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THE WEYL TRANSFORM AND ITS GENERALIZATION [*]

Abstract. Procedures that allow one to associate operators to ordinary functions are called correspondence rules, rules of association, or just transform. There have been a number of such rules studied, among them the Weyl transform and the symmetrization rule. We present a generalization that allows one to generate an infinite number of such transforms. The advantage of the formulation is that all such transforms can be studied in a simple and consistent fashion.

1. Introduction

The concept of associating ordinary functions with operators arose in many areas of analysis but took particular importance with the discovery of quantum mechanics. For the two most fundamental quantities, position and momentum, it became clear that the operators are x and $\frac{\hbar}{i}\frac{\partial}{\partial x}$ respectively (in the position representation) where $\hbar$ is the Planck constant. Procedures to construct other operators developed into the subject now known as correspondence rules, that is, rules to associate an ordinary function with an operator. It is clear that there is an infinite number of ways to associate an ordinary function with a corresponding operator because ordinary variables commute but operators do not. Some correspondence rules that have been studied are the Weyl [4, 12, 15], normal ordering, and symmetrization rule, among others [2, 4, 11]. It is the aim of this paper to develop a methodology where all correspondence rules can be characterized and studied in a unified way.

1.1. Notation, terminology, and conventions

“Symbol”, “classical function”, and “c-function”, are terms used in different fields to signify the same things, namely an ordinary function, $a(x,\xi)$, of two variables x and ξ. This is the common notation used in mathematics, while in physics it is position and momentum signified by $a(q,p)$ [8], and in time-frequency analysis [3,4] one generally writes $a(t,\omega)$. In this paper we will use the mathematics notation and as is standard define the conjugate operator to x by D, where

$$D = \frac{1}{i}\frac{d}{dx}$$

The commutator between x and D is denoted by $[x,D]$ and is given by

$$[x,D] = xD - Dx = i$$

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[*] It is a pleasure to dedicate this paper to Prof. Luigi Rodino on the occasion of his 60th birthday.

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Also, we will occasionally use $D_y = \frac{1}{i}\frac{d}{dy}$, where y is an arbitrary real variable.

The phrases *"operator transform", "correspondence rule", "rule of association"*, will all mean the same thing, namely the association of an operator $\mathcal{A}_a(x,D)$ with a symbol $a(x,\xi)$. The association is symbolized by

$$\mathcal{A}_a(x,D) \leftrightarrow a(x,\xi)$$

Generally speaking operators such as $\mathcal{A}_a(x,D)$ will be denoted by script letters although there will be exceptions, D being one of them.

The word *"transform"* by itself will mean the operation of $\mathcal{A}_a(x,D)$ on an arbitrary function, say $u(x)$, and will be denoted by $A_a[u(x)]$ or just by $A_a[u]$ when it is clear what the variable is. That is,

$$A_a[u] = \mathcal{A}_a(x,D)u(x) \tag{1}$$

The Fourier transform of the symbol, $a(x,\xi)$, will be denoted by $\widehat{a}(\theta,\tau)$ and the normalization is taken so that

$$\begin{aligned}\widehat{a}(\theta,\tau) &= \frac{1}{4\pi^2}\iint a(x,\xi)\, e^{-i\theta x - i\tau\xi}\, dx d\xi \\ a(x,\xi) &= \iint \widehat{a}(\theta,\tau)\, e^{i\theta x + i\tau\xi}\, d\theta\, d\tau\end{aligned}$$

Integrals without limits imply integration over the reals,

$$\int = \int_{\mathbb{R}}$$

2. The generalized operator transform

We define the generalized operator transform associated with the symbol $a(x,\xi)$ by

$$\mathcal{A}_a^{\Phi}(x,D) = \iint \widehat{a}(\theta,\tau)\Phi(\theta,\tau)\, e^{i\theta x + i\tau D}\, d\theta\, d\tau \tag{2}$$

where $\Phi(\theta,\tau)$ is a two dimensional function called the kernel [2]. The kernel characterizes a specific transform and its properties. Since [14]

$$e^{i\theta x + i\tau D} = e^{i\theta\tau/2}\, e^{i\theta\, x}\, e^{i\tau D} = e^{-i\theta\tau/2}\, e^{i\tau D}\, e^{i\theta x}$$

we have that

$$\begin{aligned}\mathcal{A}_a^{\Phi}(\, x,D) &= \iint \widehat{a}(\theta,\tau)\Phi(\theta,\tau)\, e^{i\theta\tau/2}\, e^{i\theta x}\, e^{i\tau D} d\theta\, d\tau \qquad (3)\\ &= \iint \widehat{a}(\theta,\tau)\Phi(\theta,\tau)\, e^{-i\theta\tau/2}\, e^{i\tau D}\, e^{i\theta x} d\theta\, d\tau \qquad (4)\end{aligned}$$

Equivalently,

$$\begin{aligned}\mathcal{A}_a^{\Phi}(\, x,D) &= \frac{1}{4\pi^2}\iiiint a(x,\xi)\, \Phi(\theta,\tau)\, e^{i\theta(x-x') + i\tau(D-\xi')}\, d\theta\, d\tau\, dx'\, d\xi' \\ &= \frac{1}{4\pi^2}\iiiint a(x',\xi')\, \Phi(\theta,\tau)\; e^{i\theta\tau/2} e^{i\theta(x-x')}\; e^{i\tau(D-\xi')} d\theta\, d\tau\, dx'\, d\xi'\end{aligned}$$

3. The generalized transform

We now consider the operation of $\mathcal{A}_a^{\Phi}(x,D)$ on an arbitrary function, $u(x)$. Using Eq. (3) we have

$$A_a^{\Phi}[u]=\mathcal{A}_a^{\Phi}(x,D)\,u(x)=\iint\widehat{a}(\theta,\tau)\Phi(\theta,\tau)\,e^{i\theta\tau/2}\,e^{i\theta x}\,u(x+\tau)d\theta\,d\tau$$

where we have used the fact that $e^{i\tau D}u(x)=u(x+\tau)$. Equivalently,

$$A_a^{\Phi}[u]=\iint\widehat{a}(\theta,\tau-x)\Phi(\theta,\tau-x)\,e^{i\theta(\tau+x)/2}\,u(\tau)d\theta\,d\tau \tag{5}$$

We call $A_a^{\Phi}[u]$ the generalized transform. Writing $A_a^{\Phi}[u]$ in terms of the symbol $a(x',\xi')$ directly one obtains

$$A_a^{\Phi}[u]\;=\;\frac{1}{4\pi^2}\iiiint a\left(q+\frac{\tau+x}{2},\xi\right)\,e^{-i\theta q-i(\tau-x)\xi}\,\Phi(\theta,\tau-x)\,u(\tau)\,d\tau dq\,d\xi d\theta \tag{6}$$

Notice $\widehat{a}(\theta,\tau-x)\Phi(\theta,\tau-x)\,e^{i\theta(\tau+x)/2}$ is a function of $\tau-x$ and $\tau+x$ and hence we write

$$A_a^{\Phi}[u]=\int k(\tau+x,\tau-x)\,u(\tau)\,d\tau \tag{7}$$

with

$$k(x,\tau)=\int\widehat{a}(\theta,\tau)\Phi(\theta,\tau)\,e^{i\theta x/2}\,d\theta \tag{8}$$

$$=\frac{1}{4\pi^2}\iiint a(q+x/2,\xi)\,e^{-i\theta q-i\tau\xi}\Phi(\theta,\tau)\;dq\,d\xi\,d\theta \tag{9}$$

3.1. From transform to symbol

We now describe how starting with the operator transform, $\mathcal{A}_a^{\Phi}$, one can obtain the corresponding symbol, $a(x,\xi)$. First we describe a notation that is helpful. For any operator $\mathcal{A}(x,D)$ we define $R_A(x,\xi)$ by the following procedure

$R_A(x,\xi)=$ the rearrangement of $\mathcal{A}(x,D)$, so that all the x factors
are to the left of the D operators; then one replaces D by ξ.

The rearrangement is achieved by using $[x,D]=i$. Applying this procedure to $\mathcal{A}_a^{\Phi}(x,D)$ as given by Eq. (3) and noting that it is already in the appropriate form, (since the x factors are already to the left of the D factors), we immediately have

$$R_A^{\Phi}(x,\xi)=\iint\widehat{a}(\theta,\tau)\Phi(\theta,\tau)\,e^{i\theta\tau/2}e^{i\theta x}\,e^{i\tau\xi}d\theta\,d\tau \tag{10}$$

Inverting we have,

$$\widehat{a}(\theta,\tau)\Phi(\theta,\tau)e^{i\theta\tau/2}=\frac{1}{4\pi^2}\iint R_A^{\Phi}(x,\xi)e^{-i\theta x-i\tau\xi}dx\,d\xi$$

which gives

$$\widehat{a}(\theta,\tau)=\frac{1}{4\pi^2}\frac{e^{-i\theta\tau/2}}{\Phi(\theta,\tau)}\iint R_A^{\Phi}(x,\xi)e^{-i\theta x-i\tau\xi}dx\,d\xi$$

and from which $a(x,\xi)$ can be obtained by inversion.

We now derive an alternate form. We first state the following theorem. Suppose we have two functions $f(\theta,\tau)$ and $\widehat{a}(\theta,\tau)$, then it is easily shown that

$$\iint f(\theta,\tau)\,\widehat{a}(\theta,\tau)\;e^{i\theta x+i\tau z}d\theta\,d\tau=f\left(\frac{1}{i}\frac{\partial}{\partial x},\frac{1}{i}\frac{\partial}{\partial\xi}\right)a(x,\xi) \tag{11}$$

Taking $f(\theta,\tau)=\Phi(\theta,\tau)\,e^{i\theta\tau/2}$ in Eq. (11) we have

$$R_A^{\Phi}(x,\xi)\;=\exp\left(\frac{1}{2i}\frac{\partial}{\partial x}\frac{\partial}{\partial\xi}\right)\Phi\left(\frac{1}{i}\frac{\partial}{\partial x},\frac{1}{i}\frac{\partial}{\partial\xi}\right)a(x,\xi) \tag{12}$$

and hence

$$a(x,\xi)=\exp\left[-\frac{1}{2i}\frac{\partial}{\partial x}\frac{\partial}{\partial\xi}\right]\Phi^{-1}\left(\frac{1}{i}\frac{\partial}{\partial x},\frac{1}{i}\frac{\partial}{\partial\xi}\right)R_A^{\Phi}(x,\xi) \tag{13}$$

These can be writen in a somewhat more compact way,

$$R_A^{\Phi}(x,\xi)\;=\exp\left(\frac{i}{2}D_xD_\xi\right)\Phi\left(D_x,D_\xi\right)a(x,\xi) \tag{14}$$

$$a(x,\xi)=\exp\left[\frac{1}{2i}D_xD_\xi\right]\Phi^{-1}\left(D_x,D_\xi\right)R_A^{\Phi}(x,\xi) \tag{15}$$

4. Constraints on the kernel

The advantage of the above formulation is that one can readily obtain conditions on the kernel corresponding to properties we desire in the transform. We now list some possible properties and the constraints on the kernel to assure the requirement is met.

(i) Hermiticity. If the symbol is real and $\Phi(\theta,\tau)\;=\Phi^*(-\theta,-\tau)$ then $\mathcal{A}_a^{\Phi}(x,D)$ is a Hermitian operator. That is, for any two functions $u(x)$ and $v(x)$

$$\int v^*(x)\mathcal{A}_a^{\Phi}u(x)dx=\int u(x)\left(\mathcal{A}_a^{\Phi}v(x)\right)^*dx\qquad\text{if}\qquad\Phi(\theta,\tau)\;=\Phi^*(-\theta,-\tau)$$

(ii) Unit correspondence. If we want the correspondence between the number one and the unit operator $1\leftrightarrow\mathbf{I}$ then we must take $\Phi(0,0)=1$. That is

$$\mathbf{I}\leftrightarrow 1\qquad\text{if}\qquad\Phi(0,0)=1$$

(iii) Symbols of x or ξ only. Suppose we want to be certain that for a symbol that is a function of x or ξ only the operator should be the same function of x and ξ then the condition on the kernel is,

$$\mathcal{A}_a^{\Phi}(x,D) = a(x) \leftrightarrow a(x) \qquad \text{if} \qquad \Phi(0,\tau) = 1$$
$$\mathcal{A}_a^{\Phi}(x,D) = a(\xi) \leftrightarrow a(D) \qquad \text{if} \qquad \Phi(\theta,0) = 1$$

(iv) Translation invariance. Consider the symbol $a_0(x,\xi) = a(x-x_0,\xi-\xi_0)$, then

$$\widehat{a}_0(\theta,\tau) = e^{i\theta x_0 + i\tau\xi_0}\,\widehat{a}(\theta,\tau)$$

and substituting in Eq. (3) we have that

$$\mathcal{A}_{a_0}^{\Phi}(x,D) = \mathcal{A}_a^{\Phi}(x+x_0,D+\xi_0)$$

This is true for all kernels that are functions of only θ and τ.

5. The Fourier, polynomial, and delta function associations

The above formulation can be viewed profitably from different perspectives in ways that we now discuss.

5.1. The Fourier association

One can think of $e^{i\theta x + i\tau\xi}$ as a symbol with parameters θ and τ and associate $e^{i\theta x+i\tau\xi}$ to $\Phi(\theta,\tau)\,e^{i\theta x+i\tau D}$. That is

$$\mathcal{M}\,(\theta,\tau) = \Phi(\theta,\tau)\,e^{i\theta x+i\tau D} \leftrightarrow e^{i\theta x+i\tau\xi} \tag{16}$$

where $\mathcal{M}\,(\theta,\tau)$ is called the characteristic function operator. We call this the Fourier association. Hence, one argues, for a general symbol, expand the symbol in terms of its Fourier transform

$$a(x,\xi) = \iint \widehat{a}(\theta,\tau)\; e^{i\theta x+i\tau\xi}\,d\theta\,d\tau \tag{17}$$

and then one substitutes Eq. (16) into this to obtain Eq. (2).

We note that

$$\begin{aligned}\mathcal{M}\,(\theta,\tau)\mathcal{M}\,(\theta',\tau') &= \Phi(\theta,\tau)\,\Phi(\theta',\tau')\,e^{i\theta\tau/2}\,e^{i\theta'\tau'/2}e^{i\theta\,x}\,e^{i\tau D}e^{i\theta'\,x}\,e^{i\tau' D}\\ &= \frac{\Phi(\theta,\tau)\,\Phi(\theta',\tau')}{\Phi(\theta+\theta',\tau+\tau')}\,e^{i(\theta'\tau-\theta\tau')/2}\,\mathcal{M}\,(\theta+\theta',\tau+\tau')\end{aligned} \tag{18}$$

5.2. The Taylor series association

Suppose that there is an operator correspondence for $x^n\xi^m$ and we denote it by $A_{nm}(x,D)$,

$$P_{nm}(\,x,D) \leftrightarrow x^n\xi^m$$

We call this the polynomial association. Now, expand the symbol, $a(x,\xi)$, in a Taylor series

$$a(x,\xi) = \sum_{n,m=0}^{\infty} \frac{1}{n!m!} \left\{ \frac{\partial^{n+m}}{\partial x^n \partial \xi^m}\, a(x,\xi)|_{x,\xi=0} \right\} x^n \xi^m$$

and define the operator transform by

$$\mathcal{A}_a\,(\,x,D) = \sum_{n,m=0}^{\infty} \frac{\left\{ \frac{\partial^{n+m}}{\partial x^n \partial \xi^m}\, a(x,\xi)|_{x,\xi=0} \right\}}{n!m!} P_{nm}(x,D) \tag{19}$$

To make $\mathcal{A}_a\,(\,x,D)$ as given by Eq. (19) equal to $\mathcal{A}_a^{\Phi}(x,D)$ as given by Eq. (2) the same one takes

$$P_{nm}(\,x,D) \;=\; \frac{1}{i^n i^m} \frac{\partial^{n+m}}{\partial \theta^n \partial \tau^m} \Phi(\theta,\tau)\, e^{i\theta\tau/2}\, e^{i\theta\, x}\, e^{i\tau D}\Big|_{\theta,\tau=0} \leftrightarrow x^n \xi^m \tag{20}$$

5.3. The Delta function association

Starting with the identity

$$a(x,\xi) = \iint a(x',\xi')\delta(x-x')\delta(\xi-\xi')dx'd\xi'$$

we write the correspondence between $\delta(x)\delta(\xi)$ and the corresponding operator by $A_\delta(x,D)$ as

$$A_\delta(\,x,D) \leftrightarrow \delta(x)\delta(\xi)$$

Hence, we define

$$\mathcal{A}_a\,(\,x,D) = \iint a(x',\xi')A_\delta(x-x',D-\xi')dx'd\xi'$$

We call this the delta function association [7, 11, 16]. From Eq. (2) we immediately have that we must take

$$A_\delta(\,x,D) \;=\; \frac{1}{4\pi^2} \iint \Phi(\theta,\tau)\, e^{i\theta\, x + i\tau D}\, d\theta\, d\tau \leftrightarrow \delta(x)\delta(\xi)$$

5.4. General association

A more general approach that encompasses the above is to consider an orthogonal complete set of functions, $v(x;\theta)$, and expand an arbitrary symbol as

$$a(x,\xi) = \iint \widehat{a}(\theta,\tau) v(x;\theta) v(x;\xi)\, d\theta\, d\tau \tag{21}$$

Since we assume that $v(x;\theta)$ are complete and orthogonal we have

$$\widehat{a}(\theta,\tau) = \iint a(x,\xi) v^*(x;\theta) v^*(\xi;\tau)\, dx d\xi$$

Now, suppose the operator association for $v(x;\theta)v(x;\tau)$ is $\mathcal{V}(\theta,\tau;x,D)$,

$$\mathcal{V}(\theta,\tau;x,D) \leftrightarrow v(x;\theta)v(x;\tau)$$

Substitting in Eq. (21) we have

$$\mathcal{A}_a^{\Phi}(x,D) = \iint \widehat{a}(\theta,\tau)\mathcal{V}(\theta,\tau;x,D)\,d\theta\,d\tau$$

This general approach will be developed in a future paper.

6. Transformation between transforms

Suppose we have two different transforms characterized by kernels $\Phi_2(\theta,\tau)$ and $\Phi_1(\theta,\tau)$

$$A_a^{\Phi_2}[u] = \int k_2(\tau+x,\tau-x)\,u(\tau)\,d\tau$$
$$A_a^{\Phi_1}[u] = \int k_1(\tau+x,\tau-x)\,u(\tau)\,d\tau$$

Simple manipulation of Eq. (8) leads to

$$k_2(x,\tau) = \frac{1}{2\pi}\iint \frac{\Phi_2(2\theta,\tau)}{\Phi_1(2\theta,\tau)}e^{i\theta(x-x')}\,k_1(x',\tau)d\theta dx'$$

Also, using Eq. (10) one can show that the corresponding operators transfroms are related by

$$R_A^{\Phi_2}(x,\xi) = \frac{\Phi_2\left(\frac{1}{i}\frac{\partial}{\partial x},\frac{1}{i}\frac{\partial}{\partial \xi}\right)}{\Phi_1\left(\frac{1}{i}\frac{\partial}{\partial x},\frac{1}{i}\frac{\partial}{\partial \xi}\right)}R_A^{\Phi_1}(x,\xi) \tag{22}$$

which can be writen as

$$R_A^{\Phi_2}(x,\xi) = \frac{\Phi_2\left(D_x,D_\xi\right)}{\Phi_1\left(D_x,D_\xi\right)}R_A^{\Phi_1}(x,\xi) \tag{23}$$

7. Relation between transforms and phase-space distributions

Shortley after the invention of quantum mechanics, Wigner [13] and Kirkwood [6] addressed the issue of quantum statical mechanics in the following way. They devised a distribution function (different ones) aimed to calculate quantum averages by way of phase space averaging. Here is the fundamental idea [2]. Suppose we have the association

$$\mathcal{A}_a^{\Phi}(x,D) \leftrightarrow a(x,\xi)$$

We want to find a distribution, $C(x,\xi)$ so that

$$\int u^*(x)\,\mathcal{A}_a^{\Phi}(x,D)\,u(x)dx = \iint a(x,\xi)\,C(x,\xi)dxd\xi \tag{24}$$

where $u(x)$ is an arbitrary function, which in quantum mechanics is called the wave function. The left hand side is the quantum mechanical way of calculating expectation values and the right hand side is the standard probabilistic method. This formulation has become known as the phase space of quantum mechanics. Moyal [10] was the first to understand the relationship between the Wigner distribution and the Weyl rule and Cohen [2] gave the general formulation for arbitrary rules and arbitrary phase space distributions.

Starting with Eq. (6) multiply it by $u^*(x)$ and integrate both sides. After some manipulation one derives that we must take [2]

$$C(x,\xi) = \frac{1}{4\pi^2}\iiint u^*(x'-\tau/2)\Phi(\theta,\tau)\,e^{i\theta x'-i\theta x-i\tau\xi}u(x'+\tau/2)d\theta\,d\tau dx'$$

to satisfy Eq. (24). Also, one can readily prove that for two arbitrary functions $v(x)$ and $u(x)$

$$\int v^*(x)\,\mathcal{A}_a^{\Phi}(\,x,D)\,u(x)\,dx = \iint a(x,\xi)\,C_{hg}(x,\xi)dxd\xi$$

if we indeed take

$$C_{vu}(x,\xi) = \frac{1}{4\pi^2}\iiint v^*(x'-\tau/2)\Phi(\theta,\tau)\,e^{i\theta x'-i\theta x-i\tau\xi}u(x'+\tau/2)d\theta\,d\tau dx'$$

We also point out that if we have two distributions characterized by Φ_1 and Φ_2 then the corresponding distributions are related by

$$C_2(x,\xi) = \frac{1}{4\pi^2}\iint \frac{\Phi_2(\theta,\tau)}{\Phi_1(\theta,\tau)}e^{-i\theta(x-x')-i\tau(\xi-\xi')}C_1(x,\xi)\,d\theta\,d\tau dx'd\xi'$$

This can be written in operational form,

$$C_2(x,\xi) = \frac{\Phi_2\left(i\frac{\partial}{\partial x},i\frac{\partial}{\partial \xi}\right)}{\Phi_1\left(i\frac{\partial}{\partial x},i\frac{\partial}{\partial \xi}\right)}C_1(x,\xi)$$

8. Examples

We now give a number of examples and in particular we consider some of the historical rules.

8.1. Weyl transform

The Weyl case is obtained by taking

$$\Phi_W(\theta,\tau) = 1$$

giving

$$e^{i\theta x + i\tau\xi} \leftrightarrow e^{i\theta\tau/2} e^{i\theta\, x}\, e^{i\tau D} \qquad \text{Weyl} \qquad \Phi_W(\theta,\tau) = 1$$

From Eq. (4) and (6) we have

$$\mathcal{A}_a^W\,(\,x, D) = \iint \widehat{a}(\theta,\tau)\, e^{i\theta\tau/2}\, e^{i\theta\, x}\, e^{i\tau D} d\theta\, d\tau$$

$$A_a^{\Phi}[u] = \frac{1}{2\pi}\iint a\left(\frac{\tau + x}{2}, p\right)\, e^{ip(x-\tau)}\, u(\tau)\, d\tau\, dp$$

and also from Eq. (9) we obtain

$$k(x,\tau) = \frac{1}{2\pi}\int a(x/2,\xi)\, e^{-i\tau\xi}\, d\xi$$

We also mention that from Eq. (18) we have that

$$\mathcal{M}\,(\theta,\tau)\mathcal{M}\,(\theta',\tau') = e^{i(\theta'\tau - \theta\tau')/2}\,\mathcal{M}\,(\theta+\theta',\tau+\tau')$$

which is well known. We also point out that and using Eq. (12) we have

$$\mathcal{A}_a^W\,(\,x, D) = \frac{1}{i^n i^m}\; \frac{\partial^{n+m}}{\partial\theta^n \partial\tau^m} e^{i\theta\tau/2}\, e^{i\theta x}\, e^{i\tau D}\bigg|_{\theta,\tau=0} = \frac{1}{2^m}\sum_{\ell=0}^{m}\binom{m}{\ell} D^{m-\ell}\, x^n D^{\ell}$$

which was first derived by McCoy. Also, using Eq. (22) we have

$$R_A(x,\xi) = e^{\frac{1}{2i}\frac{\partial^2}{\partial x \partial\xi}} a(x,\xi)$$

8.2. Margenou-Hill, normal, and antinormal

Before we discuss these specific rules we consider taking the following kernel

$$\Phi_c(\theta,\tau) = e^{ic\theta\tau/2}$$

where c is a real number. Boggiatto, De Donno, and Oliaro [1] have made a careful study of this kernel and showed the relationship with other kernels. We rederive some of their results. Using Eq. (4) and Eq. (6) we obtain

$$\mathcal{A}_a\,(\,x, D) = \frac{1}{2\pi}\iint a\left(q + \frac{\tau(1+c)}{2}, p\right)\, e^{-i\tau p'}\, e^{i\tau D}\, dp'\, d\tau$$

$$A_a[u] = \frac{1}{2\pi}\iint a\left(\frac{x+\tau}{2} + \frac{\tau - x}{2}c, p\right)\, e^{-i(\tau - x)p}\, u(\tau)\, dp\, d\tau$$

Consider now the case where the symbol is

$$a\,(x,\xi) = f(x)h(\xi)$$

then it follows that

$$\mathcal{A}_a\,(x,D) = \sum_{n=0}^{\infty}\left(\frac{1+c}{2i}\right)^n \frac{1}{n!}\frac{\partial^n f(x)}{\partial x^n}\frac{\partial^n h(D)}{\partial \xi^n}$$

where $\frac{\partial^n h(D)}{\partial \xi^n}$ means that after we differentiate $h(\xi)$ we set $\xi = D$ If we further take

$$f(x)h(\xi) = e^{i\tau x}\,e^{i\tau\xi}$$

then

$$\mathcal{A}_a\,(x,D) = e^{i(c+1)\theta\tau/2}e^{i\tau x}\,e^{i\tau D} = e^{i(c-1)\theta\tau/2}\,e^{i\tau D}e^{i\tau x}$$

If we take $c = -1$ and $c = 1$ we obtain the socalled normal and antinormal cases,

$e^{i\theta x+i\tau\xi} \leftrightarrow e^{i\theta x}\,e^{i\tau D}$ and $x^n\xi^m \leftrightarrow x^nD^m$ normal: $\Phi_N(\theta,\tau) = e^{-i\theta\tau/2}$

$e^{i\theta x+i\tau\xi} \leftrightarrow e^{i\tau D}e^{i\theta x}$ and $x^n\xi^m \leftrightarrow D^mx^n$ antinormal: $\Phi_A(\theta,\tau) = e^{i\theta\tau/2}$

Now consider the case $\Phi(\theta,\tau) = \cos(c\theta\tau/2)$. We obtain

$$\mathcal{A}_a\,(x,D) = \frac{1}{2}\left[e^{i(c+1)\theta\tau/2}e^{i\tau x}\,e^{i\tau D} + e^{-i(c+1)\theta\tau/2}\,e^{i\tau D}e^{i\tau x}\right]$$

and if we take $c = 1$ then we obtain the Margenau-Hill [9] or symmetrization rule

$e^{i\theta x+i\tau\xi} \leftrightarrow \frac{1}{2}\left[e^{i\tau D}\,e^{i\theta x} + e^{i\theta x}\,e^{i\tau D}\right]$ symmetrization $\Phi_{MH}(\theta,\tau) = \cos\theta\tau/2$

and

$x^n\xi^m \leftrightarrow x^nD^m + D^mx^n$ symmetrization $\Phi_{MH}(\theta,\tau) = \cos\theta\tau/2$

8.3. Born and Jordan association

Perhaps the first rule that was proposed historically was that of Born and Jordan ,

$$x^n\xi^m \leftrightarrow A_{nm}(x,D) = \frac{1}{m+1}\sum_{\ell=0}^{m} D^{m-\ell}x^nD^\ell$$

Using Eq. (20) on obtains the kernel [2],

$$\Phi(\theta,\tau) = \frac{\sin\theta\tau/2}{\theta\tau/2}$$

8.4. Choi-Williams kernel

Choi and Williams [5] devised a kernel that mitigates the so-called cross terms of the Winger distribution but non the less satisfies the important conditions for a representation. The kernel is

$$\Phi_{CW}(\theta,\tau) = e^{-\theta^2\tau^2/\sigma} \tag{25}$$

Where σ is a positive constant. Note that as $\sigma \to \infty$, the kernel approaches one, $\Phi(\theta,\tau) \to 1$, which is the kernel for the Weyl case. Using Eq. (12) we have

$$\begin{aligned} R_A^{\Phi_{CW}}(x,\xi) &= \exp\left[\frac{i}{2}D_xD_\xi\right]\exp\left[-\frac{1}{\sigma}D_x^2D_\xi^2\right]a(x,\xi) \\ &= \exp\left[-\frac{1}{\sigma}D_x^2D_\xi^2\right]R_A^{\Phi_W}(x,\xi) \end{aligned} \tag{26}$$

where $R_A^{\Phi_W}(x,\xi)$ is the Weyl case. As an example consider the case

$$a(x,\xi) = x\xi$$

Then, we have

$$R_A^{\Phi_W}(x,\xi)x\xi = \exp\left[\frac{i}{2}D_xD_\xi\right]x\xi = x\xi - i/2$$

and also

$$\mathcal{A}_a^{\Phi_W}(\ x,D) = xD - \frac{i}{2} = \frac{1}{2}[xD + Dx]$$

It is clear from Eq. (26) that

$$R_A^{\Phi_{CW}}(x,\xi) = \exp\left[-\frac{1}{\sigma}D_x^2D_\xi^2\right](x\xi - i/2) = (x\xi - i/2) \tag{27}$$

Thus the Choi-Williams association is the same as the Weyl association for $a(x,\xi) = x\xi$. Now consider the case

$$a(x,\xi) = x^2\xi^2$$

then

$$R_A^{\Phi_W}(x,\xi)x^2\xi^2 = \exp\left[\frac{i}{2}D_xD_\xi\right]x^2\xi^2 = x^2\xi^2 - 2ix\xi - 1/2$$

and

$$R_A^{\Phi_{CW}}(x,\xi) = \exp\left[-\frac{1}{\sigma}D_x^2D_\xi^2\right][x^2\xi^2 - 2ix\xi - 1/2] = x^2\xi^2 - 2ix\xi - 1/2 - \frac{4}{\sigma}$$

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