Rend. Sem. Mat. Univ. Pol. Torino - Vol. 66, 4 (2008)
Second Conf. on Pseudo-Diff. Operators: Invited Lectures

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HYPOELLIPTICITY, SOLVABILITY AND CONSTRUCTION OF SOLUTIONS WITH PRESCRIBED SINGULARITIES FOR SEVERAL CLASSES OF PDE HAVING SYMPLECTIC CHARACTERISTICS*

Abstract. This paper deals with the hypoellipticity, local solvability and construction of solutions with prescribed singularities for several classes of PDE having double symplectic characteristics. We not only propose a short survey but we investigate several instructive model examples as well. As our results are obtained in the C^∞ category, it is interesting to study the same operators in the Gevrey category too.

1. Several definitions and formulation of the main results

1. In the paper under consideration we denote by $L^m(X)$ the set of all classical scalar properly supported pseudodifferential operators of order m and $D'(X)$ stands for the set of all Schwartz distributions on the smooth manifold X. As usual the closed conic in ξ set $WF(u)$, $u \in D'(X)$ (wave front set of u) is defined by

$$WF(u) = \left\{\rho \in T^*X\backslash 0 : a \in L^0(X),\, a(x,D)u \in C^\infty(X) \Rightarrow a^0(\rho) = 0\right\}.$$

We have denoted by $a^0(\rho) = \tau(a)$ the principal symbol of the operator $a(x,D) \in L^0(X)$.

The s-wave front set of $u \in D'(X)$, $s \in \mathbb{R}^1$ is given by

$$WF_s(u) = \left\{\rho \in T^*(X)\backslash 0 : a \in L^0(X),\, a(x,D)u \in H^s(X) \Rightarrow a^0(\rho) = 0\right\}.$$

Certainly, $\rho = (x,\xi)$, $\xi \neq 0$ and $WF_s(u)$ is a closed conical in ξ set.

Evidently, $s' < s \Rightarrow WF_s(u) \subset WF_{s'}(u)$.

Let $V \subset T^*(X)\backslash 0$ be an open conical in ξ set and N is a closed cone in ξ contained in $T^*(X)\backslash 0$, $N \subset V$.

THEOREM 1. *[11] Assume that the operator $P \in L^m(X)$, $s' < s$. Suppose that it does not exist a function $u \in H^{s'}(X)$ such that*

$$(*) \qquad V \cap WF(Pu) = \emptyset, \quad V \cap WF_s(u) = V \cap WF(u) = N.$$

Then there exists $\rho^0 \in N$, pseudodifferential operators ψ, φ, $\varphi' \in L^0(X)$, cone supp $\varphi \subset V\backslash N$, cone supp $\varphi' \subset V$, $\psi(\rho) \equiv 1$ in a tiny neighborhood of ρ^0, $C = const > 0$, $\mu \in \mathbb{Z}_+$ and such that

$$(1) \qquad \|\psi w\|_s \leq C\left[\|\varphi' P w\|_\mu + \|\varphi w\|_\mu + \|w\|_{s'}\right], \quad \forall w \in C_0^\infty(X).$$

*It is a pleasure to dedicate this paper to Prof. Luigi Rodino on the occasion of his 60th birthday.

REMARK 1. Instead of (1) we can write

$$\|w\|_s \leq C\left[\|Pw\|_\mu + \|Aw\|_0 + \|w\|_{s'}\right], \quad \forall w \in C_0^\infty(X), \tag{2}$$

where the full symbol of A is identically 0 near ρ^0.

It is evident that $N \subset \operatorname{Char} P = \{\rho \in T^*X\backslash 0 :\ p_m^0(\rho) = 0\}$ is the non-trivial case in Theorem 1. In fact, if $\rho^0 \in N$, $\rho^0 \notin \operatorname{Char} P$ then $p_m^0(\rho^0) \neq 0$ and consequently $\rho^0 \notin WF(u)$, i. e. $(*)$ does not hold.

This way we conclude that the problem of existence of solution of the equation $Pu = f$ with given (prescribed singularity) $(*)$ is reduced to the violation of the a-priori estimate (1) /(2)/ for some $w \in C_0^\infty(X)$.

We shall illustrate Theorem 1 by the following example.

EXAMPLE 1. Let $P \in L^m(X)$, $p_m^0(\rho^0) = \nabla_{x,\xi} p_m^0(\rho^0) = 0$, $\rho^0 \in T^*X\backslash 0$ and let

$$C_{2m-1}^0(\rho) \leq -\alpha|\rho - \rho^0|^2, \quad \alpha = \text{const} > 0, \quad \forall \rho^0 \in V, \tag{3}$$

where V is an open conical neighborhood of ρ^0, while

$$C_{2m-1}^0 = \tau[P^*, P] = \frac{1}{i}\left\{\overline{p_m^0},\ p_m^0\right\} = 2\Im \sum_{j=1}^n \frac{\partial \overline{p_m^0}}{\partial \xi_j}\frac{\partial p_m^0}{\partial x_j}.$$

Then it is proved in [13] that the L_2 adjoint operator P^* of P is locally and even microlocally non solvable at $x_0(\rho^0)$. Applying Theorem 1 we conclude that for each closed cone $N \subset \operatorname{Char} P \cap V$ and for each $s' < s$ one can find a distribution $u \in H^{s'}(X)$ for which $WF(Pu) \cap V = \emptyset$, $V \cap WF(u) = V \cap WF_s(u) = N$.

Assume now that again $p_m^0(\rho^0) = \nabla_{x,\xi} p_m^0(\rho^0) = 0$ but contrary to (3) the following inequality holds:

$$C_{2m-1}^0(\rho) \geq C|\nabla_{x,\xi} p_m^0(\rho)|^2, \quad \forall \rho \in V,\ C = \text{const} > 0.$$

More precisely, $\left|\nabla_{x,\xi} p_m^0(\rho)\right|^2 = \left|\nabla_x p_m^0\right|^2 |\xi|^{-1} + \left|\nabla_\xi p_m^0\right|^2 |\xi|$. Suppose also that the spectrum of the Hamilton map (fundamental matrix) $F_{C_{2m-1}^0}(\rho^0)$ is non trivial.

Then the operator P is microlocally hypoelliptic at ρ^0 with sharp loss of regularity 1 and without any importance of the lower order terms. Thus, $Pu \in H_{\text{mcl}}^s(\rho^0) \Rightarrow u \in H_{\text{mcl}}^{s+m-1}(\rho^0)$ $\left(u \in H_{\text{mcl}}^s(\rho^0) \Leftrightarrow \rho^0 \notin WF_s(\rho^0)\right)$. The proof can be found in [10]. Below we shall discuss the fundamental matrix and its spectrum.

2. Consider now the symplectic manifold Σ, $\operatorname{codim}\Sigma = 2\nu < 2n$, written in canonical coordinates:

$$\Sigma = \{(x,\xi), \xi \neq 0 :\ x_1 = \cdots = x_\nu = 0,\ \xi_1 = \cdots = \xi_\nu = 0\}, 1 \leq \nu < n, \tag{i}$$

and suppose that for $\nu \geq 2$ the principal symbol p_m^0 of $P \in L^m(X)$ vanishes of sharp order 2 on Σ and $p_m^0(\rho) \in \Gamma = \left\{z \in \mathbb{C}^1 : |\Im z| \leq \gamma \Re z\right\}, \gamma = \text{const} > 0$, i. e. $p_m^0(\rho)$ describes a closed angle in $\mathbb{C}^1$ with vertex at 0 and an opening strictly less than π.

This is the definition of the subprincipal symbol:

$$p'_{m-1}(\rho) = p_{m-1}(\rho) + \frac{i}{2}\sum_{j=1}^{n} \frac{\partial^2 p_m^0(\rho)}{\partial x_j \partial \xi_j}.$$

It is well known that p'_{m-1} is symplectic invariant on Σ.

From geometrical reasons it is clear that one can define winding number of $p_m^0(\rho)$ on Σ.

(ii) We shall suppose that in the special case $\nu = 1$ the winding number of p_m^0 is 0. Then with some constant $c \neq 0$ we have $cp_m^0 \in \Gamma$. The Hamilton map (fundamental matrix) $F_{p_m^0}$ is symplectic invariant on Σ and is defined by

$$F_{p_m^0} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} p^{0''}_{m,\xi\xi} & p^{0''}_{m,x\xi} \\ {}^t p^{0''}_{m,x\xi} & \frac{1}{2} p^{0''}_{m,xx} \end{pmatrix} = \begin{pmatrix} {}^t p^{0''}_{m,x\xi} & \frac{1}{2} p^{0''}_{m,xx} \\ -\frac{1}{2} p^{0''}_{m,\xi\xi} & -p^{0''}_{m,x\xi} \end{pmatrix},$$

I_n being the unit matrix in $\mathbb{C}^n$.

The eigenvalues of $F_{p_m^0(\rho)}$, $\rho \in \Sigma$ are denoted by $\mu_j(\rho)$, $1 \leq j \leq 2\nu$; $\mu_j \in i\Gamma$, $1 \leq j \leq \nu$, i.e. $\mu_j = i\lambda_j$, $\lambda_j \in \Gamma$. In the special case $p_m^0(\rho) \geq 0$: $\operatorname{spec} F_{p_m^0(\rho)} = \{i\lambda_1, \dots, i\lambda_\nu,\ \lambda_j \geq 0, 1 \leq j \leq \nu\} \cup \{-i\lambda_1, \dots, -i\lambda_\nu\}$.

THEOREM 2. *(Grušin [3, 4], B. de Monvel, Trèves [5, 6], Hörmander [14]). Under the assumptions (i), (ii) and*

$$(iii) \qquad p'_{2m-1}(\rho) + \sum_{j=1}^{\nu} (2\alpha_j + 1)\lambda_j(\rho) \neq 0,\ \forall \rho \in \Sigma,\ \forall \alpha_j \in \mathbb{Z}_+$$

the operator $P(x, D)$ is hypoelliptic and even microhypoelliptic with sharp loss of regularity 1.

According to Sjöstrand [7] if (iii) is violated at some point $\rho^0 \in \Sigma$ and for some $\alpha_j \in \mathbb{Z}_+$, $\lambda_j(\rho^0)$ then the loss of regularity r of the operator P is $\geq 3/2$. We remind of the reader that $r \geq 0$ is called loss of regularity of P if $u \in D'(\omega)$, $Pu \in H^s_{\text{loc}}(\omega) \Rightarrow u \in H^{s+m-r}_{\text{loc}}(\omega)$. Certainly, s is arbitrary and r is fixed.

A rather interesting question is to study the hypoellipticity of the operator P, (i), (ii), when (iii) is violated.

THEOREM 3. *(Helffer [8]). Consider the operator P under the conditions (i), (ii) and in the special case $\nu = 1$. Suppose that there exists $\rho^0 \in \Sigma$, $\exists j \in \mathbb{Z}_+$ such that*

$$(iv) \qquad p'_{m-1}(\rho^0) + (2j+1)\lambda_1(\rho^0) = 0.$$

Define now on Σ the function

$$\widetilde{p}_{m-1}(\rho) = p'_{m-1}(\rho) + (2j+1)\lambda_1(\rho).$$

Then $P(x,D)$ is microlocally hypoelliptic at ρ^0 with sharp loss of regularity $r = 3/2$ iff

$$\frac{1}{i}\left\{\widetilde{p}_{m-1}, \overline{\widetilde{p}}_{m-1}\right\}_\Sigma < 0 \text{ at } \rho^0.$$

REMARK 2. It is well known that $\lambda_1(\rho) \in C^\infty(\Sigma)$ and both $p'_{m-1}(\rho)$ and $\lambda_1(\rho)$ are symplectic invariant on Σ, i. e. $\widetilde{p}_{m-1}(\rho)$ is well defined on Σ. As it concerns the Poisson bracket $\{a,b\}_\Sigma$ it is better to write $\{a,b\}_{T(\Sigma)}$, i. e. the Poisson bracket is taken along the canonical vector fields tangential to Σ and consequently belonging to $T(\Sigma)$.

EXAMPLE 2. Let

$$P = D_1^2 + x_1^2 D_2^2 + \lambda D_2 + D_1 + ix_1 D_2 \tag{4}$$

be a differential operator in $\mathbb{R}^2$. Then $\Sigma = \{x_1 = \xi_1 = 0, \xi_2 \neq 0\}$, $m = 2$, $\nu = 1$, the winding number of $p_2^0 \geq 0$ is 0, $p'_1 = \lambda\xi_2 + \xi_1 + ix_1\xi_2$, spec$F_{p_2^0} = \{\pm i\xi_2\} \neq 0$, $\lambda_1 = \xi_2$ if $\xi_2 > 0$. Let $\xi_2^0 > 0$, i. e. $\rho^0 = (0,0;0,\xi_2^0 > 0)$. Then $(iv) \Leftrightarrow \lambda = -(2j+1)$ for some $j \in \mathbb{Z}_+$. Thus $\widetilde{p}_1 = \xi_1 + ix_1\xi_2 \Rightarrow \widetilde{p}_1|_\Sigma = 0$, $\left\{\widetilde{p}_1, \overline{\widetilde{p}}_1\right\}_\Sigma = 0$ and therefore P is not microlocally hypoelliptic for $\lambda = -(2j+1)$ with a loss of regularity $r \leq 3/2$.

In Helffer [8] nothing is mentioned about the local (non) solvability of the operator P (4) at the origin, about the existence of solution of the equation $P^* u = f \in C^\infty$ with fixed singularity $WFu = \{\rho^0\}$ etc.

PROPOSITION 1. *The operator* (4) *and with $\lambda = -(2j+1)$ for some $j \in \mathbb{Z}_+$ is locally nonsolvable at the origin and in D'. Moreover, P^* is not microlocally hypoelliptic and possesses a distribution solution with fixed singularity at $\rho^0 = (0,0;0,\xi_2^0 > 0) \in Char P$. More precisely, let $t < s$ and s be fixed. Then one can find $u \in D'$ and such that $P^* u = f \in C^\infty$, $WF(u) = \{\rho^0\}$, $u \in H^t_{\mathrm{mcl}}(\rho^0)$ but $u \notin H^s_{\mathrm{mcl}}(\rho^0)$.*

EXAMPLE 3. As we saw in Example 2 the operator (4) does not enter in the frames of Theorem 3. Because of this reason we study in $\mathbb{R}^3$ the operator

$$P = D_1^2 + x_1^2 D_2^2 + \lambda D_2 + D_3 + ix_3 D_2,\ \lambda = -(2j+1),\ j \in \mathbb{Z}_+. \tag{5}$$

and its L_2 adjoint operator $S = P^* = D_1^2 + x_1^2 D_2^2 + \lambda D_2 + D_3 - ix_3 D_2$, $\lambda = -(2j+1)$.

Our investigation will be microlocally near the point $\rho^0 = (0,0,0;\xi_1 = 0, \xi_2^0 > 0, \xi_3 = 0)$. Evidently, $p'_1 = \lambda\xi_2 + \xi_3 + ix_3\xi_2$, spec $F_{p^0} = \{\pm i\xi_2\}$; we take $\lambda_1 = \xi_2 > 0$ and therefore $\widetilde{p}_1 = \xi_3 + ix_3\xi_2$, $\frac{1}{i}\left\{\widetilde{p}_1, \overline{\widetilde{p}}_1\right\}_\Sigma = -2\xi_2 < 0$. Thus the operator P (5) is microhypoelliptic at ρ^0 with sharp loss of regularity $r = 3/2$. Assume that $\lambda \neq 2j+1$, $\forall j \in \mathbb{Z}_+$, $\lambda \in \mathbb{R}$. Then in a conical neighborhood of ρ^0 we have that $p'_1(\rho) + (2j+1)\xi_2 = (\lambda + 2j + 1)\xi_2 + \xi_3 + ix_3\xi_2 \neq 0$ for each $j \in \mathbb{Z}_+$ and according to Theorem 2 ((iii) is satisfied) we have that the operator P (5) is microhypoelliptic at ρ^0 with sharp loss of regularity 1.

PROPOSITION 2. *The operator* (5), $\lambda = -(2j+1)$, $j \in \mathbb{Z}_+$ *is not locally solvable at the origin, while $S = P^*$ is not (micro)locally hypoelliptic (at ρ^0) at the origin* *in $\mathbb{R}^3$. More precisely, let $t < s$ and s be fixed. Then there exists $u \in D'$ such that $Su = f \in C^\infty$, $WF(u) = \{\rho^0\}$, $u \in H^t_{\mathrm{mcl}}(\rho^0)$, while $u \notin H^s_{\mathrm{mcl}}(\rho^0)$.*

We point out that there are possible generalizations of the results formulated in Propositions 1, 2. To do this we must use the definitions of symplectic manifold, the properties of Fourier integral operators and impose such conditions on the full symbol $p(x,\xi)$ that the corresponding microlocal form of p will be of the type (4), (5). Other possible generalizations are in the case $\nu \geq 2$. We omit the corresponding results as they are purely technical and we prefer to fix the ideas on the level of instructive examples that can be considered as appropriate microlocal forms of operators with double symplectic characteristics.

The paper is organized as follows. In Section 2 we propose some useful results about Hermite polynomials and Hermite functions. In Section 3 we construct distribution solution of the equation: $P^*u = f \in C^\infty$ with prescribed singularity of u: $WF(u) \cap V = WF_s(u) \cap V = N$. Here P is defined by (4) or (5). In Section 4 it is proved the local nonsolvability of (4), (5). "Grosso modo" the proof of Proposition 2 imitates the proof of Proposition 1. As it concerns Theorem 3, we give in a small Appendix a short sketch of the proof, avoiding the use of Hermite pseudodifferential operators. Our proof is elementary as it is based on a simple identity in L_2.

At the end of this section we shall mention that there is a renewed interest to the problems of hypoellipticity and subellipticity for operators with double symplectic characteristics (see for example [17], [18]).

2. Some useful results about Hermite polynomials and Hermite functions

1. We shall begin with several definitions.

DEFINITION 1. *Hermite polynomials are defined by the formula*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right),\ n \in \mathbb{Z}_+,\ \deg H_n = n.$$

Evidently, $H_n(-x) = (-1)^n H_n(x)$, i. e. $H_{2n+1}(x)$ is an odd function $\Rightarrow H_{2n+1}(0) = 0$, while $H_{2n}(x)$ is even. Moreover, $H_{2n}(0) \neq 0$. One can see that $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$. It is known that the following recurrent formulas hold:

$$H_n'(x) = 2nH_{n-1}(x),\ n \geq 1,\ H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0. \tag{6}$$

Evidently, $H_3(x) = 8x^3 - 12x$ (see (6) for $n = 2$).

The Hermite polynomials satisfy the ODE:

$$H_n'' - 2xH_n' + 2nH_n = 0,\ n \geq 0,$$

i. e. $\frac{d}{dx}\left(e^{-x^2}\frac{dH_n}{dx}\right) + 2ne^{-x^2}H_n = 0.$

Integrating by parts one obtains:

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx = \begin{cases} 0, \ n \neq m \\ 2^n n!\sqrt{\pi}, \ n = m. \end{cases}$$

DEFINITION 2. *Hermite functions are defined by*

$$\psi_n(x) = h_n(x)e^{-\frac{x^2}{2}}, \tag{7}$$

where $h_n(x) = \dfrac{H_n(x)}{\|H_n(x)\|}$, $\|H_n\| = \sqrt{2^n n!\sqrt{\pi}}$.

Therefore, $\int_{-\infty}^{\infty} h_n(x)h_m(x) = \delta_{n,m}$ *and* $\delta_{n,m}$ *is the Kronecker symbol.*

2. It is well known that $\{\psi_n\}$ form an orthonormal basis in $L_2(\mathbb{R}^1)$ and bases in the Schwartz spaces $\mathcal{S}(\mathbb{R}^1)$, $\mathcal{S}'(\mathbb{R}^1)$ ([1,2]).

Consider now the Hermite series $u = \sum_{n=0}^{\infty} c_n\psi_n$, where $c_n = \langle u, \psi_n\rangle$.

Then $u \in \mathcal{S}(\mathbb{R}^1) \Leftrightarrow \forall m \in \mathbb{Z}_+$ there exists a constant $\widetilde{c}_m > 0$ and such that $|c_n| \leq \widetilde{c}_m(1+n)^{-m}$, for each $n \in \mathbb{Z}_+$ ([2]).

Similarly, $u \in \mathcal{S}'(\mathbb{R}^1) \Leftrightarrow \exists m_0$ and $\widetilde{c}_0 > 0$ such that $|c_n| \leq \widetilde{c}_0(1+n)^{m_0}$ for each $n \in \mathbb{Z}_+$.

One can easily see that for each fixed $\xi_2 > 0$ the system $\left\{\psi_n\left(x\xi_2^{\frac{1}{2}}\right)\xi_2^{\frac{1}{2}}\right\}$ forms an orthonormal basis in $L_2(\mathbb{R}^1)$.

Below we propose the very important inequality of Cramer (Cramer – Charlier):

$$|H_n(x)|\, e^{-\frac{x^2}{2}} \leq k\sqrt{2^n n!},$$

where the constant $k = 1,086435\ldots$ (see [1]).

Define now the following differential operators:

$$M_1 = \frac{d}{dx} + x, \ M_2 = \frac{d}{dx} - x. \tag{8}$$

The one guesses that

$$M_1\psi_n = \sqrt{2n}\psi_{n-1}, \ \ n \geq 1. \tag{9}$$

Let $n = 0$. Then $M_1\psi_0 = 0$, i. e. (9) holds for each $n \geq 0$ if we define $\psi_{-1} = 0$.

In a similar way we obtain

$$M_2\psi_n = -\sqrt{2(n+1)}\psi_{n+1}, \ \ n \geq 0. \tag{10}$$

Combining (9), (10) we get:

$$M_1M_2\psi_n = -2(n+1)\psi_n \Rightarrow$$

$$\left(\frac{d^2}{dx^2}-x^2\right)\psi_n=-(2n+1)\psi_n,\ \ \forall n\geq 0. \tag{11}$$

Iterating the formula $M_2\psi_0=-\sqrt{2}\psi_1$ we obtain $M_2^n\psi_0=(-1)^n2^{\frac{n}{2}}\sqrt{n!}\psi_n$, $n\geq 0$, i. e.

$$\psi_n=(-1)^n\frac{M_2^n\psi_0}{\sqrt{2^nn!}}=\frac{(-1)^n}{\sqrt{2^nn!}}\left(\frac{d}{dx}-x\right)^n\psi_0,\ \ n\geq 0.$$

Having in mind that $\psi_0=\dfrac{e^{-\frac{x^2}{2}}}{\sqrt[4]{\pi}}$ we have

$$\psi_n=\frac{(-1)^n}{\sqrt{2^nn!\sqrt{\pi}}}\left(\frac{d}{dx}-x\right)^n e^{-\frac{x^2}{2}},\ \ n\geq 0. \tag{12}$$

Combining (12) and the fact that $e^{-\frac{x^2}{2}}\in\mathcal{S}(\mathbb{R}^1)$ we conclude that the Fourier transform $\widehat{\psi}_n$ is given by

$$\widehat{\psi}_n(\xi)=\sqrt{2\pi}(-i)^n\psi_n(\xi). \tag{13}$$

3. Construction of solutions of the equations $Qu=f_1$, $Sv=f_2\in C^\infty$ with prescribed singularity

1. We shall deal at first with the operator (4) and its L_2 adjoint operator $Q=P^*=D_1^2+x_1^2D_2+\lambda D_2+D_1-ix_1D_2$, $\lambda\in\mathbb{R}^1$. We shall denote by $\widehat{u}(x_1,\xi_2)$ the partial Fourier transformation of $u\in\mathcal{S}'(\mathbb{R}^2)$ with respect to x_2 $(x_2\to\xi_2)$. Then $Qu=0$ implies

$$\widehat{\widehat{Q}}\,\widehat{u}=\left(D_1^2+x_1^2\xi_2^2+\lambda\xi_2+D_1-ix_1\xi_2\right)\widehat{u}=0 \tag{14}$$

and for $\xi_2>0$ we make the following change of the variable x_1 in the homogeneous ODE (14): $y_1=x_1\xi_2^{\frac{1}{2}}$. Then (14) takes the form:

$$-\xi_2\left(\frac{d^2}{dy_1^2}-y_1^2-\lambda+i\xi_2^{-\frac{1}{2}}\left(\frac{d}{dy_1}+y_1\right)\right)\widehat{u}(y_1,\xi_2)=0. \tag{15}$$

Certainly, $M_1=\dfrac{d}{dy_1}+y_1$ according to (8).

In a similar way we obtain:

$$\widehat{\widehat{P}}\,\widehat{u}=0\Leftrightarrow-\xi_2\left(\frac{d^2}{dy_1^2}-y_1^2-\lambda+i\xi_2^{-\frac{1}{2}}M_2\right)\widehat{u}=0,\ \xi_2>0,\ M_2=\frac{d}{dy_1}-y_1.$$

In the case $\lambda=-(2j+1)$, $j\in\mathbb{Z}_+$ we are looking for the kernel of (14) in $\mathcal{S}'(\mathbb{R}^1)$ and for $\xi_2>0$ being a fixed parameter. As we know from Section 2, $\{\psi_n(y_1)\}$ form bases in $L_2(\mathbb{R}^1)$ and $\mathcal{S}'(\mathbb{R}^1)$. Therefore,

$$\widehat{u}=\sum_{n=0}^{\infty}c_n\psi_n(y_1)$$

and according to (15) $\widehat{\widehat{Q}}\,\widehat{\widehat{u}} = 0$, i. e.

$$\sum_{n=0}^{\infty} c_n \left[-(2n+1)\psi_n + (2j+1)\psi_n + i\xi_2^{-\frac{1}{2}}\sqrt{2n}\psi_{n-1} \right] = 0,\ \psi_{-1} \equiv 0.$$

This way we obtain the following infinite linear system for the unknown coefficients c_n:

$$\begin{array}{ll}
(0): & 2c_0 j + ic_1\xi_2^{-\frac{1}{2}}\sqrt{2\cdot 1} = 0 \\
(1): & 2(j-1)c_1 + ic_2\xi_2^{-\frac{1}{2}}\sqrt{2\cdot 2} = 0 \\
(2): & 2(j-2)c_2 + ic_3\xi_2^{-\frac{1}{2}}\sqrt{2\cdot 3} = 0 \\
 & \dots \\
(n-1): & 2(j-n+1)c_{n-1} + ic_n\xi_2^{-\frac{1}{2}}\sqrt{2\cdot n} = 0 \\
(n): & 2(j-n)c_n + ic_{n+1}\xi_2^{-\frac{1}{2}}\sqrt{2(n+1)} = 0 \\
(n+1): & 2(j-n-1)c_{n+1} + ic_{n+2}\xi_2^{-\frac{1}{2}}\sqrt{2(n+2)} = 0 \\
 & \dots
\end{array}$$

If $j = n$ (see equation (n)) the constant $c_n = c_j$ is arbitrary but $c_{n+1} = 0$. Then the $(n+1)$ equation implies that $c_{n+2} = 0$, etc. Therefore, $c_{j+k} = 0$ for each $k \geq 1$. We conclude that for each fixed $\xi_2 > 0$, $\dim \operatorname{Ker} \widehat{\widehat{Q}} = 1$, $\operatorname{Ker} \widehat{\widehat{Q}} \subset \mathcal{S}(\mathbb{R}^1)$ and

$$\widehat{\widehat{u}}(y_1) = \sum_{k=0}^{j} c_k \psi_k(y_1),$$

where

$$\begin{aligned}
c_{j-1} &= -\frac{ic_j\xi_2^{-\frac{1}{2}}\sqrt{2j}}{2\cdot 1}, \\
c_{j-2} &= -\frac{ic_{j-1}\xi_2^{-\frac{1}{2}}\sqrt{2(j-1)}}{2\cdot 2} = \left(-i\xi_2^{-\frac{1}{2}}\right)^2 \frac{\sqrt{2j}\sqrt{2(j-1)}}{2^2\cdot 1\cdot 2} c_j, \\
&\dots \\
c_{j-l} &= \left(-i\xi_2^{-\frac{1}{2}}\right)^l \frac{\sqrt{2j}\sqrt{2(j-1)}\dots\sqrt{2(j-l+1)}}{2^l l!} c_j,\ j \geq l \geq 1, \\
&\dots \\
c_0 &= \left(-i\xi_2^{-\frac{1}{2}}\right)^j \frac{1}{2^{\frac{j}{2}}\sqrt{j!}} c_j.
\end{aligned}$$

We shall take $c_j = 1$. Then for $k = j-l$, $1 \leq l \leq j$ we have:

$$c_k = \left(-i\xi_2^{-\frac{1}{2}}\right)^{j-k} \frac{\sqrt{j(j-1)\dots(j-(j-k-1))}}{2^{\frac{j-k}{2}}(j-k)!},\ k = 0, 1, \dots, j-1.$$

To simplify the notations we write

$$c_k = \left(-i\xi_2^{-\frac{1}{2}}\right)^{j-k} \widetilde{c}_k,\ \widetilde{c}_k \neq 0,\ 0 \leq k \leq j-1,\ \widetilde{c}_j = c_j = 1.$$

Going back to the old coordinate x_1 we have:

$$\begin{aligned}\widehat{u}(x_1,\xi_2) &= \sum_{k=0}^{j} c_k\psi_k(y_1) = \sum_{k=0}^{j}\widetilde{c}_k\left(-i\xi_2^{-\frac{1}{2}}\right)^{j-k}\psi_k\left(x_1\xi_2^{\frac{1}{2}}\right)\\ &= \Sigma_{k=0}^{j}\widetilde{c}_k\left(-i\xi_2^{-\frac{1}{2}}\right)^{j-k}\frac{H_k\left(x_1\xi_2^{\frac{1}{2}}\right)}{\|H_k\|}e^{-\frac{1}{2}x_1^2\xi_2},\ \xi_2>0.\end{aligned}$$

Let $\psi(\xi_2)\in C^\infty(\mathbb{R}^1)$, $\psi=0$ for $\xi_2\in(-\infty,1]$, $\psi(\xi_2)=1$ for $\xi_2\geq 2$, $0\leq\psi(\xi_2)\leq 1$ for $\xi_2\in[1,2]$.

One can easily see that for each constant $a\in\mathbb{R}^1$ the function

$$(16)\qquad \begin{aligned}&u_a(x_1,x_2)=\int_{-\infty}^{\infty}e^{ix_2\xi_2}\psi(\xi_2)\xi_2^a\widehat{u}(x_1,\xi_2)d\xi_2=\\ &=\sum_{k=0}^{j}\widetilde{c}_k\int_{-\infty}^{\infty}e^{ix_2\xi_2-\frac{1}{2}x_1^2\xi_2}\psi(\xi_2)\xi_2^a\left(-i\xi_2^{-\frac{1}{2}}\right)^{j-k}\frac{H_k\left(x_1\xi_2^{\frac{1}{2}}\right)}{\|H_k\|}d\xi_2.\end{aligned}$$

satisfies the equation $Qu=0$.

The integral (16) is rapidly oscillating and it enters in the Hörmander scheme from Vol. I of his monograph [14] (see Chapter VII, Section 7.8, Th. 7.8.2, Th. 7.8.3 and Section 8.1, Th. 8.1.9). In fact, the amplitudes are of the type

$$\psi(\xi_2)\xi_2^a\xi_2^{\frac{k-j}{2}}H_k\left(x_1\xi_2^{\frac{1}{2}}\right)\in S_{1,0}^{a-\frac{j}{2}+k},\ \ 0\leq k\leq j.$$

The phase function of (16)

$$i\phi=ix_2\xi_2-\frac{x_1^2}{2}\xi_2=i\xi_2\left(x_2+\frac{i}{2}x_1^2\right),\ \ \Im\phi=\frac{\xi_2x_1^2}{2}\geq 0,$$

as $\phi=\xi_2\left(x_2+\frac{i}{2}x_1^2\right)$ and evidently, $d_{x,\xi}\phi=(ix_1\xi_2,\ \xi_2\neq 0;\ 0,x_2+\frac{i}{2}x_1^2)\neq 0$, as $\xi_2\geq 1$. Certainly, $\phi(x,t\xi)=t\phi(x,\xi)$, $\forall t>0$. Then

$$WF(u_a)\subset\left\{(x,\phi_x):\ \phi_{\xi_2}=0\right\}=\{(x_1=0,\ x_2=0;\ \xi_1=0,\ \xi_2>0)\}.$$

Thus, $u_a\in C^\infty(\mathbb{R}^2\backslash(0,0))$ as sing supp $u_a\subset\{(0,0)\}$.

2. We shall study now the behavior of $x^\alpha D^\beta u_a(x)$ for $|x|\geq\varepsilon_0>0$ and ε_0 is arbitrary small.

Evidently,

$$(17)\qquad \begin{aligned}&x^\beta D_1^{\alpha_1}D_2^{\alpha_2}u_a(x)=x^\beta\sum_{k=0}^{j}\int_{-\infty}^{\infty}e^{ix_2\xi_2}\xi_2^{a+\frac{\alpha_1}{2}+\alpha_2}\times\\ &\times\psi(\xi_2)\widetilde{c}_k\left(-i\xi_2^{-\frac{1}{2}}\right)^{j-k}\left(D_1^{\alpha_1}\psi_k\right)\left(x_1\xi_2^{\frac{1}{2}}\right)d\xi_2.\end{aligned}$$

On the other hand,

$$e^{i\phi} = \partial_{\xi_2}^N \left(e^{i\phi}\right) \Big/ i^N \left(x_2 + \frac{i}{2}x_1^2\right)^N \text{ for } |x| \geq \varepsilon_0$$

and for arbitrary $N \in \mathbb{N}$.

According to the theory of Fourier integral operators with complex phase function, we can integrate by parts with respect to ξ_2 in (17) as its phase function is $i\phi$ and its amplitude belongs to some class $S^m_{1,0}$.

This way we conclude that

$$\left|x^\beta D^\alpha u_a(x)\right| \leq C_{\alpha\beta} \text{ for } |x| \geq \varepsilon_0,\ \forall(\alpha,\beta) \in \mathbb{Z}_+^2. \tag{18}$$

Of course, $C_{\alpha\beta} > 0$ are appropriate constants.

Introduce now the cut off function $\eta(x) \in C_0^\infty(\mathbb{R}^2)$, $\eta \equiv 1$ near $(0,0)$, $0 \leq \eta \leq 1$. According to (18), $u = \eta u + (1-\eta)u$ and $(1-\eta)u \in \mathcal{S}(\mathbb{R}^2)$.

Having in mind that $\rho^0 = (0,0;\, 0, \xi_2^0 > 0) \in WF(u_a)$ we conclude that $\widehat{\eta u_a}(\xi)$ is rapidly decreasing in the angle $\Gamma_1 = \{(\xi_1,\xi_2):$ either $\xi_2 < 0$ or $0 \leq \varepsilon_0\xi_2 \leq |\xi_1|\}$, $0 < \varepsilon_0 \ll 1$ and is not decreasing in the angle $\Gamma_2 = \{(\xi_1,\xi_2):\ \varepsilon_0\xi_2 \geq |\xi_1|\}$.

The above mentioned words enable us to conclude that $u_a \in H^t(\mathbb{R}^2) \Leftrightarrow u_a \in H^t_{\mathrm{mcl}}(\rho^0)$, i. e.

$$\iint \left(1+|\xi|^2\right)^t |\widehat{u}_a(\xi)|^2\, d\xi < \infty \Leftrightarrow \iint_{\Gamma_2} \left(1+|\xi|^2\right)^t |\widehat{u}_a(\xi)|^2\, d\xi < \infty.$$

But in Γ_2 we have that $\xi_2^2 \leq |\xi|^2 \leq (1+\varepsilon_0^2)\xi_2^2$ and consequently $(1+|\xi|)^t \sim (1+\xi_2^2)^t$ in Γ_2. So

$$u_a \in H^t(\mathbb{R}^2) \Leftrightarrow \iint_{\Gamma_2} (1+\xi_2^2)^t\, |\widehat{u}_a(\xi)|^2\, d\xi < \infty. \tag{19}$$

Then the definition (16) of u_a gives us that

$$\widehat{u}_a(\xi_1,\xi_2) = \int e^{-ix_2(\xi_2-\theta)} \sum_{k=0}^{j} \theta^{a-\frac{1}{2}} \psi(\theta) \widetilde{c}_k \left(-i\theta^{-\frac{1}{2}}\right)^{j-k} \widehat{\psi}_k\left(\frac{\xi_1}{\theta^{\frac{1}{2}}}\right) dx_2 d\theta.$$

Applying (13) to the previous integral we get

$$\begin{aligned}\widehat{u}_a(\xi_1,\xi_2) = {} & \sqrt{2\pi}(-i)^j \int \theta^{a-\frac{1}{2}} \psi(\theta) \textstyle\sum_{k=0}^j \widetilde{c}_k \theta^{\frac{k-j}{2}} \psi_k\left(\frac{\xi_1}{\theta^{\frac{1}{2}}}\right) \times \\ & \times \left[\int e^{-ix_2(\xi_2-\theta)} dx_2\right] d\theta = (-i)^j \sqrt{2\pi} \xi_2^{a-\frac{1}{2}} \psi(\xi_2) \times \\ & \times \sum_{k=0}^{j} \widetilde{c}_k \xi_2^{\frac{k-j}{2}} \psi_k\left(\frac{\xi_1}{\xi_2^{\frac{1}{2}}}\right).\end{aligned}$$

In fact, $\int e^{-ix_2(\xi_2-\theta)} dx_2 = \delta(\xi_2 - \theta)$, δ being the Dirac delta function.

Applying (19) we get that

$$u_a \in H^t(\mathbb{R}^2) \Leftrightarrow \int_{-\infty}^{\infty} (1+\xi_2^2)^{t+a-\frac{1}{2}} \left[\int_{-\infty}^{\infty} \left| \sum_{k=0}^{j} \widetilde{c}_k \xi_2^{\frac{k-j}{2}} \psi_k \left(\frac{\xi_1}{\xi_2^{\frac{1}{2}}} \right) \right|^2 d\xi_1 \right] d\xi_2 < \infty.$$

On the other hand, $\{\psi_k(y_1)\}$ form an orthonormal basis in $L_2(\mathbb{R}^1)$ and

$$\int_{-\infty}^{\infty} \left| \sum_{k=0}^{j} \widetilde{c}_k \xi_2^{\frac{k-j}{2}} \psi_k \left(\frac{\xi_1}{\xi_2^{\frac{1}{2}}} \right) \right|^2 d\xi_1 = \xi_2^{\frac{1}{2}} \int_{-\infty}^{\infty} \left| \sum_{k=0}^{j} \widetilde{c}_k \xi_2^{\frac{k-j}{2}} \psi_k(y) \right|^2 dy =$$
$$= \xi_2^{\frac{1}{2}} \sum_{k=0}^{j} |\widetilde{c}_k|^2 \xi_2^{k-j}, \text{ i. e.}$$

(20) $$u_a \in H^t(\mathbb{R}^2) \Leftrightarrow \sum_{k=0}^{j} \int_{-\infty}^{\infty} (1+\xi_2^2)^{t+a-\frac{1}{4}} \psi^2(\xi_2) \, |\widetilde{c}_k|^2 \, \xi_2^{k-j} d\xi_2 < \infty.$$

We have taken $\widetilde{c}_j = 1$ and therefore the integral participating in (20) is $< \infty$ iff $2t + 2a - \frac{1}{2} < -1 \Leftrightarrow t + a < -\frac{1}{4} \Leftrightarrow t < -a - \frac{1}{4}$. Put $s = -a - \frac{1}{4}$. Evidently, the integral (20) is divergent for $t + a = -\frac{1}{4}$, i. e. for $t = s$.

CONCLUSION. $u_a \in H^t_{\text{mcl}}(\rho^0) \Leftrightarrow t < s$, while $u_a \notin H^s_{\text{mcl}}(\rho^0)$. This way we have constructed the solution of $Qu = 0$ (4) with $WF(u_a) = \{\rho^0\}$, $u_a \in H^t_{\text{mcl}}(\rho^0)$ for each $t < s$, $u_a \notin H^s_{\text{mcl}}(\rho^0)$ and s is arbitrary real number.

3. To prove the existence of a solution with prescribed singularity of the equation $P^* u = f \in C^\infty$, where P is given by (5), we make the partial Fourier transformation with respect to x_2 in $Su = 0$, $u = u(x_1, x_2, x_3)$, $S = P^*$. Thus,

$$\widehat{\widehat{S}}\, \widehat{\widehat{u}} = \left(-\partial_{x_1}^2 + x_1^2 \xi_2^2 + \lambda \xi_2 - i\,(\partial_3 + x_3 \xi_2)\right) \widehat{\widehat{u}} = 0,$$

where $\widehat{\widehat{u}} = \widehat{\widehat{u}}(x_1, \xi_2, x_3)$, $x_2 \to \xi_2$, $-\lambda = 2j+1$. Our investigation will be microlocal near the point $\rho^0 = (0, 0, 0; \, \xi_1 = 0, \xi_2^0 > 0, \xi_3 = 0) \in \Sigma$ where

$$\Sigma = \left\{ (x, \xi) : \, x_1 = \xi_1 = 0, \, \xi \neq 0, \, (x, \xi) \in \mathbb{R}^6 \right\}.$$

The change $\left| \begin{array}{l} y_1 = x_1 \xi_2^{\frac{1}{2}} \\ y_3 = x_3 \xi_2^{\frac{1}{2}} \end{array} \right.$, $\xi_2 > 0$ in the equation $\widehat{\widehat{S}}\, \widehat{\widehat{u}} = 0$ leads to the following PDE:

(21) $$\left[\partial_{y_1}^2 - y_1^2 - \lambda + i \xi_2^{-\frac{1}{2}} \left(\partial_{y_3} + y_3 \right) \right] \widehat{\widehat{u}}(y_1, \xi_2, y_3) = 0$$

and (21) is an equation with separate variables. We are looking for $\widehat{\widehat{u}}(y_1, \xi_2, y_3) = C \psi_j(y_1) \psi_0(y_3)$ with $C = \text{const} > 0$ and $\psi_j(y_1)$, $\psi_0(y_3)$ are the corresponding Hermite functions defined in Section 2 (see (7), (11)):

$$\left(\partial_{y_1}^2 - y_1^2 + 2j + 1\right)\psi_j(y_1) = 0,\ \psi_0(y_3) = \frac{1}{\sqrt[4]{\pi}} e^{-\frac{1}{2}y_3^2},\ (\partial_{y_3} + y_3)\psi_0 = 0.$$

Taking $C = 1$ we have that the function

$$\begin{aligned}\widehat{\widehat{u}}(x_1, \xi_2, x_3) &= \psi_j\left(x_1\xi_2^{\frac{1}{2}}\right)\psi_0\left(x_3\xi_2^{\frac{1}{2}}\right) = \\ &= \frac{H_j\left(x_1\xi_2^{\frac{1}{2}}\right)}{\sqrt[4]{\pi}\|H_j\|} e^{-\frac{1}{2}\left(x_1^2 + x_3^2\right)\xi_2} \in \mathcal{S}\left(\mathbb{R}^2_{x_1, x_3}\right)\end{aligned}$$

belongs to the kernel of the operator $\widehat{\widehat{S}}$ for each fixed value $\xi_2 > 0$ of the parameter ξ_2. To prove the existence of a solution with prescribed singularity of $Su = f \in C^\infty$ we use instead of the function (16) the following function

$$v_a(x_1, x_2, x_3) = \int_{-\infty}^{\infty} e^{ix_2\xi_2 - \frac{\xi_2}{2}\left(x_1^2 + x_3^2\right)} \psi(\xi_2)\xi_2^a \frac{H_j\left(x_1\xi_2^{\frac{1}{2}}\right)}{\sqrt[4]{\pi}\|H_j\|} d\xi_2.$$

Then $Sv_a = 0$, $WF(v_a) = \{\rho^0\}$, $v_a \in H^t_{\mathrm{mcl}}(\rho^0) \Leftrightarrow t < s = -a - \dfrac{1}{4}$; $v_a \notin H^s_{\mathrm{mcl}}(\rho^0)$. The proof of these facts is the same as in the case of the operator Q and we omit the details.

The proof of the nonsolvability of the operators (4), (5), $\lambda = -(2j+1)$, $j \in \mathbb{Z}_+$ will be given in Section 4.

4. Local nonsolvability in D' of the operators (4), (5) in the case $\lambda = -(2j+1)$, $j \in \mathbb{Z}_+$

1. As is well known if the PDO $P(x, D)$ with C^∞ coefficients is locally solvable at the origin $0 \in \mathbb{R}^n$ and in D' then the following a-priori estimate holds.

There exists a neighborhood $\omega \ni 0$, an integer $N \in \mathbb{N}$ and a constant $C_N > 0$ such that

$$\left|\int f(x)v(x)dx\right| \le C_N \sum_{|\alpha| \le N} \sup|D^\alpha f| \sum_{|\alpha| \le N} \sup|D^\alpha P^* v|,\ \ \forall f, v \in C_0^\infty(\omega). \tag{22}$$

Therefore, in order to prove the local nonsolvability of the operator P (4), (5), $P^* = Q$ or $P^* = S$ we must violate (22) for arbitrary but fixed ω, N, C_N. The proof here repeats with some changes the proof of Theorem 1 from [12]. Because of this reason we shall not give everywhere the details. Moreover, we shall concentrate on the case (4), i. e. $P^* = Q$ in (4.1).

2. Introduce now the function $\eta(\rho) \in C_0^\infty(\mathbb{R}^1)$, $\eta \ge 0$, $\int \eta(\rho)d\rho = 1$, $\operatorname{supp}\eta \subset [1, 2]$, $0 < \eta(\rho) < 1$ for $\rho \in (1, 2)$. We consider the function $F \in C_0^\infty(\mathbb{R}^2)$, such that $\iint F(x_1, x_2)dx_1dx_2 = 1$ and define

$$f_\lambda(x_1, x_2) = F(\lambda^2 x_1, \lambda^2 x_2),\ \lambda \ge 1,\ \lambda - \text{parameter},$$

in the case j – even.

In the case j – odd we define

$$f_\lambda(x) = \frac{\partial}{\partial x_1}\left(F\left(\lambda^2 x_1, \lambda^2 x_2\right)\right). \tag{23}$$

Evidently, supp $f_\lambda \Subset \omega$ for $\lambda \geq \lambda_0 \gg 1$.

From the considerations in Section 3 we know that

$$u(x_1, x_2) = \int_{-\infty}^{\infty} \eta(\xi_2) e^{ix_2\xi_2} \sum_{k=0}^{j} \widetilde{c}_k \left(-i\xi_2^{-\frac{1}{2}}\right)^{j-k} \psi_k\left(x_1 \xi_2^{\frac{1}{2}}\right) d\xi_2 \in C^\infty(\mathbb{R}^2)$$

satisfies the equation $Qu = 0$.

Put

$$u_\lambda(x_1, x_2) = \int_{-\infty}^{\infty} \eta(\xi_2) e^{i\lambda x_2\xi_2} \sum_{k=0}^{j} \widetilde{\widetilde{c}}_k (\lambda\xi_2)^{-\frac{j-k}{2}} \psi_k\left(x_1 \sqrt{\lambda\xi_2}\right) d\xi_2,$$

where $\widetilde{\widetilde{c}}_k = (-i)^{j-k}\widetilde{c}_k$.

Then $u_\lambda(x) \in C^\infty(\mathbb{R}^2)$ and $Qu_\lambda = 0$, $\forall x \in \mathbb{R}^2$, $\lambda \geq 1$. Let $\omega' \Subset \omega'' \Subset \omega$ be neighborhoods of the origin and the function $\varphi \in C_0^\infty(\mathbb{R}^2)$ be equal to 1 on ω', $0 \leq \varphi(x) \leq 1$ for $x \in \omega$ and $\varphi = 0$ outside ω''. Evidently, $\sum_{|\alpha| \leq N} \sup |D^\alpha f_\lambda| \leq \text{const}\,\lambda^{N+2}$.

Define now

$$v_\lambda(x) = \varphi(x) u_\lambda \Rightarrow \text{supp}\, v_\lambda \Subset \omega.$$

Then $Qv_\lambda = \varphi Q u_\lambda + [Q, \varphi] u_\lambda = [Q, \varphi] u_\lambda$ and $\text{supp}\,[Q, \varphi] \subset \omega'' \backslash \omega'$.

As we mentioned before there are two cases to be studied: a) j – even $\Rightarrow \psi_j(0) \neq 0$, b) j – odd $\Rightarrow H_j(0) = 0 \Rightarrow \psi_j(0) = 0$. In the case b) we have that according to (6)

$$\partial_{x_1} \psi_j(0) = \frac{H_j'(0)}{\|H_j\|} = \frac{2jH_{j-1}(0)}{\|H_j\|} \neq 0, \; j \geq 1.$$

We shall investigate the case a) only. Let us estimate the left hand side of (22), namely $I_\lambda = \iint f_\lambda(x) v_\lambda(x) dx$. The standard change of the variables $y_1 = \lambda^2 x_1$, $y_2 = \lambda^2 x_2$ in the previous integral gives us that

$$\lim_{\lambda\to\infty} \lambda^4 I_\lambda = \varphi(0, 0) \iiint F(y_1, y_2) \eta(\xi_2) \widetilde{\widetilde{c}}_j \psi_j(0) dy_1 dy_2 d\xi_2 = \psi_j(0) \neq 0$$

as $\widetilde{\widetilde{c}}_j = \widetilde{c}_j = 1$, i. e.

$$I_\lambda = \lambda^{-4}\left(\psi_j(0) + o(1)\right), \; \lambda \to \infty. \tag{24}$$

In estimating $\sup |D^\alpha(Qv_\lambda)|$, $|\alpha| \leq N$ we have in mind that $Qv_\lambda = [Q, \varphi] u_\lambda = \varphi_1 u_\lambda$ and the function $\varphi_1 \in C_0^\infty(\mathbb{R}^2)$, $\text{supp}\,\varphi_1 \subset \omega'' \backslash \omega'$, does not depend on λ.

The Leibnitz rule gives us the typical term participating in $D^{\alpha}(Qv_{\lambda}) = D^{\alpha}(\varphi_1 u_{\lambda})$:

$$D^{\alpha_1}_{x_1} D^{\alpha_2}_{x_2} u_{\lambda} = \int_{-\infty}^{\infty} \eta(\xi_2) e^{i\lambda x_2 \xi_2} \xi_2^{\alpha_2 + \frac{\alpha_1}{2}} \sum_{k=0}^{j} \widetilde{\widetilde{c}}_k (\lambda\xi_2)^{-\frac{j-k}{2}} \times \times \left(D^{\alpha_1}_{x_1}\psi_k\right)\left(x_1\sqrt{\lambda\xi_2}\right) d\xi_2 \cdot \lambda^{\alpha_2+\frac{\alpha_1}{2}}. \tag{25}$$

Certainly, $\lambda^{\alpha_2+\frac{\alpha_1}{2}} \leq \lambda^N$.

We shall consider two different cases in estimating $\sup|D^{\alpha}(\varphi_1 u_{\lambda})|$: I) $|x_1| \geq \varepsilon_0 > 0$ and II) $|x_2| \geq \varepsilon_0$, $0 < \varepsilon_0 \ll 1$. In fact, $\varphi_1 \equiv 0$ near the origin.

Case I. The Hermite function $\psi_k \in \mathcal{S}(\mathbb{R}^1) \Rightarrow D^{\alpha_1}_{x_1}\psi_k \in \mathcal{S}(\mathbb{R}^1)$ and therefore for each integer $M \geq 1$ there exists a constant $C_M > 0$ and such that

$$\left|\left(D^{\alpha_1}_{x_1}\psi_k\right)\left(x_1\sqrt{\lambda\xi_2}\right)\right| \leq \frac{C_M}{\left(1+\left|x_1\sqrt{\lambda\xi_2}\right|\right)^{2M}} \leq \varepsilon_0^{-2M}\frac{C_M}{\lambda^M}$$

as $\left|x_1\sqrt{\lambda\xi_2}\right| \geq \varepsilon_0\sqrt{\lambda}$ for $\xi_2 \geq 1$.

From (25) we obtain in the case I and for $|\alpha| \leq N$ that

$$\left|D^{\alpha_1}_{x_1} D^{\alpha_2}_{x_2} u_{\lambda}\right| \leq \widetilde{C}_M \lambda^{N-M} \int_{-\infty}^{\infty} \eta(\xi_2) d\xi_2 = \widetilde{C}_M \lambda^{N-M} \tag{26}$$

as $j \geq k$, $\lambda \geq 1$, $1 \leq \xi_2 \leq 2 \Rightarrow (\lambda\xi_2)^{-\frac{j-k}{2}} \leq 1$. Assume now that $|x_2| \geq \varepsilon_0 > 0$. Evidently, for each integer $M \geq 1$ we have

$$\frac{\partial^M}{\partial\xi_2^M}\left(e^{i\lambda x_2\xi_2}\right) = (i\lambda x_2)^M e^{i\lambda x_2 \xi_2}$$

and we can integrate by parts in (25) with respect to ξ_2.

Thus, in the case II we get:

$$D^{\alpha_1}_{x_1} D^{\alpha_2}_{x_2} u_{\lambda} = \frac{\lambda^{\frac{\alpha_1}{2}+\alpha_2}(-1)^M}{(i\lambda x_2)^M} \int_{-\infty}^{\infty} e^{i\lambda x_2\xi_2} \frac{\partial^M}{\partial\xi_2^M} \times \times \left[\eta(\xi_2)\xi_2^{\frac{\alpha_1}{2}+\alpha_2} \sum_{k=0}^{j} \widetilde{\widetilde{c}}_k \lambda^{\frac{k-j}{2}} \xi_2^{\frac{k-j}{2}} \left(D^{\alpha_1}_{x_1}\psi_k\right)\left(x_1\sqrt{\lambda\xi_2}\right)\right] d\xi_2, \tag{27}$$

$|i\lambda x_2|^M \geq \lambda^M \varepsilon_0^M$.

Using the fact that we are integrating in the interval $1 \leq \xi_2 \leq 2$ we conclude that the "most dangerous term" in the previous integral (i. e. the term containing the highest power of λ for $0 \leq k \leq j$) is: $\frac{\partial^M}{\partial\xi_2^M}\left[\left(D^{\alpha_1}_{x_1}\psi_j\right)\left(x_1\sqrt{\lambda\xi_2}\right)\right]$. Having in mind that $\left|\left(\frac{\partial^l}{\partial\xi_2^l}D^{\alpha_1}_{x_1}\psi_j\right)\left(x_1\sqrt{\lambda\xi_2}\right)\right| \leq C_{l,\alpha_1} = \text{const}$ we obtain that the highest power of

λ arising in the integral of (27) is $\lambda^{\frac{M}{2}}$ as $x_1\sqrt{\lambda\xi_2} = x_1\sqrt{\lambda}\sqrt{\xi_2}$. More precisely, "the most dangerous term" is:

$$(28)\qquad (-1)^M\left(\frac{x_1\sqrt{\lambda}}{2}\right)^M \xi_2^{-\frac{M}{2}}\left(\frac{\partial^M}{\partial\xi_2^M}D_{x_1}^{\alpha_1}\psi_j\right)\left(x_1\sqrt{\lambda\xi_2}\right).$$

Consequently, in the case II and for $|\alpha| \le N$ (28) $|D^\alpha u_\lambda| \le D_M\lambda^{N-M/2}$, $D_M = \text{const} > 0$. Combining (24), (26) and (28) we violate (22) for $\lambda \to \infty$ as M is arbitrary integer. This way we complete the proof of the nonsolvability of the operator (4), $-\lambda = 2j+1$ in the case j – even.

3. The case j – odd is studied in a similar way. In fact, then f_λ is given by (23) and therefore $|I_\lambda| = \left|\int f_\lambda v_\lambda\right| = \left|\int F(\lambda^2 x)\frac{\partial v_\lambda}{\partial x_1}\right|$, $\frac{\partial\psi_j}{\partial x_1}(0) \ne 0$, etc.

To prove the local nonsolvability of the operator (5) at the origin we violate (22) by using the following functions: $f_\lambda(x) = F(\lambda^2 x)$, $x \in \mathbb{R}^3$ and $w_\lambda = \varphi(x)v_\lambda$, where $\varphi \in C_0^\infty(\mathbb{R}^3)$, $\varphi \equiv 1$ near the origin and

$$v_\lambda = \int_{-\infty}^{\infty}\eta(\xi_2)e^{i\lambda x_2\xi_2}\frac{H_j\left(x_1\sqrt{\lambda\xi_2}\right)}{\sqrt[4]{\pi}\|H_j\|}e^{-\frac{\lambda}{2}\left(x_1^2+x_3^2\right)\xi_2}d\xi_2.$$

We assume j – even and as in the previous case we estimate $D^\alpha v_\lambda$ in two situations: I) $|x_2| \ge \varepsilon_0 > 0$ and II) $|x_2| \le \varepsilon_0 (\Rightarrow x_1^2 + x_3^2 \ge \varepsilon_0^2$ on $\operatorname{supp} D\varphi)$, etc.

5. Appendix

Short sketch of the proof of the microhypoellipticity of the operator (5), $\lambda = -(2j+1)$ will be given here.

1. We are working in a conical neighborhood of the point $\rho^0 = (0,0,0;0,1,0)$, i. e. in the cone $\Gamma = \left\{\xi \in \mathbb{R}^3\backslash 0 :\ \xi_2 \ge \varepsilon_0\sqrt{\xi_1^2+\xi_3^2}\right\}$, $\varepsilon_0 > 0$. Consider now the identity $\|Pu\|_0^2 = \|P^*u\|_0^2 + ([P^*,P]u,u)$. In our case $P = D_1^2 + x_1^2D_2^2 - (2j+1)D_2 + D_3 + ix_3D_2$. Put $Q = Q^* = D_1^2 + x_1^2D_2^2 - (2j+1)D_2$; $R = D_3 + ix_3D_2$, $R^* = D_3 - ix_3D_2$.

Thus, $[P^*,P] = [Q+R^*, Q+R] = [Q, R-R^*] + [R^*,R] = 2i[Q, x_3D_2] + 2D_2 = 2D_2$, as $[Q,D_2] = 0$. Therefore

$$(5.1)\qquad \|Pu\|_0^2 \ge c\|u\|^2_{H^{\frac{1}{2}}_{\text{mcl}}(\rho^0)},\ \ c = \text{const} > 0,$$

as $u \in H^{\frac{1}{2}}_{\text{mcl}}(\rho^0) \Leftrightarrow \int_\Gamma\left(1+\xi_2^2\right)^2|\hat{u}(\xi)|^2 d\xi < \infty$.

Assume now that $Pu = f \in H^s_{\text{mcl}}(\rho^0)$. Then $|D_2|^s f \in L_2$, $|D_2|^s f = |D_2|^s Pu = P(|D_2|^s u) \Rightarrow |D_2|^s u \in H^{\frac{1}{2}}_{\text{mcl}}(\rho^0)$ according to (5.1) and consequently $u \in H^{s+\frac{1}{2}}_{\text{mcl}}(\rho^0)$. The estimate (5.1) holds for each $\lambda \in \mathbb{R}^1$ in (5) too.

2. It is interesting to study the operator $P = D_1^2 + x_1^2 D_2^2 + \lambda D_2 + D_3 + i x_3^{2k+1} D_2$. By using the repeated Poisson brackets technique one can expect to prove microlocal hypoellipticity of P at ρ^0 with loss of regularity $r = 1 + \frac{2k+1}{2k+2}$. For $k = 0$ this is (5.1).

We mentioned above that the examples here proposed can be generalized in the frames of the C^∞ category. On the other hand side, it is very interesting to investigate the same operators in the Gevrey spaces. A precise microlocal analysis of several classes of ψdo with multiple characteristics and in Gevrey spaces is given in the Rodino's monographs [15], [16]. We hope that a combination of the approach there and the technique of the Hermite operators will enlarge the scope of the microlocal analysis and its applications.

References

[1] SANSONE G., *Orthogonal functions*, Interscience publishers, New York 1959.

[2] ANTOSIK P., MIKUSINSKI J. AND SIKORSKI R., *Theory of distributions – The sequential approach*, Elsevier and Polish Scientific publishers, Warsaw 1973.

[3] GRUŠIN V., *On a class of hypoelliptic operators*, Math. USSR Sb. **12** (1970), 458–476.

[4] GRUŠIN V., *On a class of elliptic pseudodifferential operators degenerate on a submanifold*, Math. USSR Sb. **13** (1971), 155–185.

[5] BOUTET DE MONVEL L., *Hypoelliptic operators with double characterizatistics and related pseudodifferential operators*, Comm. Pure Appl. Math. **27** (1974), 585–639.

[6] BOUTET DE MONVEL L. AND TRÈVES F., *On a class of systems of pseudodifferential equations with double characteristics*, Comm. Pure Appl. Math. **27** (1974), 59–89.

[7] SJÖSTRAND J., *Parametrices for pseudodifferential operators with multiple characteristics*, Ark. Math. **12** (1974), 85–130.

[8] HELFFER B., *Sur l'hypoellipticité des opérateurs pseudodifferentiels à caractéristique multiple (perte de* $3/2$ *derivées)*, Bull. Soc. Math. France **51–52** (1977), 13–61.

[9] B. DE MONVEL L., GRIGIS A. AND HELFFER B., *Parametrixes d'opérateurs pseudodifférentiels à caractéristiques multiples*, Journée EDP de Rennes (1975), 93–121, Astèrisque **34–35**, Soc. Math. France, Paris 1976.

[10] POPIVANOV P., *Elements of the symplectic differential geometry and partial differential equations, Part II*, Akad. der Wissenschaften der DDR, Zentralinst. für Math. und Mech., Berlin 1977, 1–40.

[11] IVRII V., *Wave front sets of the solutions of some classes of pseudodifferential equations*, Trudi Mosc. Math. Obsc. **39** (1979), 49–82.

[12] POPIVANOV P., *Some classes of differential operators with multiple characteristics having no solutions*, Pliska – Bulg. Math. J. **3** (1981), 47–60.

[13] EGOROV J., RANGELOV C., *On a class of presudodifferential equations with multiple characterisics*, AMS Translations, series 2, 16 papers on Differential equations **118** (1982), 185–190.

[14] HÖRMANDER L., *The analysis of linear partial differential operators*, vol. I–IV, Springer-Verlag, Berlin, Heidelberg 1983–1985.

[15] RODINO L., *Linear partial differential operators in Gevrey spaces*, World Scientific, Singapore 1993.

[16] MASCARELLO M. AND RODINO L., *Partial differential equations with multiple characteristics*, Akad. Verlag, Berlin 1997.

[17] PARENTI C. AND PARMEGGIANI A., *On the hypoellipticity with a big loss of derivatives*, Kyushu J. Math. **59** (2005), 155–230.

[18] PARENTI C. AND PARMEGGIANI A., *A note on Kohn's and Christ's examples*, in: "Hyperbollic problems and regularity questions" (Eds. Padula M. and Zanghirati L.), Birkhäuser 2007, 151–158.

AMS Subject Classification: 35A27, 35A20, 35A18, 35H10, 35H20.

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