In Section 8, Gelfand–Shilov spaces are first recalled and then used to have another look at the global hypoellipticity of the twisted Laplacian.

## 2. Weyl transforms

Let  $f$  and  $g$  be functions in the Schwartz space  $\mathcal{S}(\mathbb{R})$  on  $\mathbb{R}$ . Then the Fourier–Wigner transform  $V(f, g)$  of  $f$  and  $g$  is defined by

$$V(f, g)(q, p) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{iqy} f\left(y + \frac{p}{2}\right) \overline{g\left(y - \frac{p}{2}\right)} dy$$

for all  $q$  and  $p$  in  $\mathbb{R}$ . It can be proved that  $V(f, g)$  is a function in the Schwartz space  $\mathcal{S}(\mathbb{R}^2)$  on  $\mathbb{R}^2$ . We define the Wigner transform  $W(f, g)$  of  $f$  and  $g$  by

$$W(f, g) = V(f, g)^\wedge,$$

where  $\hat{F}$  is the Fourier transform of  $F$ , which we choose to define by

$$\hat{F}(\zeta) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-iz \cdot \zeta} F(z) dz, \quad \zeta \in \mathbb{R}^n,$$

for all  $F$  in the Schwartz space  $\mathcal{S}(\mathbb{R}^n)$  on  $\mathbb{R}^n$ . It can be shown that

$$W(f, g)(x, \xi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-i\xi p} f\left(x + \frac{p}{2}\right) \overline{g\left(x - \frac{p}{2}\right)} dp$$

for all  $x$  and  $\xi$  in  $\mathbb{R}$ . It is obvious that

$$W(f, g) = \overline{W(g, f)}, \quad f, g \in \mathcal{S}(\mathbb{R}).$$

Now, let  $\sigma \in L^p(\mathbb{R}^2)$ ,  $1 \leq p \leq \infty$ , and let  $f \in \mathcal{S}(\mathbb{R})$ . Then we define  $W_\sigma f$  to be the tempered distribution on  $\mathbb{R}$  by

$$(W_\sigma f, g) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(x, \xi) W(f, g)(x, \xi) dx d\xi$$

for all  $g$  in  $\mathcal{S}(\mathbb{R})$ , where  $(F, G)$  is defined by

$$(F, G) = \int_{\mathbb{R}^n} F(z) \overline{G(z)} dz$$

for all measurable functions  $F$  and  $G$  on  $\mathbb{R}^n$ , provided that the integral exists. We call  $W_\sigma$  the Weyl transform associated to the symbol  $\sigma$ . It should be noted that if  $\sigma$  is a symbol in  $\mathcal{S}(\mathbb{R}^2)$ , then  $W_\sigma f$  is a function in  $\mathcal{S}(\mathbb{R})$  for all  $f$  in  $\mathcal{S}(\mathbb{R})$ .

### 3. Fourier–Wigner transforms of Hermite functions

For  $k = 0, 1, 2, \dots$ , the Hermite function  $e_k$  of order  $k$  is the function on  $\mathbb{R}$  defined by

$$e_k(x) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} e^{-x^2/2} H_k(x)$$

for all  $x$  in  $\mathbb{R}$ , where  $H_k$  is the Hermite polynomial of degree  $k$  given by

$$H_k(x) = (-1)^k e^{x^2} \left( \frac{d}{dx} \right)^k (e^{-x^2})$$

for all  $x$  in  $\mathbb{R}$ .

For  $j, k = 0, 1, 2, \dots$ , we define the function  $e_{j,k}$  on  $\mathbb{R}^2$  by

$$e_{j,k}(x, y) = V(e_j, e_k)(x, y)$$

for all  $x$  and  $y$  in  $\mathbb{R}$ . Then we have the following fact, which is Theorem 21.2 in [11].

**THEOREM 1.**  $\{e_{j,k} : j, k = 0, 1, 2, \dots\}$  is an orthonormal basis for  $L^2(\mathbb{R}^2)$ .

The spectral analysis of the twisted Laplacian  $L$  is based on the following result, which is Theorem 22.1 in [11].

**THEOREM 2.** For  $j = 0, 1, 2, \dots$ ,  $k = 1, 2, \dots$ ,

$$Z e_{j,k} = i(2k)^{1/2} e_{j,k-1}$$

and for  $j, k = 0, 1, 2, \dots$ ,

$$\bar{Z} e_{j,k} = i(2k+2)^{1/2} e_{j,k+1}.$$

**REMARK 1.** In view of Theorem 2, we call  $Z$  and  $\bar{Z}$  the annihilation operator and the creation operator, respectively, for the functions  $e_{j,k}$ ,  $j, k = 0, 1, 2, \dots$ , on  $\mathbb{R}^2$ .

An immediate consequence of Theorem 2 is the following theorem.

**THEOREM 3.** For  $j, k = 0, 1, 2, \dots$ ,

$$L e_{j,k} = (2k+1) e_{j,k}.$$

**REMARK 2.** Theorem 3 says that for  $k = 0, 1, 2, \dots$ , the number  $2k+1$  is an eigenvalue of the twisted Laplacian  $L$ , and the functions  $e_{j,k}$ ,  $j = 0, 1, 2, \dots$ , on  $\mathbb{R}^2$  are eigenfunctions of  $L$  corresponding to the eigenvalue  $2k+1$ .

### 4. Global hypoellipticity

Using Weyl transforms and Fourier–Wigner transforms of Hermite functions, we get the heat kernel  $\kappa_t$ ,  $t > 0$ , of the twisted Laplacian  $L$  given by

$$\kappa_t(z, w) = \frac{1}{4\pi \sinh t} e^{-\frac{1}{4}|z-w|^2 \coth t} e^{i\frac{1}{4}[z,w]},$$

for all  $z$  and  $w$  in  $\mathbb{C}$ , where  $[z, w]$  is the symplectic form of  $z$  and  $w$  given by

$$[z, w] = 2\operatorname{Im}(z\bar{w}).$$

The Green function  $G$  of the twisted Laplacian  $L$  can then be obtained simply by integrating the heat kernel  $\kappa_t$  from 0 to  $\infty$  with respect to  $t$ . The result is

$$G(z, w) = e^{i\frac{1}{4}[z, w]} \frac{1}{4\pi} K_0\left(\frac{1}{4}|z - w|^2\right)$$

for all  $z$  and  $w$  in  $\mathbb{C}$ , where  $K_0$  is the modified Bessel function of order 0 given by

$$K_0(x) = \int_0^\infty e^{-x\cosh t} dt, \quad x > 0.$$

The Green function can then be used to prove that the twisted Laplacian  $L$  is globally hypoelliptic. More precisely, we have the following theorem.

**THEOREM 4.** *The twisted Laplacian  $L$  is globally hypoelliptic in the sense that*

$$u \in S'(\mathbb{R}^2), Lu \in S(\mathbb{R}^2) \Rightarrow u \in S(\mathbb{R}^2).$$

The results in this section can be found in [13].

## 5. Essential self-adjointness

Since  $-\bar{Z}$  is the formal adjoint of  $Z$ , it follows that the twisted Laplacian  $L$  is a symmetric operator from  $L^2(\mathbb{R}^2)$  into  $L^2(\mathbb{R}^2)$  with dense domain  $S(\mathbb{R}^2)$ . As such, it is closable and we denote the closure by  $L_0$ .

**PROPOSITION 1.**  *$L_0$  is closed and symmetric.*

*Proof.* That  $L_0$  is closed is obvious. Now, let  $u$  and  $v$  be in the domain  $\mathcal{D}(L_0)$  of  $L_0$ . Then let  $\{\varphi_l\}_{l=1}^\infty$  and  $\{\psi_l\}_{l=1}^\infty$  be sequences in  $S(\mathbb{R}^2)$  be such that

$$\varphi_l \rightarrow u,$$

$$L\varphi_l \rightarrow L_0u,$$

$$\psi_l \rightarrow v$$

and

$$L\psi_l \rightarrow L_0v$$

in  $L^2(\mathbb{R}^2)$  as  $l \rightarrow \infty$ . So,

$$(L_0u, v) = \lim_{l \rightarrow \infty} (L\varphi_l, \psi_l) = \lim_{l \rightarrow \infty} (\varphi_l, L\psi_l) = (u, L_0v),$$

and  $L_0$  is symmetric. □

In fact, we can prove a stronger result than Proposition 1.

**THEOREM 5.**  $L_0$  is self-adjoint.

*Proof.* Since  $L_0$  is closed and symmetric, it follows that  $L_0$  is self-adjoint if we can prove that the resolvent set  $\rho(L_0)$  of  $L_0$  contains a real number. (See, for instance, page 137 of [7].) It is sufficient to prove that the range  $R(L_0)$  of  $L_0$  is dense in  $L^2(\mathbb{R}^2)$  and there exists a positive constant  $C$  such that

$$(1) \quad \|L_0 u\|_2 \geq C \|u\|_2, \quad u \in \mathcal{D}(L_0).$$

The density follows from the global hypoellipticity of  $L$  in the sense of Schwartz functions and distributions given in Theorem 4. To prove the inequality (1), let  $\varphi \in \mathcal{S}(\mathbb{R}^2)$ . Then, by Parseval's identity,

$$\begin{aligned} \|L\varphi\|_2^2 &= \left\| \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (2k+1) (\varphi, e_{jk}) e_{jk} \right\|^2 \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (2k+1)^2 |(\varphi, e_{jk})|^2 \\ &\geq \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |(\varphi, e_{jk})|^2 \\ &= \|\varphi\|_2^2. \end{aligned}$$

□

**COROLLARY 1.** The twisted Laplacian  $L$  from  $L^2(\mathbb{R}^2)$  into  $L^2(\mathbb{R}^2)$  with dense domain  $\mathcal{S}(\mathbb{R}^2)$  is essentially self-adjoint.

## 6. Sobolev spaces

For  $s \in \mathbb{R}$ , we define  $H^{s,2}$  by

$$H^{s,2} = \left\{ u \in \mathcal{S}'(\mathbb{R}^2) : \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (1+k^2)^s |(u, e_{jk})|^2 < \infty \right\}.$$

It is easy to see that  $H^{s,2}$  is an inner product space with inner product  $(\cdot, \cdot)_{s,2}$  and norm  $\|\cdot\|_{s,2}$  given by

$$(u, v)_{s,2} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (1+k^2)^s (u, e_{jk})(e_{jk}, v)$$

and

$$\|u\|_{s,2}^2 = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (1+k^2)^s |(u, e_{jk})|^2$$

for all  $u$  and  $v$  in  $H^{s,2}$ .

THEOREM 6.  $H^{s,2}$  is a Hilbert space with respect to the inner product  $(\cdot, \cdot)_{s,2}$ .

*Proof.* First we look at the case when  $s \geq 0$ . Then the domain  $\mathcal{D}(L_0^s)$  of the self-adjoint operator  $L_0^s$  from  $L^2(\mathbb{R}^2)$  into  $L^2(\mathbb{R}^2)$  is a Banach space with respect to the graph norm  $\|\cdot\|_s$  given by

$$\|u\|_s^2 = \|L_0^s u\|_2^2 + \|u\|_2^2, \quad u \in \mathcal{D}(L_0^s).$$

But for all  $u$  in  $\mathcal{D}(L_0^s)$ ,

$$\|L_0^s u\|_2^2 = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (2k+1)^{2s} |(u, e_{jk})|^2 \sim \|u\|_{s,2}^2.$$

So,  $\|\cdot\|_{s,2}$  is a norm in  $H^{s,2}$ . For  $s < 0$ , we see easily that  $H^{s,2}$  can be identified with the dual space of  $H^{-s,2}$  via the pairing  $\langle \cdot, \cdot \rangle$  given by

$$\langle u, v \rangle = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (1+k^2)^{s/2} (u, e_{jk}) (1+k^2)^{-s/2} (e_{jk}, v)$$

for all  $u$  in  $H^{s,2}$  and  $v$  in  $H^{-s,2}$ . So,  $H^{s,2}$  is complete.  $\square$

From the proof of Theorem 6, we obtain the following result.

THEOREM 7.  $\mathcal{D}(L_0) = H^{1,2}$ .

## 7. Sobolev inequalities

The following theorem gives another measure of the global hypoellipticity of the twisted Laplacian  $L$  using the Sobolev spaces in the preceding section.

THEOREM 8. Let  $s \in \mathbb{R}$ . Then

$$u \in \mathcal{S}'(\mathbb{R}^2), Lu \in H^{s,2} \Rightarrow u \in H^{s+1,2}.$$

In fact, there exist positive constants  $C_1$  and  $C_2$  such that

$$C_1 \|u\|_{s+1,2} \leq \|Lu\|_{s,2} \leq C_2 \|u\|_{s+1,2}, \quad u \in H^{s+1,2}.$$

*Proof.* For all  $u \in H^{s+1,2}$ ,

$$\begin{aligned} \|Lu\|_{s,2}^2 &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (1+k^2)^s (2k+1)^2 |(u, e_{jk})|^2 \\ &\sim \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (1+k^2)^{s+1} |(u, e_{jk})|^2 \\ &= \|u\|_{s+1,2}^2. \end{aligned}$$

$\square$

The inequalities in Theorem 8 are the analogues for the twisted Laplacian  $L$  of the Agmon–Douglis–Nirenberg inequalities for elliptic boundary value problems in [1] and elliptic pseudo-differential operators, *i.e.*, Proposition 13.13, in [12].

REMARK 3. The Sobolev space  $H^{s,2}$  is the same as the Shubin Sobolev space  $Q^{s,2}$  given by

$$Q^{s,2} = \{u \in S'(\mathbb{R}^2) : L^s u \in L^2(\mathbb{R}^2)\}$$

and equipped with the norm  $\|\cdot\|_{Q^{s,2}}$ , where

$$\|u\|_{Q^{s,2}} = \|L^s u\|_2, \quad u \in Q^{s,2}.$$

Let  $u \in H^{s,2}$ . Then  $L^s u \in L^2(\mathbb{R}^2)$ . Since  $L^s$  is a locally elliptic operator of order  $2s$ , it follows from the standard theory of elliptic regularity that  $u$  is locally in the standard Sobolev space of order  $2s$ .

## 8. Gelfand–Shilov spaces

Let  $\mu$  and  $\nu$  be positive real numbers such that  $\mu + \nu \geq 1$ . Then the Gelfand–Shilov space  $S_V^\mu(\mathbb{R}^n)$  is defined to be the set of all functions  $\varphi$  in  $C^\infty(\mathbb{R}^n)$  for which there exists a positive constant  $C$  such that for all multi-indices  $\alpha$  and  $\beta$ ,

$$|x^\alpha (\partial^\beta \varphi)(x)| \leq C^{|\alpha|+|\beta|+1} (\alpha!)^\nu (\beta!)^\mu, \quad x \in \mathbb{R}^n.$$

It can be shown that a function  $\varphi$  is in  $S_V^\mu(\mathbb{R}^n)$  if and only if there exist positive constants  $C$  and  $\varepsilon$  such that for all multi-indices  $\alpha$ ,

$$|(\partial^\alpha \varphi)(x)| \leq C^{|\alpha|+1} (\alpha!)^\mu e^{-\varepsilon|x|^{1/\nu}}, \quad x \in \mathbb{R}^n.$$

This characterization tells us that a function in a Gelfand–Shilov space has exponential decay at infinity. Another interesting characterization is that a function is in a Gelfand–Shilov space if and only if the function itself and its Fourier transform have exponential decay at infinity. More precisely, a function  $\varphi$  is in the Gelfand–Shilov space  $S_V^\mu(\mathbb{R}^n)$  if and only if there exist positive constants  $C$  and  $\varepsilon$  such that

$$|\varphi(x)| \leq C e^{-\varepsilon|x|^{1/\nu}}, \quad x \in \mathbb{R}^n,$$

and

$$|\hat{\varphi}(\xi)| \leq C e^{-\varepsilon|\xi|^{1/\mu}}, \quad \xi \in \mathbb{R}^n.$$

Moreover, the Gelfand–Shilov space  $S_1^1(\mathbb{R}^n)$  is the same as the space  $\mathcal{F}$  of test functions for Fourier hyperfunctions. In fact,  $\mathcal{F}$  is the set of all functions  $\varphi$  in  $C^\infty(\mathbb{R}^n)$  for which there exist positive constants  $C$ ,  $\varepsilon$  and  $\delta$  such that for all multi-indices  $\alpha$ ,

$$|(\partial^\alpha \varphi)(x)| \leq C \delta^{|\alpha|} \alpha! e^{-\varepsilon|x|}, \quad x \in \mathbb{R}^n.$$

See Chapter 4 of [6] and the paper [5] for details on Gelfand–Shilov spaces. Recent applications of Gelfand–Shilov spaces in the regularity and exponential decay of solutions of partial differential equations can be found in [3, 4].

The following theorem tells us that the global hypoellipticity of the twisted Laplacian can also be measured by the Gelfand–Shilov spaces.

**THEOREM 9.** *The twisted Laplacian  $L$  is globally hypoelliptic in the Gelfand–Shilov spaces. More precisely, let  $\mu$  and  $\nu$  be positive real numbers such that  $\mu + \nu \geq 1$ . Then*

$$u \in S'(\mathbb{R}^2), Lu \in S_\nu^\mu(\mathbb{R}^2) \Rightarrow u \in S_\nu^\mu(\mathbb{R}^2).$$

For a proof of Theorem 9, we make use of an estimate in [13] for the modified Bessel function  $K_0$  of order 0, which comes up in the Green function for the twisted Laplacian described in Section 4. For the sake of completeness, we recall it here.

**PROPOSITION 2.** *For every positive number  $\eta$ , there exists a positive constant  $C_\eta$  such that*

$$|K_0(x)| \leq C_\eta x^{-\eta}, \quad x > 0.$$

*Proof of Theorem 9.* Since  $f \in S_\nu^\mu(\mathbb{R}^2)$ , it follows that there exists a positive constant  $C$  such that for all multi-indices  $\alpha$  and  $\beta$ ,

$$(2) \quad |z^\alpha (\partial^\beta f)(z)| \leq C^{|\alpha|+|\beta|+1} (\alpha!)^\nu (\beta!)^\mu, \quad z \in \mathbb{C}.$$

Let  $f = Lu$ . Then

$$u(z) = (L^{-1}f)(z) = \int_{\mathbb{C}} G(z, w) f(w) dw = \int_{\mathbb{C}} g(w) e^{i\frac{1}{4}[z, w]} f(z - w) dw$$

for all  $z$  in  $\mathbb{C}$ , where

$$g(w) = \frac{1}{4\pi} K_0\left(\frac{1}{4}|w|^2\right), \quad w \in \mathbb{C}.$$

For all multi-indices  $\beta$ , we write for all  $z$  in  $\mathbb{C}$ ,

$$(\partial^\beta u)(z) = \int_{\mathbb{C}} g(w) \partial_z^\beta \left( e^{i\frac{1}{4}[z, w]} f(z - w) \right) dw.$$

To see that the interchange of differentiation and integration is permitted, we write

$$\int_{\mathbb{C}} |g(w)| \left| \partial_z^\beta \left( e^{i\frac{1}{4}[z, w]} f(z - w) \right) \right| dw = I_1(z) + I_2(z),$$

where

$$I_1(z) = \int_{|w| \leq 1} |g(w)| \left| \partial_z^\beta \left( e^{i\frac{1}{4}[z, w]} f(z - w) \right) \right| dw$$

and

$$I_2(z) = \int_{|w| \geq 1} |g(w)| \left| \partial_z^\beta \left( e^{i\frac{1}{4}[z, w]} f(z - w) \right) \right| dw.$$

To estimate  $I_1(z)$ , we use the inequality (2) and Leibniz' formula to obtain a positive constant  $C_1$  such that

$$I_1(z) \leq C_1^{|\beta|+1} (\beta!)^\mu \int_{|w| \leq 1} |g(w)| dw, \quad z \in \mathbb{C}.$$

By Proposition 2, we see that

$$\int_{|w| \leq 1} |g(w)| dw < \infty$$

and hence we get a positive constant  $C_2$  for which

$$I_1(z) \leq C_2^{|\beta|+1} (\beta!)^\mu, \quad z \in \mathbb{C}.$$

Similarly, we see that there exists a positive constant  $C_3$  such that

$$I_2(z) \leq C_3^{|\beta|+1} (\beta!)^\mu \int_{|w| \geq 1} |w|^{|\beta|} |g(w)| dw, \quad z \in \mathbb{C}.$$

By Proposition 2, we see that there exists a positive constant  $C_4$  such that

$$I_2(z) \leq C_4^{|\beta|+1} (\beta!)^\mu, \quad z \in \mathbb{C}.$$

Now, let  $\alpha$  and  $\beta$  be arbitrary multi-indices with  $\alpha \neq 0$ . Then for all  $z$  in  $\mathbb{C}$ ,

$$|z^\alpha (\partial^\beta u)(z)| \leq 2^{|\alpha|} (J_1(z) + J_2(z)),$$

where

$$J_1(z) = \int_{\mathbb{C}} |w|^{|\alpha|} |g(w)| \left| \partial_z^\beta \left( e^{i\frac{1}{4}[z,w]} f(z-w) \right) \right| dw$$

and

$$J_2(z) = \int_{\mathbb{C}} |g(w)| |z-w|^{|\alpha|} \left| \partial_z^\beta \left( e^{i\frac{1}{4}[z,w]} f(z-w) \right) \right| dw.$$

As in the case when  $\alpha = 0$ , we can get a positive constant  $C_5$  such that

$$J_1(z) \leq C_5^{|\alpha|+|\beta|+1} (\alpha!)^\nu (\beta!)^\mu, \quad z \in \mathbb{C}.$$

Now, we note that by Leibniz' formula and the inequality (2), we get a positive constant  $C_6$  such that

$$J_2(z) \leq C_6^{|\alpha|+|\beta|+1} (\alpha!)^\nu (\beta!)^\mu \int_{\mathbb{C}} |w|^{|\beta|} |g(w)| dw, \quad z \in \mathbb{C}.$$

So, as in the case when  $\alpha = 0$ , we can estimate  $\int_{\mathbb{C}} |w|^{|\beta|} |g(w)| dw$  by breaking  $\mathbb{C}$  into  $|w| \leq 1$  and  $|w| \geq 1$ . Hence there exists a positive constant  $C_7$  such that

$$J_2(z) \leq C_7^{|\alpha|+|\beta|+1} (\alpha!)^\nu (\beta!)^\mu, \quad z \in \mathbb{C}.$$

This completes the proof.

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