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AN APPLICATION OF THE MONTGOMERY IDENTITY TO QUADRATURE RULES

Abstract. We use the Montgomery identity to obtain an optimal quadrature rule. It turns out that this rule is the well-known compound trapezoidal rule.

1. Introduction

The Montgomery identity is recently considered by many authors ([1]-[5]). This identity has the form

$$f(x)=\frac{1}{b-a}\int_a^b f(t)dt+\int_a^b K(x,t)f'(t)dt,$$

where

$$K(t)=\begin{cases} t-a, & t\in[a,x] \\ t-b, & t\in(x,b] \end{cases}.$$

From this identity we can obtain the well known integral Ostrowski inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right]\|f'\|_\infty(b-a).$$

This inequality is considered in really many recently published papers. It gives an error bound for a simple quadrature rule.

In this paper we use the Montgomery identity and derive an optimal quadrature rule with respect to a given way of estimation of its error. It turns out that this optimal quadrature rule is in fact the well-known compound trapezoid rule. In such a way we proved that the compound trapezoidal rule is the best possible in a given class of quadrature rules if we a priory give the way of estimation of remainder terms for these rules. In this sense the results of this paper can be considered as a generalization of results obtained in [6].

2. Quadrature rules

First we consider a problem in the interval $[0,1]$. Let $\{0=x_0<x_1<\cdots<x_n=1\}$ be a partition of $[0,1]$ and

$$\sum_{i=0}^{n} w_i=1,\ w_i\in R. \tag{1}$$

If $x_i \in [0,1]$ then, using the Montgomery identity, we have

$$f(x_i) = \int_0^1 f(t)dt + \int_0^1 K(x_i,t) f'(t)dt, \tag{2}$$

where

$$K_0(t) = t-1, t \in [0,1],$$

$$K_i(t) = \begin{cases} t, & t \in [0,x_i] \\ t-1, & t \in (x_i,1] \end{cases},$$

for $i = 1,2,...,n-1$, $x_i \in [0,1]$ and

$$K_n(t) = t, t \in [0,1].$$

From (2) we get

$$\sum_{i=0}^{n} w_i f(x_i) = \int_0^1 f(t)dt + \int_0^1 K(t) f'(t)dt,$$

where

$$K(t) = \sum_{i=0}^{n} w_i K(x_i,t). \tag{3}$$

The function $K(t)$ can be written in the equivalent form

$$K(t) = \begin{cases} t - w_0, & t \in [0,x_1] \\ t - \sum\limits_{j=0}^{1} w_j, & t \in (x_1,x_2] \\ t - \sum\limits_{j=0}^{2} w_j, & t \in (x_2,x_3] \\ \cdots \\ t - \sum\limits_{j=0}^{n-1} w_j, & t \in (x_{n-1},1] \end{cases}.$$

We have

$$\left| \int_0^1 f(t)dt - \sum_{i=0}^{n} w_i f(x_i) \right| \leq \left\| f' \right\|_\infty \int_0^1 |K(t)|\, dt.$$

We now consider the minimizing problem

$$\int_0^1 |K(t)|\, dt \to \min. \tag{4}$$

If we introduce the notation

$$\sigma_i = \sum_{j=0}^{i} w_j$$

then the problem (4) can be written in the form

$$\int_0^{x_1} |t-\sigma_0|\,dt + \cdots + \int_{x_{n-1}}^{1} |t-\sigma_{n-1}|\,dt \to \min. \tag{5}$$

We required that $\sigma_i \in [x_i, x_{i+1}]$, $i = 0,1,...,n-1$. Then we have

$$\begin{aligned} \int_{x_i}^{x_{i+1}} |t-\sigma_i|\,dt &= \int_{x_i}^{\sigma_i} (\sigma_i - t)\,dt + \int_{\sigma_i}^{x_{i+1}} (t-\sigma_i)dt \\ &= \frac{1}{2}\left[(\sigma_i - x_i)^2 + (x_{i+1}-\sigma_i)^2\right]. \end{aligned} \tag{6}$$

Using the relation (6) we can write the problem (5) in the form

$$F(w_0,...,w_{n-1}) = \frac{1}{2}\sum_{i=0}^{n-1}\left[(\sigma_i - x_i)^2 + (x_{i+1}-\sigma_i)^2\right] \to \min.$$

We now solve this problem. For that purpose we calculate

$$\frac{\partial F}{\partial w_j} = \sum_{i=0}^{n-1}\left[(\sigma_i - x_i)\frac{\partial \sigma_i}{\partial w_j} - (x_{i+1}-\sigma_i)\frac{\partial \sigma_i}{\partial w_j}\right], \tag{7}$$

for $j = 0,1,2,...,n-1$. Since

$$\frac{\partial \sigma_i}{\partial w_j} = \begin{cases} 1, & j \le i \\ 0, & j > i \end{cases}$$

from (7) we get

$$\begin{aligned} \frac{\partial F}{\partial w_0} &= \sum_{i=0}^{n-1}\left[2\sigma_i - (x_i + x_{i+1})\right] \\ \frac{\partial F}{\partial w_1} &= \sum_{i=1}^{n-1}\left[2\sigma_i - (x_i + x_{i+1})\right] \\ &\cdots \\ \frac{\partial F}{\partial w_{n-1}} &= 2\sigma_{n-1} - (x_{n-1} + x_n). \end{aligned}$$

If we wish to find the minimum we have to require that

$$\frac{\partial F}{\partial w_j} = 0,\ j = 0,1,2,...,n-1.$$

We also introduce the notations

$$\begin{aligned} p_j &= \sum_{i=j}^{n-1}(x_i + x_{i+1}), \\ q_j &= \sum_{i=j}^{n-1}\sigma_i, \end{aligned}$$

for $j = 0,1,2,...,n-1$. The we can write the above system in the form

$$\begin{aligned} 2q_0 - p_0 &= 0 \\ 2q_1 - p_1 &= 0 \\ &\cdots \\ 2q_{n-1} - p_{n-1} &= 0. \end{aligned}$$

Since

$$\begin{aligned} p_j - p_{j+1} &= x_j + x_{j+1}, \\ q_j - q_{j+1} &= \sigma_j, \end{aligned}$$

we have

$$\begin{aligned} 2\sigma_0 - (x_0 + x_1) &= 0 \\ 2\sigma_1 - (x_1 + x_2) &= 0 \\ &\cdots \\ 2\sigma_{n-1} - (x_{n-1} + x_n) &= 0. \end{aligned}$$

It is not difficult to solve the above system. We have

$$\sigma_0 = w_0 = \frac{x_0 + x_1}{2},$$

$$\sigma_1 = w_0 + w_1 = \frac{x_1 + x_2}{2}$$

such that

$$w_1 = \frac{x_2 - x_0}{2}.$$

Generally, we have

$$w_j = \frac{x_{j+1} - x_{j-1}}{2},\ j = 2,3,..,n-1,$$

and

$$w_n = \frac{1 - x_{n-1}}{2},$$

since (1) holds.

In fact, we get the next formula

$$\int_0^1 f(t)dt = \sum_{i=0}^{n} w_i f(x_i) - \int_0^1 K(t) f'(t)dt,$$

where $K(t)$ is given by (3) and w_i are given above. Hence, we got the following result.

THEOREM 1. *Let $f \in C^1(0,1)$. Then*

$$\int_0^1 f(t)dt = \frac{h_0 f(0) + h_{n-1} f(1)}{2} + \sum_{i=1}^{n-1} \frac{h_{i-1} + h_i}{2} f(x_i) + R(f), \tag{8}$$

where

$$R(f) = -\int_0^1 K(t) f'(t) dt,$$

$K(t)$ *is given by (3),* $h_i = x_{i+1} - x_i$, $i = 0, 1, 2, ..., n-1$ *and*

$$|R(f)| \leq \frac{\|f'\|_\infty}{4} \sum_{i=0}^{n-1} h_i^2. \tag{9}$$

Proof. The relation (8) is proved above. We have

$$\begin{aligned} \int_0^1 |K(t)|\, dt &= \sum_{i=0}^{n-1} \left[\int_{x_i}^{\sigma_i} (\sigma_i - t)\, dt + \int_{\sigma_i}^{x_{i+1}} (t - \sigma_i) dt \right] \\ &= \frac{1}{2} \sum_{i=0}^{n-1} \left[(\sigma_i - x_i)^2 + (x_{i+1} - \sigma_i)^2 \right] \\ &= \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^2}{4}, \end{aligned}$$

since

$$\sigma_i = \frac{x_i + x_{i+1}}{2},\ i = 0, 1, 2, ..., n-1.$$

Hence, the relation (9) holds, too. □

COROLLARY 1. *Let* $f \in C^1(0,1)$. *Then*

$$\int_0^1 f(t) dt = \frac{f(0) + f(1)}{2} h + h \sum_{i=1}^{n-1} f(x_i) + R_1(f), \tag{10}$$

where

$$R_1(f) = -\int_0^1 K(t) f'(t) dt,$$

$K(t)$ *is given by (3),* $h = x_{i+1} - x_i = 1/n$, $i = 0, 1, 2, ..., n-1$ *and*

$$|R_1(f)| \leq \frac{h^2}{4} \|f'\|_\infty = \frac{\|f'\|_\infty}{4n}.$$

REMARK 1. Note that the quadrature rule given in Corollary 1 is the well-known compound trapezoidal quadrature rule. From the above considerations we can also conclude that this rule is optimal in the sense that it has a minimal error bound when we estimate this error in the described way.

In fact, in both above cases we can obtain a better estimation of the error.

THEOREM 2. *Let the assumptions of Theorem 1 and Corollary 1 hold. If* $\gamma, \Gamma \in R$ *are numbers such that* $\gamma \leq f'(t) \leq \Gamma$, $t \in [0,1]$ *then*

$$|R(f)| \leq \frac{\Gamma - \gamma}{8} \sum_{i=0}^{n-1} h_i^2, \tag{11}$$

$$|R_1(f)| \leq \frac{h^2}{8}(\Gamma-\gamma) \tag{12}$$

Proof. We have

$$\begin{aligned}\int_0^1 K(t)dt &= \int_0^{x_1}(t-\sigma_0)\,dt+\cdots+\int_{x_{n-1}}^1(t-\sigma_{n-1})\,dt\\ &= \frac{1}{2}\sum_{i=0}^{n-1}\left[(\sigma_i-x_{i+1})^2-(x_i-\sigma_i)^2\right]\\ &= 0,\end{aligned}$$

since

$$\sigma_i=\frac{x_i+x_{i+1}}{2},\ i=0,1,2,...,n-1.$$

Thus,

$$\int_0^1 K(t)(f'(t)-\frac{\Gamma+\gamma}{2})dt=\int_0^1 K(t)f'(t)$$

and

$$\int_0^1\left|K(t)(f'(t)-\frac{\Gamma+\gamma}{2})\right|\leq\frac{\Gamma-\gamma}{2}\int_0^1|K(t)|\,dt,$$

since

$$\left\|f'-\frac{\Gamma+\gamma}{2}\right\|_\infty\leq\frac{\Gamma-\gamma}{2}.$$

Now it is not difficult to see that (11) and (12) hold. □

If we write the above rule in the interval $[a,b]$ then we get the next result.

THEOREM 3. *Let $f\in C^1(a,b)$. Then*

$$\int_a^b f(t)dt=\frac{h_0f(a)+h_{n-1}f(b)}{2}+\sum_{i=1}^{n-1}\frac{h_{i-1}+h_i}{2}f(x_i)+R(f),$$

where $h_i=x_{i+1}-x_i$, $i=0,1,2,...,n-1$ and

$$|R(f)|\leq\frac{\Gamma-\gamma}{8}\sum_{i=0}^{n-1}h_i^2,$$

for $\gamma,\Gamma\in R$ such that $\gamma\leq f'(t)\leq\Gamma$, $t\in[a,b]$.

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