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M. Bernardara

CALABI-YAU COMPLETE INTERSECTIONS WITH INFINITELY MANY LINES

Abstract. We give two new examples of families of Calabi-Yau complete intersection threefolds whose generic element contains infinitely many lines. We get some results about the normal bundles of these lines and the Hilbert scheme of lines on the threefolds. In particular, the surface swept out by such a family is not a cone.

1. Calabi-Yau complete intersections and lines on them

Throughout the paper, CY is used instead of Calabi-Yau.

The Clemens conjecture originally states that on the generic quintic threefold the number of rational curves in a fixed homology class is finite. More generally, the conjecture is expected to hold also for CY complete intersection threefolds in ordinary projective spaces (see [8]). In particular, all lines on a CY threefold lie in the same homology class, hence the conjecture states that the number of lines on the generic such threefold is finite.

Moreover, the expected number of lines on a generic CY complete intersection threefold can be computed with algebraic geometric techniques such as Schubert calculus in the Grassmannians.

We get the same result about CY manifolds in mirror symmetry: there is a way to predict correctly the number n_d of rational curves of a given degree d lying on the generic CY threefold.

Recall that a CY threefold is a complex compact Kähler threefold X with trivial canonical bundle:

$$K_X \simeq \mathcal{O}_X.$$

We will call a complete intersection of type $(d_1,\dots,d_k)$ a threefold which is a complete intersection of k hypersurfaces in $\mathbb{P}^{k+3}$ of degrees $d_1,\dots,d_k$ respectively.

The adjunction formula for a complete intersection of type $(d_1,\dots,d_k)$

$$K_X \cong \mathcal{O}_X(\sum_{i=1}^{k} d_i - k - 4)$$

allows to conclude that the only projective CY threefolds that are complete intersections are of type (5), (the quintic threefold in $\mathbb{P}^4$), $(3,3)$ and $(4,2)$ in $\mathbb{P}^5$, $(3,2,2)$ in $\mathbb{P}^6$ and $(2,2,2,2)$ in $\mathbb{P}^7$.

Using Schubert calculus, we have the following results about the number of

lines on the generic threefold:

$$\begin{array}{rcl} (5) & 2875 & \text{lines} \\ (3,3) & 1053 & \text{lines} \\ (4,2) & 1280 & \text{lines} \\ (3,2,2) & 720 & \text{lines} \\ (2,2,2,2) & 512 & \text{lines.} \end{array}$$

These results agree with mirror symmetry predictions (see [5, 6] for mirror symmetry techniques, [12] for the case (3,3)).

The genericity assumption in Clemens conjecture is crucial: in fact we know examples of CY threefolds with infinitely many lines. The simplest is the Fermat quintic threefold in $\mathbb{P}^4$, defined by the equation

$$x_0^5+x_1^5+x_2^5+x_3^5+x_4^5=0.$$

The lines on the threefold are described in [2].

However, in [4], Clemens asked if we can find a continuous family of lines on a smooth quintic threefold which is not a cone, as in the Fermat quintic case. The first such example is due to van Geemen. He found infinitely many lines on the generic threefold of a family called the Dwork pencil. Its equation is

$$x_0^5+x_1^5+x_2^5+x_3^5+x_4^5-5\lambda x_0x_1x_2x_3x_4=0$$

hence we get a pencil of quintic hypersurfaces in $\mathbb{P}^4$, whose zero fiber is the Fermat.

To see how to find lines on them, see [1], for a deeper investigation, see [14]. The result is obtained by showing that on the generic threefold of the family there are more than the expcted 2875 lines. This can be done choosing a "good" automorphism of the threefold and finding lines fixed by it. In this case "good" means that its order does not divide the expected number of lines, so it has fixed lines. Under the action of the automorphisms of the threefold, the orbit of one of those contains at least 5000 lines, clearly more than 2875. Moreover, this example gives a positive answer to Clemens [4] question. Indeed, a simple calculation of the normal bundle allows us to say that the family of lines on the generic quintic of the pencil is not a cone.

In this paper, we are giving two new examples of families of CY threefolds whose generic element contains infinitely many lines. The first one is a pencil of $(3,3)$ complete intersections, the second one a 2-parameter family of $(2,2,2,2)$ complete intersections. Both provide a postive answer to Clemens [4] question, as stated in corollaries 1 and 2.

2. A $(3,3)$ complete intersection pencil

In this paper we consider exclusively projective spaces over the complex field $\mathbb{C}$.

The first example is a pencil of CY threefolds of type $(3,3)$. On this particular pencil, we are able to construct more lines than expected (1458 instead of 1053).

The equations of the generic threefold X_λ (smooth for generic λ) of the pencil are:

$$X_\lambda := \begin{cases} x_0^3 + x_1^3 + x_2^3 - 3\lambda x_3 x_4 x_5 = 0 \\ x_3^3 + x_4^3 + x_5^3 - 3\lambda x_0 x_1 x_2 = 0. \end{cases} \tag{1}$$

This pencil is invariant under a group of automorphisms of $\mathbb{P}^5$ of order 81 (see [12]).

Let ϕ be the involution of $\mathbb{P}^5$ given by the change of coordinates $(12)(45) \in S_6$, which preserves X_λ. We consider its invariant subspaces $V_\pm$:

$$\begin{aligned} V_+ &= \{(a:a:b:c:c:d)\} \\ V_- &= \{(q:-q:0:p:-p:0)\}. \end{aligned} \tag{2}$$

Consider lines either contained in one of these subspaces or intersecting both; such lines are ϕ-invariant. In this case there is no line lying on X_λ entirely contained in $V_\pm$, but we have the following result.

LEMMA 1. *On the generic threefold X_λ there are 36 lines connecting the invariant subspaces (2), hence each one is fixed by ϕ.*

Proof. It can be easily seen that there are no points in $V_\pm$ lying on X_λ if $d = 0$ or $q = 0$, so, without loss of generality, consider:

$$V_+ = \{(a:a:b:c:c:1) | a,b,c \in \mathbb{C}\}$$

$$V_- = \{(1:-1:0:p:-p:0) | p \in \mathbb{C}\}.$$

Lines joining such points have parametric equations:

$$(at+s : at-s : bt : ct+ps : ct-ps : t) \tag{3}$$

where $(s:t)$ ranges over $\mathbb{P}^1$.

Substituting the equation (3) in the equations (1) of X_λ, we obtain two cubic homogeneous polynomials in s, t. The line belongs to the threefold if and only if these polynomials vanish identically. It appears in the following cases:

$$a^3 = \frac{(2c^3+1)\lambda}{12c}$$

$$b = \frac{4ac}{\lambda^2}$$

$$p^2 = -\frac{2a}{\lambda}$$

and c satisfies:

$$64c^6 - (16\lambda^6 - 32)c^3 + \lambda^6 = 0.$$

In particular, we have 6 values for c for generic λ, then we have 18 values for a and 36 for p. □

THEOREM 1. *On the generic threefold in the pencil X_λ there are infinitely many lines.*

Proof. We know from the preceding Lemma that we have 36 lines on X_λ. Pick one of them and call it l.
Consider the action of the group $(\mathbb{C}^*)^6$ on $\mathbb{P}^5$, where an element $(a_0,\dots,a_5) \in (\mathbb{C}^*)^6$ acts componentwise by:

$$(a_0,a_1,a_2,a_3,a_4,a_5)\cdot(x_0:x_1:x_2:x_3:x_4:x_5) =$$

$$(a_0x_0 : a_1x_1 : a_2x_2 : a_3x_3 : a_4x_4 : a_5x_5)$$

on $(x_0 : \dots : x_5) \in \mathbb{P}^5$.

Let α_i in $(\mathbb{C}^*)^6$ be the elements:

$$\begin{array}{ll} \alpha_1 & = (1,\omega,\omega^{-1},1,1,1) \\ \alpha_2 & = (1,1,\omega,\zeta,\zeta,\zeta^{-2}) \\ \alpha_3 & = (1,1,1,1,\omega,\omega^{-1}) \end{array} \tag{4}$$

where ζ is a primitive ninth root of unity and $\omega = \zeta^3$.

We note that the group

$$G := < \alpha_1,\alpha_2,\alpha_3 > \subset \mathrm{Aut}X_\lambda,$$

has order 81.

Two other subgroups of $\mathrm{Aut}X_\lambda$ are given by the actions of S_3 on the first three coordinates and on the last three. We denote the product of these two groups by H.

The orbit of the line l under the action of the group G has order 81, because no element of this group fixes l.

The orbit of l under the action of H has 18 elements, because $\phi = (12)(45)$ fixes l.

Consider now the group $G \times H$ and check that if $gh(l) = l$, where $g \in G$ and $h \in H$, then $g(l) = l$ and $h(l) = l$ (consider the points of $h(l) \cap V_-$ and then the action of G on these points).

Hence, the order of the orbit of l under the action of the group $G \times H$, is $81 \cdot 18 = 1458$. This number is larger than expected. □

REMARK 1. Recall the way of counting lines proposed by S. Katz in [10]. It is based on finding a compact moduli space $\mathcal{M}$ of the curves on the manifold, then constructing a rank $r = \dim\mathcal{M}$ vector bundle with some good properties and then computing its r-th Chern class. We note that, because of their construction with automorphisms, our lines have the same behavior. If they were isolated, each would count as one; this would make the calculation fail. Then we deduce that each of these lines belongs to a continuous family. Notice that this tells nothing about the number and the geometric properties of these families, except that this excludes the case that these lines have normal bundle of the form $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

3. Normal bundle and Hilbert scheme of lines

Recall the definition of normal bundle of a line L in a manifold X, as the cokernel in the exact sequence:

$$0 \longrightarrow T_L \longrightarrow T_{X|L} \longrightarrow N \longrightarrow 0. \tag{5}$$

Now we are looking for the normal bundle of a line L on a threefold X, which is a bundle over $\mathbb{P}^1$, hence we can split it as

$$N \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b). \tag{6}$$

From the CY condition we deduce

$$a+b=-2.$$

Let X be a projective variety and $Z \subset X$ a subvariety. It is well known (see [11]), that for the Zariski tangent space to the Hilbert scheme in $[Z]$ the following isomorphism holds:

$$T_{[Z]}\mathrm{Hilb}(X) \cong \mathrm{Hom}_X(I(Z), \mathcal{O}_Z) = \mathrm{Hom}_Z(I(Z)/I(Z)^2, \mathcal{O}_Z).$$

The right side is the zeroth cohomology group of the normal bundle of Z in X (see [7]), thus:

$$T_{[Z]}\mathrm{Hilb}(X) \cong H^0(N_{Z|X}). \tag{7}$$

In our case, we are looking for the normal bundle of lines lying in a continuous family, hence the Zariski tangent space to the Hilbert scheme in the point corresponding to these lines should be positive dimensional. This gives

$$N_{l|X_\lambda} \ncong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$$

because in this case we would have $h^0(N) = 0$.

3.1. How to calculate the normal bundle

In this section, we show how to calculate the normal bundle of a line on a CY complete intersection threefold and after we will apply the calculation to the lines previously constructed.

Our aim is to calculate a and b in (6), trying to generalize slightly the calculations in [9] to the complete intersection case.

First, let X be a hypersurface in $\mathbb{P}^n$ and $L \subset X$ a line on it. Change the coordinates of $\mathbb{P}^n$ such that the line L has parametrization $(s:t:0:\cdots:0)$; in this case, the ideal I_L of L is $I_L = (x_2, \cdots, x_n)$. Let us call F_d the polynomial defining X and d its degree. $L \subset X$ and L is the intersection of the hyperplanes $x_2 = \cdots = x_n = 0$, so we can write:

$$F_d = x_2F_2 + \cdots + x_nF_n.$$

Modulo elements of I_L^2, we get

$$F_d = x_2 f_2(x_0,x_1) + \cdots + x_n f_n(x_0,x_1)$$

where each f_i is homogeneous of degree $d-1$; it can be seen as $F_{i|L}$.

Using these exact sequences

$$\begin{array}{ll} a) & 0 \longrightarrow N \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1)^{n-1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(d) \longrightarrow 0 \\ b) & 0 \longrightarrow T_X \longrightarrow T_{\mathbb{P}^n|X} \longrightarrow \mathcal{O}_X(d) \longrightarrow 0 \end{array}$$

and (5), it is possible ([9]) to get the normal bundle as the kernel of the map

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}^1}(1)^{n-1} & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(d) \\ (s_2,\cdots,s_n) & \longmapsto & \sum_{i=2}^n f_i s_i. \end{array} \tag{8}$$

Now let X be a complete intersection of two hypersurfaces of degree d and e, given respectively by $F=0$ and $G=0$. Let $L \subset X$ be parametrized as before, so we get the homogeneous polynomials f_i and g_i of degrees $d-1$ and $e-1$ respectively in (x_0,x_1).

From a direct calculation, we get

$$N_{L|X} \simeq ker(\mathcal{O}_{\mathbb{P}^1}(1)^{n-1} \xrightarrow{M} \mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathcal{O}_{\mathbb{P}^1}(e))$$

where the map is given by the $2 \times (n-1)$ matrix M with rows given by the f_i and g_i. If we let $A = \mathbb{C}[x_0,x_1]$, we can rewrite this map as a map $A^{n-1} \to A^2$. Hence we are looking at the module

$$B = ker(A^{n-1}(1) \xrightarrow{M} A(d) \oplus A(e))$$

and we know that B has a basis of vectors of homogeneous polynomials T_i of the same degree (within the vector) t_i. In the case of a line in a threefold we have $i = 1,2$, hence:

$$N = \mathcal{O}_{\mathbb{P}^1}(1-t_1) \oplus \mathcal{O}_{\mathbb{P}^1}(1-t_2).$$

In conclusion we get the following result.

THEOREM 2. *Let T a vector of homogeneous polynomials of minimal degree t in (x_0,x_1) such that*

$$M \cdot T = 0$$

where M is the matrix with rows given by the f_i and the g_i. Then the normal bundle $N_{L|X}$ splits in the following way:

$$\begin{array}{lll} N_{L|X} & = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-3) & \text{if } t = 0 \\ N_{L|X} & = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) & \text{if } t = 1 \\ N_{L|X} & = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) & \text{otherwise.} \end{array}$$

Proof. This follows easily from the above considerations, remembering that we should have, for the CY condition, $t_1 - 1 + t_2 - 1 = -2$. □

The argument is essentially the same for a generic CY complete intersection threefold in projective space.

3.2. The normal bundle of constructed lines

We calculate the normal bundle of the lines constructed in the previous section on the generic threefold X_λ.

LEMMA 2. *Let λ be generic and $l \subset X_\lambda$ be the line parametrized by*

$$(at+s : at-s : bt : ct+ps : ct-ps : t)$$

as in Lemma 1. Then its normal bundle on X_λ splits as:

$$N_{l|X_\lambda} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2).$$

Proof. Recall that

$$N_{l|X_\lambda} \ncong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

Define new coordinates:

$$\begin{cases} x_0 &= y_0 + ay_5 \\ x_1 &= -y_0 + y_1 + ay_5 \\ x_2 &= y_2 + by_5 \\ x_3 &= py_0 + y_3 + cy_5 \\ x_4 &= -py_0 + y_4 + cy_5 \\ x_5 &= y_5. \end{cases}$$

l has now parametrization $(s:0:0:0:0:t)$.
We can obtain the matrix M, with coefficients homogeneous quadratic polynomials in y_0 and y_5:

$$\begin{pmatrix} (y_0-ay_5)^2 & b^2y_5^2 & \lambda p(y_0y_5) - c\lambda y_5^2 & -\lambda p(y_0y_5) - c\lambda y_5^2 \\ -b\lambda(y_0y_5+y_5^2) & \lambda(y_0^2-a^2y_5^2) & (py_0+cy_5)^2 & (py_0-cy_5)^2 \end{pmatrix}.$$

Now we verify that there are no nonzero vectors $B \in \mathbb{C}^4$ such that $M \cdot B = 0$. This leads to:

$$N_{l|X_\lambda} \ncong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-3).$$

□

COROLLARY 1. *Let l be a line on X_λ in the family constructed in section 2. Then its normal bundle splits as:*

$$N_{l|X_\lambda} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2).$$

This means that the surface swept out by these lines on X_λ is not a cone.

Proof. Let l be as in Lemma 2. Each line constructed in Theorem 1 can be obtained by l using an automorphism of X_λ. □

Moreover we can conclude the dimension of the Hilbert scheme is positive, in particular

$$\dim T_{[l]}\mathcal{H}_\lambda = h^0(N_{l|X_\lambda}) = 1$$

for the lines l we constructed in section 2.

4. A (2,2,2,2) two-parameter Family

We now give a new example, a two-parameter family of $(2,2,2,2)$ threefolds in $\mathbb{P}^7$. Consider the family (smooth for generic (λ,μ)) obtained by the complete intersection of the four quadrics

$$X_{\lambda,\mu} := \begin{cases} x_0^2+x_1^2+x_2^2+x_3^2+x_4^2+x_5^2 - 2\mu x_6x_7 = 0 \\ x_0^2+x_1^2+x_2^2+x_3^2 + x_6^2+x_7^2 - 2\lambda x_4x_5 = 0 \\ x_0^2+x_1^2 + x_4^2+x_5^2+x_6^2+x_7^2 - 2\lambda x_2x_3 = 0 \\ x_2^2+x_3^2+x_4^2+x_5^2+x_6^2+x_7^2 - 2\lambda x_0x_1 = 0. \end{cases} \tag{9}$$

As in the previous case, on the generic threefold more than the 512 expected lines are shown.
The technique is the same: in this case we take ϕ to be the order 3 automorphism of $X_{\lambda,\mu}$ given by the permutation of coordinates $(135)(246)$ in $\mathbb{P}^7$. On $\mathbb{P}^7$ we consider its invariant subspaces:

$$V_+ = \{(a:b:a:b:a:b:c:d)\}$$
$$V_\omega = \{(p:q:\omega p:\omega q:\omega^2 p:\omega^2 q:0:0)\}$$

where $(a:b:c:d)\in\mathbb{P}^3$, $(p:q)\in\mathbb{P}^1$ and $\omega\in\mathbb{C}$ is primitive third root of the unity.

LEMMA 3. *For generic (λ,μ), there are 8 lines on $X_{\lambda,\mu}$ intersecting both V_+ and V_ω.*

Proof. First we verify that the points with $b=0$ and the ones with $p=0$ don't lie on the threefold, hence we can consider:

$$V_+ = \{(a:1:a:1:a:1:c:d)|a,c,d\in\mathbb{C}\} \tag{10}$$

$$V_\omega = \{(1:q:\omega:\omega q:\omega^2:\omega^2 q:0:0)|q\in\mathbb{C}\}. \tag{11}$$

Lines joining these points have parametric equation

$$(s+at:qs+t:\omega s+at:\omega qs+t:\omega^2 s+at:\omega^2 qs+t:ct:dt) \tag{12}$$

where $(s:t)\in\mathbb{P}^1$.Substituting these values into the equations of the generic threefold, the line lies on $X_{\lambda,\mu}$ if and only if

$$a = -\frac{\lambda+q}{\lambda q+1}$$
$$d = \frac{3a^2+3}{2\mu c}$$

where q is a root of

$$q^2+2\lambda q+1=0$$

and c is a root of

$$4\mu^2c^4+(2a^2-2\lambda a+2)c^2+(3a^2+3)^2=0.$$

We get the proof, remarking that if we have 2 values for a, then it is clear that we have 8 values for c for the generic couple (λ,μ) and that this does not depend on a and on q. □

THEOREM 3. *On the generic threefold of the pencil $X_{\lambda,\mu}$ there are infinitely many lines.*

Proof. A subgroup of $\mathrm{Aut}X_{\lambda,\mu}$ is given by the action of S_3 on the first three pairs of coordinates. The orbit of one of the constructed lines under this automorphism group consists of 2 lines, because ϕ belongs to this group.
Another subgroup of automorphisms is generated by the permutations of coordinates (12), (34), (56) and (78): the constructed lines are not fixed by any of these automorphisms, hence the orbit of each line under the action of this subgroup has 16 elements.
Let G be the product of these two groups (in particular, $G\leq S_8$).
Let H be the subgroup of $(\mathbb{C}^*)^8$, acting on $\mathbb{P}^7$ by the coordinatewise product, with generators

$$\begin{array}{l}\alpha_1=(\ -1,\ -1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1)\\ \alpha_2=(\ \ 1,\ \ 1,\ -1,\ -1,\ \ 1,\ \ 1,\ \ 1,\ \ 1)\\ \alpha_3=(\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ -1,\ -1,\ \ 1,\ \ 1).\end{array}$$

The orbit of each line under its action consists of 8 lines.
Consider now the group $G\times H$: we have to check that if $gh(l)=l$, where $g\in G$ and $h\in H$, then $g(l)=l$ and $h(l)=l$ (consider the points in the set $h(l)\cap V_\omega$ and then the action of G on these points). We get finally that the orbit of each line under the action of this group consists of 256 elements.
The key remark now is that the orbits of the lines are disjoint, and this is made making a table comparing the values obtained for the points in V_ω and V_+ starting from different values of c.
We finally get on $X_{\lambda,\mu}$ at least 2048 lines, that is more than expected. □

REMARK 2. The same argument used in Remark 1 is valid in this case, so all the constructed lines belong to a continuous family.

4.1. The normal bundle

LEMMA 4. *The line $l\subset X_{\lambda,\mu}$ parametrized by*

$$(s+at:qs+t:\omega s+at:\omega qs+t:\omega^2s+at:\omega^2qs+t:ct:dt)$$

as in Lemma 3, has normal bundle

$$N_{l|X_{\lambda,\mu}}\cong\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(-2).$$

Proof. As before:

$$N_{l|X_{\lambda,\mu}} \ncong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

Now with the change of coordinates:

$$\begin{cases} x_0 &= y_0 + ay_7 \\ x_1 &= qy_0 + y_1 + y_7 \\ x_2 &= \omega y_0 + y_2 + ay_7 \\ x_3 &= \omega q y_0 + y_3 + y_7 \\ x_4 &= \omega^2 y_0 + y_4 + ay_7 \\ x_5 &= \omega^2 q y_0 + y_5 + y_7 \\ x_6 &= y_6 + cy_7 \\ x_7 &= dy_7 \end{cases}$$

the line l gets parametrization $(s:0:0:0:0:0:0:t)$. We calculate the matrix M, with coefficients linear homogeneous polynomials in y_0 and y_7:

$$\begin{pmatrix} qy_0+y_7 & \omega y_0+ay_7 & \omega q y_0+y_7 & \omega^2 y_0+ay_7 & \omega^2 q y_0+y_7 & -\mu d y_7 \\ qy_0+y_7 & \omega y_0+ay_7 & \omega q y_0+y_7 & -\lambda(\omega^2 q y_0+y_7) & -\lambda(\omega^2 y_0+ay_7) & cy_7 \\ qy_0+y_7 & -\lambda(\omega q y_0+y_7) & -\lambda(\omega y_0+ay_7) & \omega^2 y_0+ay_7 & \omega^2 q y_0+y_7 & cy_7 \\ -\lambda(y_0+ay_7) & \omega y_0+ay_7 & \omega q y_0+y_7 & \omega^2 y_0+ay_7 & \omega^2 q y_0+y_7 & cy_7 \end{pmatrix}.$$

We now verify that for generic (λ,μ) there are no nonzero vectors B in $\mathbb{C}^6$ such that $M \cdot B = 0$ and then $N_{l|X_{\lambda,\mu}} \ncong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$. □

COROLLARY 2. *Let l be a line on $X_{\lambda,\mu}$ in the family previously constructed. Then its normal bundle splits as:*

$$N_{l|X_{\lambda,\mu}} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2).$$

This means that the surface swept out by these lines on $X_{\lambda,\mu}$ is not a cone.

Moreover, the dimension of the Hilbert scheme is positive, in particular:

$$\dim T_{[l]}\mathcal{H}_\lambda = h^0(N_{l|X_(\lambda,\mu)}) = 1$$

for the lines l we constructed.

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Marcello BERNARDARA, Laboratoire J. A. Dieudonné, Université de Nice Sophia Antipolis, Parc Valrose, 06108 Nice Cedex 2, FRANCE
e-mail: `bernamar@math.unice.fr`

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