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OPERATOR CALCULUS FOR *p*-ADIC VALUED SYMBOLS AND QUANTIZATION

Dedicated to Professor Luigi Rodino on the occasion of his 60th birthday

Abstract. The aim of this short review is to attract the attention of the pseudo-differential community to possibilities in the development of operator calculus for symbols (depending on p-adic conjugate variables) taking values in fields of p-adic numbers. Essentials of this calculus were presented in works of the authors of this paper in order to perform p-adic valued quantization. Unfortunately, this calculus still has not attracted a great deal of attention from pure mathematicians, although it opens new and interesting domains for the theory of pseudo-differential operators.

1. Introduction

Quantum formalism with wave functions valued in non-Archimedean fields was developed in a series of papers and books [1]–[13], see also related works of Vladimirov and Volovich [14]–[15] and the book [16] on quantum formalism with p-adic variables but complex-valued wave functions. In this review article, we present the essentials of this theory. We restrict attention to the fields of p-adic numbers. General quantum theory has been developed for an arbitrary non-Archimedean field K, see [11].

The basic objects of this theory are p-adic Hilbert spaces and symmetric operators acting in these spaces. Vectors of a p-adic Hilbert space which are normalized with respect to the inner product represent quantum states. In the p-adic case, the norm is not determined by the inner product. Therefore normalization with respect to the norm and the inner product, which coincides for real and complex Hilbert spaces, is different for p-adic Hilbert spaces. We shall proceed in the following way.

Consider the formal differential expression $\widehat{\mathbf{H}} = H(\partial_{x_j}, x_j)$ of operators of quantum mechanics or quantum field theory. Let us realize this formal expression as a differential operator with variables x_j belonging to the field of p-adic numbers $\mathbb{Q}_p$ and study properties of this operator in a p-adic Hilbert space. Thus we would like to perform a p-adic analogue of Schrödinger's quantization.

We remark that p-adic valued quantum theory suffers from the absence of a "good spectral theorem" for symmetric operators. At the same time, this theory is essentially simpler (mathematically) than ordinary quantum mechanics, since *operators of position and momentum are bounded in the p-adic case*, as was found by Albeverio and Khrennikov [3].

The representation theory of groups in Hilbert spaces forms one of the cornerstones of ordinary quantum mechanics. It is very natural to develop p-adic quantum mechanics in a similar way. We construct a representation of the Weyl–Heisenberg group in a p-adic Hilbert space, namely the space $L_2(\mathbb{Q}_p,\nu_b)$ of L_2-functions with respect to a p-adic valued Gaussian distribution ν_b (the symbol b indicates a p-adic analogue of dispersion), see [3].[1] Here the situation differs very much from that of ordinary quantum mechanics. If we denote by $\widehat{U}(\alpha)$ and $\widehat{V}(\beta)$ the groups of unitary operators corresponding to position and momentum operators, respectively, then these groups are defined only for parameters α and β belonging to balls $U_{R(b)}$ and $U_{r(b)}$, respectively, where $R(b)$ and $r(b)$ depend on the dispersion b of the Gaussian distribution and they are coupled by a kind of Heisenberg uncertainty relation.

We shall also study the representation of the translation group on the space $L_2(\mathbb{Q}_p,\nu_b)$. Here the result also differs from that of ordinary quantum mechanics, and is more similar to one that holds in quantum field theory where Gaussian distributions on infinite dimensional spaces are used.

Let μ be Gaussian measure on the infinite-dimensional real Hilbert space $\mathcal{H}$. It is impossible to construct a representation of translations from all of $\mathcal{H}$ in $L_2(\mathcal{H},\mu)$, because of the well-known fact that the translation μ^h of a Gaussian measure on $\mathcal{H}$ by a vector $h\in\mathcal{H}$ can be singular with respect to μ. It is well known that μ^h is equivalent to μ if and only if h belongs to a certain proper ("Cameron–Martin") subspace. In a similar way we cannot construct in the space $L_2(\mathbb{Q}_p,\nu_b)$ a representation of translations by all elements h in $\mathbb{Q}_p$; in fact, we have to restrict consideration to translations belonging to some ball (which is an additive subgroup in $\mathbb{Q}_p$) whose radius depends on the dispersion b. This fact is connected with the nonexistence of translation-invariant measures in the p-adic case (similarly for infinite-dimensional spaces over the field of real numbers), see [6].

2. Banach and Hilbert spaces

2.1. p-adic numbers and their quadratic extensions

The field of real numbers $\mathbb{R}$ is constructed as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the metric $\rho_{\mathbb{R}}(x,y)=|x-y|$, where $|\cdot|$ is the usual real valuation (absolute value). The fields of p-adic numbers $\mathbb{Q}_p$ are constructed in a corresponding way, by using other valuations. For any prime number $p>1$, the p-adic valuation $|\cdot|_p$ is defined in the following way. First we define it for natural numbers. Every natural number n can be represented as the product of prime numbers: $n=2^{r_2}3^{r_3}\cdots p^{r_p}\cdots$. Then we define $|n|_p=p^{-r_p}$, and in addition set $|0|_p=0$ and $|-n|_p=|n|_p$. We extend the definition of the p-adic valuation $|\cdot|_p$ to all rational numbers by setting $|n/m|_p=|n|_p/|m|_p$ for $m\neq 0$. The completion of $\mathbb{Q}$ with respect to the metric $\rho_p(x,y)=|x-y|_p$ is the locally compact field of p-adic numbers $\mathbb{Q}_p$. By the well-known *Ostrovsky theorem*, the real valuation (absolute value) $|\cdot|$ and the p-adic valuations $|\cdot|_p$ are the only possible valuations on $\mathbb{Q}$. Thus if one wants to construct a

[1] We remark that ν_b is not a p-adic valued measure, i.e. a bounded linear functional on the space of continuous functions. It is just a distribution, a generalized function, which is primarily defined on the space of analytic test functions. A analogue of the L_2-space can be constructed by completing the space of test functions with respect to a natural norm.

physical model starting with rational numbers, then there are only two possibilities: to proceed to real numbers or to one of the fields of p-adic numbers.[2]

The p-adic valuation satisfies the so-called strong triangle inequality: $|x+y|_p \leq \max[|x|_p, |y|_p]$, which makes ρ_p into an ultrametric. Set $U_r(a) = \{x \in \mathbb{Q}_p : |x-a|_p \leq r\}$ and $U_r^-(a) = \{x \in \mathbb{Q}_p : |x-a|_p < r\}$, with $r = p^n$ and $n = 0, \pm 1, \pm 2, \ldots$; these are ("closed" and "open") balls in $\mathbb{Q}_p$. Set $S_r(a) = \{x \in \mathbb{Q}_p : |x-a|_p = r\}$; these are spheres in $\mathbb{Q}_p$. Any p-adic ball $U_r \equiv U_r(0)$ is an additive subgroup of $\mathbb{Q}_p$. The ball $U_1(0)$ is also a ring, called the *ring of p-adic integers* and denoted by $\mathbb{Z}_p$. For any $x \in \mathbb{Q}_p$, we have a unique canonical expansion (converging in the $|\cdot|_p$-norm) of the form

$$x = \alpha_{-n}/p^n + \cdots \alpha_0 + \cdots + \alpha_k p^k + \cdots, \tag{1}$$

where $\alpha_j = 0, 1, ..., p-1$, are the "digits" of the p-adic expansion. The elements $x \in \mathbb{Z}_p$ have an expansion $x = \alpha_0 + \alpha_1 p + \cdots + \alpha_k p^k + \cdots$, i.e., they are natural generalizations of natural numbers. Moreover, even negative natural numbers can be represented as elements of $\mathbb{Z}_p$, e.g., $-1 = (p-1) + (p-1)p + (p-1)p^2 + \ldots + (p-1)p^n + \ldots$ This is the source of the terminology "p-adic integer".

For $p_1 \neq p_2$, the fields of p-adic numbers $\mathbb{Q}_{p_1}$ and $\mathbb{Q}_{p_2}$ are not isomorphic as topological fields. Thus by moving into the p-adic domain one obtains, in fact, an infinite series of fields for the modeling of, e.g., space geometry. None of these fields is isomorphic to the field of real numbers $\mathbb{R}$. The crucial difference is in the topology.

Fields of p-adic numbers are *disordered*. It is impossible to introduce a linear order on $\mathbb{Q}_p$ (at least in a natural way, e.g., matching algebraic operations). This fact induces interesting departures from the real case. It also plays a fundamental role in the application of p-adic numbers to string theory and cosmology. For a long time, physicists discussed the idea that at Planck distances (which are extremely small) space-time is disordered. In particular, it cannot be described by real numbers. On the other hand, p-adic numbers provide an excellent possibility for the mathematical formulation of this physical idea.

In applications to physics, the following complicated problem arises: "Which p should be used for modeling?" There are various opinions. Igor Volovich proved that some amplitudes used in "ordinary string theory", i.e., based on the real model of space-time, can be reproduced in the limit $p \to \infty$ from the corresponding amplitudes of p-adic string theory [16]. The authors of this paper think that this is not crucial for the new geometry. Therefore the p selected for physical modeling (at least in a theoretical model) does not play an important role. One can switch from one scale to another as one does in the real case by switching in the expansion (1) from one p to another, see [11] for a detailed presentation of this ideology. Of course, each physical phenomenon has its own scale. One can discuss concrete scales, e.g., in the p-adic approach to quantum physics. The authors of this paper proposed selecting $p = [1/\alpha]$: the integer part of the fine structure constant α. However, all such physical discussions have no direct relation to the present paper. For a mathematician, it may be more important to

[2] We remark that experimental data is always rational. It is a consequence of the finite precision of any measurement.

know that typically the case $p = 2$ should be treated separately, and proofs obtained for $p > 2$ typically do not work for $p = 2$.

Let $\tau \in \mathbb{Q}_p$ and suppose that $x^2 = \tau$ have no solution in $\mathbb{Q}_p$. The symbol $\mathbb{Q}_p(\sqrt{\tau})$ denotes the corresponding quadratic extension of $\mathbb{Q}_p$. Its elements have the form $z = x + \sqrt{\tau}y$, where $x, y \in \mathbb{Q}_p$. The operation of conjugation is defined by $\overline{z} = x - \sqrt{\tau}y$. We remark that $z\overline{z} = x^2 - \tau y^2$ for $z \in \mathbb{Q}_p(\sqrt{\tau})$, and that $z\overline{z} \in Q_p$ for any $z \in \mathbb{Q}_p(\sqrt{\tau})$. The extension of the p-adic valuation from $\mathbb{Q}_p$ onto $\mathbb{Q}_p(\sqrt{\tau})$ is denoted by the same symbol $|\cdot|_p$. We have $|z|_p = \sqrt{|z\overline{z}|_p}$ for $z \in \mathbb{Q}_p(\sqrt{\tau})$. Besides quadratic extensions, we shall also operate with the field of complex p-adic numbers $\mathbb{C}_p$. Its construction is very complicated. Unlike in the real case, we cannot obtain an algebraically closed field by taking a quadratic extension, nor indeed by taking an algebraic extension of any finite order. The algebraic closure $\mathbb{Q}_p^a$ of $\mathbb{Q}_p$ is constructed as an infinite tower of finite extensions. In particular, it is an infinite-dimensional linear space over $\mathbb{Q}_p$ (compare with the real case where the algebraic closure $\mathbb{C}$ is just two dimensional over $\mathbb{R}$). The p-adic valuation is defined on the tower of finite extensions in a consistent way. In this way we obtain the p-adic valuation on $\mathbb{Q}_p^a$. However, this is not the end of the story concerning a p-adic analogue of complex numbers. The field $\mathbb{Q}_p^a$ is not complete with respect to such an extension of the p-adic valuation. Finally, we complete it and obtain that its completion, denoted by $\mathbb{C}_p$, is *algebraically closed!* The latter is a notrivial result, Krasner's theorem. As the reader has seen, the construction of p-adic complex numbers is quite complicated. However, it might be even worse – if Krasner's theorem were not true.

2.2. Banach spaces

Essentials of non-Archimedean functional analysis can be found in, e.g., the book of van Rooji [18].

The symbol K denotes a non-Archimedean field with the valuation (absolute value) $|\cdot|_K$. It is a map from K to $[0, +\infty)$ such that

(1) $|x|_K = 0 \Leftrightarrow x = 0$;

(2) $|xy|_K = |x|_K\,|y|_K$;

(3) $|x + y|_K \le \max(|x|_K, |y|_K)$.

The latter feature of the valuation is the strong triangle inequality. It plays a fundamental role in the determination of special features of the corresponding non-Archimedean topology. Such terminology is common in so-called non-Archimedean analysis, see e.g. [18]. However, in other domains of mathematics, a non-Archimedean field is a totally (or partially) ordered field containing nonzero infinitesimals, e.g., the field of nonstandard numbers $\mathbb{R}^*$. We emphasize that this paper has nothing to do with the latter case!

Let E be a linear space over a non-Archimedean field K. A *non-Archimedean norm* on E is a mapping $\|\cdot\| : E \to [0, +\infty)$ satisfying the following conditions:

(a) $\|x\| = 0 \Leftrightarrow x = 0$;

(b) $\|\alpha x\| = |\alpha|_K\,\|x\|,\ \alpha \in K$;

(c) $\|x+y\| \leq \max(\|x\|, \|y\|)$.

As usual, we define non-Archimedean Banach space E as a complete normed space over K. The metric $\rho(x,y) = \|x-y\|$ is ultrametric. Hence every non-Archimedean Banach space is zero-dimensional and totally disconnected. All balls $W_r(a) = \{x \in E : \|x-a\| \leq r\}$ are clopen.

The dual space E' is defined as the space of continuous K-linear functionals $l : E \to K$. Let us introduce the usual norm on $E' : \|l\| = \sup_{x \neq 0} |l(x)|_K / \|x\|$. The space E' endowed with this norm is a Banach space.

The simplest example of a non-Archimedean Banach space is the space $K^n = K \times \cdots \times K$ (n times) with the non-Archimedean norm $\|x\| = \max_{1 \leq j \leq n} |x_j|_K$. More interesting examples are infinite-dimensional non-Archimedean Banach spaces realized as spaces of sequences: set $c_0 \equiv c_0(K) = \{x \in K^\infty : \lim\limits_{n\to\infty} x_n = 0\}$ and $\|x\| = \max_n |x|_K$.

2.3. Hilbert spaces

We take a sequence of p-adic numbers $\lambda = (\lambda_n) \in Q_p^\infty$, $\lambda_n \neq 0$. We set

$$l^2(p,\lambda) = \Big\{ f = (f_n) \in \mathbb{Q}_p^\infty : \text{ the series } \sum f_n^2 \lambda_n \text{ converges in } \mathbb{Q}_p \Big\}.$$

It turns out that $l^2(p,\lambda) = \{ f = (f_n) \in \mathbb{Q}_p^\infty : \lim_{n\to\infty} |f_n|_p \sqrt{|\lambda_n|_p} = 0\}$. In the space $l^2(p,\lambda)$ we introduce the norm $\|f\|_\lambda = \max_n |f_n|_p \sqrt{|\lambda_n|_p}$. The space $l^2(p,\lambda)$ endowed with this norm is non-Archimedean Banach space. On the space $l^2(p,\lambda)$ we also introduce the p-adic valued inner product $(\,\cdot\,,\,\cdot\,)_\lambda$ by setting $(f,g)_\lambda = \sum f_n g_n \lambda_n$.

We remark that $\|f\|_\lambda \in \mathbb{R}$, but $(f,f)_\lambda \in \mathbb{Q}_p$. The norm is not determined by the inner product. Nevertheless, the p-adic inner product $(\,\cdot\,,\,\cdot\,)_\lambda : l^2(p,\lambda) \times l^2(p,\lambda) \to \mathbb{Q}_p$ is continuous and the *Cauchy–Bunyakovsky–Schwarz inequality* holds, namely $|(f,g)_\lambda|_p \leq \|f\|_\lambda \, \|g\|_\lambda$.

DEFINITION 1. *A triplet* $(l^2(p,\lambda),\ (\cdot,\cdot)_\lambda,\ \|\cdot\|_\lambda)$ *is called a p-adic coordinate Hilbert space.*

More generally, we shall define a p-adic inner product on $\mathbb{Q}_p$-linear space E as an arbitrary non-degenerate symmetric bilinear form $(\cdot,\cdot) : E \times E \to \mathbb{Q}_p$.

REMARK 1. We cannot introduce a p-adic analogue of positive definiteness of a bilinear form. For instance, any element $\gamma \in \mathbb{Q}_p$ can be represented as $\gamma = (x,x)_\lambda$, with $x \in l^2(p,\lambda)$ (this is a simple consequence of properties of bilinear forms over $\mathbb{Q}_p$).

The triplets $(E_j,\ (\cdot,\cdot)_j,\ \|\cdot\|_j)$, $j = 1,2$, where E_j are non-Archimedean Banach spaces, $\|\cdot\|_j$ are norms and $(\cdot,\cdot)_j$ are inner products satisfying the Cauchy–Buniakovski–Schwarz inequality, are isomorphic if the spaces E_1 and E_2 are algebraically isomorphic and the algebraic isomorphism $I : E_1 \to E_2$ is a unitary isometry, i.e., $\|Ix\|_2 = \|x\|_1$ and $(Ix, Iy)_2 = (x,y)_1$.

DEFINITION 2. *The triplet* $(E, (\cdot,\cdot), \|\cdot\|)$ *is a* p*-adic Hilbert space if it is isomorphic to the coordinate Hilbert space* $(l^2(p,\lambda), (\cdot,\cdot)_\lambda, \|\cdot\|_\lambda)$ *for some sequence of weights* λ.

The isomorphism relation splits the family of p-adic Hilbert spaces into equivalence classes. An equivalence class is characterized by some coordinate representative $l^2(p,\lambda)$. The classification of p-adic Hilbert spaces is an open mathematical problem.

Hilbert spaces over quadratic extensions $\mathbb{Q}_p(\sqrt{\tau})$ of $\mathbb{Q}_p$ can be introduced in the same way. For a given sequence $\lambda = (\lambda_n) \in \mathbb{Q}_p^\infty$, $\lambda_n \neq 0$, we set

$$l^2(p,\lambda,\sqrt{\tau}) = \{f = (f_n) \in \mathbb{Q}_p(\sqrt{\tau})^\infty : \text{ the series} \sum f_n \bar{f}_n \lambda_n \text{ converges}\},$$

with $\|f\|_\lambda = \max_n |f_n|_p \sqrt{|\lambda_n|_p}$ and $(f,g)_\lambda = \sum f_n \bar{g}_n \lambda_n$.

The triplet $(l^2(p,\lambda,\sqrt{\tau}), (\cdot,\cdot)_\lambda, \|\cdot\|_\lambda)$ is the coordinate Hilbert space over the quadratic extension $\mathbb{Q}_p(\sqrt{\tau})$. In general, a Hilbert space $(E, (\cdot,\cdot), \|\cdot\|)$ over the quadratic extension $\mathbb{Q}_p(\sqrt{\tau})$, is by definition isomorphic to some coordinate Hilbert space. We denote a p-adic Hilbert space over $\mathbb{Q}_p(\sqrt{\tau})$ by

$$\mathcal{H}_p \equiv \mathcal{H}_p(\sqrt{\tau}).$$

3. Groups of unitary isometric operators in p-adic Hilbert space

As usual, we introduce unitary operators $\widehat{U} : \mathcal{H}_p \to \mathcal{H}_p$ as operators which preserve the inner product, so $(\widehat{U}x, \widehat{U}y) = (x,y)$ for all $x,y \in \mathcal{H}_p$, with image $\mathrm{Im}\,\widehat{\mathrm{U}} = \widehat{\mathrm{U}}(\mathcal{H}_\mathrm{p}) = \mathcal{H}_\mathrm{p}$. Isometric operators are operators which preserve the norm, so $\|\widehat{U}x\| = \|x\|$, and have $\mathrm{Im}\,\widehat{\mathrm{U}} = \mathcal{H}_\mathrm{p}$. Denote the space of all bounded linear operators $\widehat{A} : \mathcal{H}_p \to \mathcal{H}_p$ by $\mathcal{L}(\mathcal{H}_p)$. It is a Banach space with respect to the operator norm $\|\widehat{A}\| = \sup_{x\neq 0} \|\widehat{A}x\|/\|x\|$. A unitary operator need not be isometric.[3] Indeed, it could even be unbounded. Denote the group of linear isometries of the p-adic Hilbert space $\mathcal{H}_p$ by $IS(\mathcal{H}_p)$, and the group of all bounded unitary operators in $\mathcal{H}_p$ by $UN(\mathcal{H}_p)$. Set $UI(\mathcal{H}_p) = UN(\mathcal{H}_p) \cap UI(\mathcal{H}_p)$.

An operator $\widehat{A} \in \mathcal{L}(\mathcal{H}_p)$ is said to be symmetric if $(\widehat{A}x, y) = (x, \widehat{A}y)$ for all x,y. The following simple fact will be useful later.

THEOREM 1. *The eigenvalue* α *of a symmetric operator* $\widehat{A} : \mathcal{H}_p \to \mathcal{H}_p$ *corresponding to an eigenvector u with nonzero square,* $(u,u) \neq 0$, *belongs to* $\mathbb{Q}_p$. *Eigenvectors corresponding to different eigenvalues of such type are orthogonal.*

The proof is similar to the standard one for complex Hilbert space $\mathcal{H}$.

As usual, we introduce the resolvent set $\mathrm{Res}(\widehat{\mathrm{A}})$ of an operator $\widehat{A} \in \mathcal{L}(\mathcal{H}_p)$; it consists of $\lambda \in \mathbb{Q}_p(\sqrt{\tau})$ such that the operator $(\lambda I - \widehat{A})^{-1}$ exists. The spectrum $\mathrm{Spec}(\widehat{\mathrm{A}})$ of $\widehat{A}$ is the complement of the resolvent set.

[3] Recall that the norm on the p-adic Hilbert space is not determined by the inner product. The only condition of consistency between them is the Cauchy–Bunyakovsky–Schwarz inequality.

Note that every ball U_r in $\mathbb{Q}_p$ is an additive subgroup of $\mathbb{Q}_p$. A map $\widehat{F} : U_r \to \mathcal{L}(\mathcal{H}_p)$ with the properties $\widehat{F}(t+s) = \widehat{F}(t)\widehat{F}(s)$, $t,s \in U_r$, and $\widehat{F}(0) = I$, where I is the unit operator in $\mathcal{H}_p$, is said to be a one-parameter group of operators. If we consider $IS(\mathcal{H}_p), UN(\mathcal{H}_p), UI(\mathcal{H}_p)$ instead of $\mathcal{L}(\mathcal{H}_p)$, we obtain definitions of the parametric groups of isometric, unitary, and isometric unitary operators, respectively. If the map $F : \mathcal{U}_r \to \mathcal{L}(\mathcal{H}_p)$ is analytic the one-parameter group is called analytic.

We recall that any p-adic ball is, in fact, a ball with radius $r = p^k$, with $k = 0, \pm 1, \ldots$ (since the p-adic valuation takes only such values). On the other hand, in a normed space over $\mathbb{Q}_p$ or its quadratic extension, the norm can take any value belonging to $[0, +\infty)$. To match these two ranges of values, we invent the following quantity. Let a be a positive real number. We define

(2) $$[a]_p^- = \sup\{\lambda = p^k,\ k \in \mathbb{Z} : \lambda < a\}.$$

This number approximates (from below) the real number a by numbers from the range of values of the p-adic valuation.

For a bounded operator $\widehat{A}$, we define

(3) $$\gamma(\widehat{A}) = \frac{1}{p^{1/(p-1)}\|\widehat{A}\|}\ .$$

It is a real number, the reciprocal of the norm $\|\widehat{A}\|$ multiplied by the factor $p^{1/(p-1)}$. The latter appears in connection with convergence of the exponential series in the p-adic case. The series e^y, where in general y belongs to $\mathbb{C}_p$, converges on the ball of radius $r_{\exp} = p^{-1/(p-1)}$.

THEOREM 2. *Let $\widehat{A}$ be a bounded symmetric operator in $\mathcal{H}_p \equiv \mathcal{H}_p(\sqrt{\tau})$. The map*

$$t \mapsto e^{\sqrt{\tau}\, t\widehat{A}}, \quad t \in U_r,\ r = [\gamma(\sqrt{\tau}\widehat{A})]_p^-,$$

is an analytic one-parameter group of isometric unitary operators.

Thus every symmetric operator $\widehat{A} \in \mathcal{L}(\mathcal{H}_p(\sqrt{\tau}))$ generates the one-parameter operator group of isometric unitary operators $t \mapsto \widehat{U}(t) = e^{\sqrt{\tau}\, t\widehat{A}}$. This theorem is a natural generalization of the standard theorem for $\mathbb{C}$-Hilbert space. The following result has no analogue in functional analysis over $\mathbb{C}$.

THEOREM 3. *Suppose that an operator $\widehat{A}$ belongs to $\mathcal{L}(\mathcal{H}_p)$. The map $\alpha \mapsto e^{\alpha\widehat{A}}$, $\alpha \in U_r,\ r = [\gamma(\widehat{A})]_p^-$, is an analytic one-parameter group of isometric operators.*

4. Gaussian integral and spaces of square integrable functions

As already remarked, the mathematical formalism of p-adic quantization does not depend on the choice of a quadratic extension $\mathbb{Q}_p(\sqrt{\tau})$ of $\mathbb{Q}_p$. To make considerations symbolically closer to ordinary complex quantization, we shall proceed for the quadratic extension $\mathbb{Q}_p(i)$. Of course, this choice restricts in an essential way the class of prime numbers under consideration.

To provide the pointwise realization of elements of the p-adic analogue of the L_2-space, we shall consider analytic functions over the field of complex p-adic numbers $\mathbb{C}_p$. In $\mathbb{C}_p$ we denote the ball of radius $s \in \mathbb{R}_+$ with center at $z = 0$ by the symbol $\mathcal{U}_s$. We denote the space of analytic functions $f : \mathcal{U}_s \to \mathbb{C}_p$ by $\mathcal{A}(\mathcal{U}_s)$.

In [2], the general definition of a p-adic valued Gaussian integral was proposed on the basis of distribution theory. In this context, the Gaussian distribution was defined as the distribution having Laplace transform of the form $\exp\{bx^2/2\}$, where $b \in \mathbb{R}$. We recall that in the real case if $b > 0$ then Gaussian distribution is simply a countably additive measure – Gaussian measure with dispersion b. If b is negative or even complex then the Gaussian distribution cannot be realized as a measure.

For our present applications to quantization, we can use a simpler approach based on the definition of Gaussian distribution through the definition of its moments. Roughly speaking, we know moments of Gaussian distribution over the reals. Suppose now that dispersion is a rational number, $b \in \mathbb{Q}$. Then moments can equally well be interpreted as elements of any $\mathbb{Q}_p$. We now can extend by continuity our definition of moments to any "dispersion" $b \in \mathbb{Q}_p$.

Let b be a p-adic number, $b \neq 0$. The p-adic Gaussian distribution ν_b is defined by its moments $(n = 0, 1, \ldots)$:

$$M_{2n} = \int_{\mathbb{Q}_p} x^{2n} \nu_b(dx) \equiv \frac{(2n)!\, b^n}{n!\, 2^n}, \quad M_{2n+1} = \int_{\mathbb{Q}_p} x^{2n+1} \nu_b(dx) \equiv 0.$$

We define the Gaussian integral for polynomial functions by linearity. Then we can define it for some classes of analytic functions. The analytic function $f(x) = \sum_{n=0}^{\infty} c_n x^n$, with $c_n \in \mathbb{C}_p$, is said to be integrable with respect to the Gaussian distribution ν_b if the series

$$\int_{\mathbb{Q}_p} f(x) \nu_b(dx) \equiv \sum_{n=0}^{\infty} c_n M_n = \sum_{n=0}^{\infty} c_{2n} M_{2n} \tag{4}$$

converges. It was shown in [11] that all entire analytic functions on $\mathbb{C}_p$ are integrable. In fact, we do not need analyticity on the whole of $\mathbb{C}_p$ to be able to define the Gaussian integral. The following (real) constant

$$\theta_b \equiv p^{\frac{1}{2(1-p)}} \sqrt{|b/2|_p}$$

will play a fundamental role. If $p \neq 2$, then $\theta_b = p^{\frac{1}{2(1-p)}} \sqrt{|b|_p}$. If $p = 2$, then $\theta_b = \sqrt{|b|_p}$.

PROPOSITION 1. *Let $f(x)$ belong to the class $\mathcal{A}(\mathcal{U}_s)$. If $s > \theta_b$, then the integral (4) converges.*

REMARK 2. There exist functions which are analytic on the ball $\mathcal{U}_{\theta_b}$ but are not integrable, see [11].

In fact, we have proved that the Gaussian distribution is a continuous linear functional on the space of analytic functions $\mathcal{A}(\mathcal{U}_s)$, i.e., it is an analytic generalized function (distribution); for the details see [2]. We shall use the symbol $\int$ to represent the duality between the space of test functions $\mathcal{A}(\mathcal{U}_s)$ and the space of generalized functions $\mathcal{A}'(\mathcal{U}_s)$ by setting $(\mu\prime, f) \equiv \int f(x)\mu(dx)$ for $f \in \mathcal{A}(\mathcal{U}_s)$ and $\mu \in \mathcal{A}'(\mathcal{U}_s)$. As usual, we define the derivative of a generalized function μ by means of the equality $\int f(x)\mu(dx) = -\int f'(x)\mu(dx)$.

It should be remarked that the distribution ν_b is not a bounded measure on any ball of $\mathbb{Q}_p$. (This was proved for the case $p \neq 2$; in the case $p = 2$ the question is still open), see Endo and Khrennikov [19]. Thus we could not integrate continuous functions with respect to the p-adic Gaussian distribution.

We introduce Hermite polynomials over $\mathbb{Q}_p$ by substituting a p-adic variable, in place of a real one, into the ordinary Hermite polynomials over the reals:

$$H_{n,b}(x) = \frac{n!}{b^n} \sum_{k=0}^{[n/2]} \frac{(-1)^k x^{n-2k} b^k}{k!(n-2k)!2^k}.$$

We shall use also the following representation for the Hermite polynomials: $H_{n,b}(x) = (-1)^n e^{x^2/2b} \frac{d^n}{dx^n} e^{-x^2/2b}$. This representation holds on a ball of sufficiently small radius with center at zero. As a consequence, we obtain the following equality in the space of generalized functions $\mathcal{A}'(\mathcal{U}_s)$, with $s > \theta_b$:

$$H_{n,b}(x)\nu_b(dx) = (-1)^n \frac{d^n}{dx^n} \nu_b(dx), \tag{5}$$

i.e., multiplication of the Gaussian distribution by a Hermite polynomial is equivalent to evaluating the corresponding derivative (in the sense of distribution theory).

In the space $\mathcal{P}(\mathbb{Q}_p)$ of polynomials on $\mathbb{Q}_p$ with coefficients belonging to $\mathbb{Q}_p(i)$, we introduce the inner product $(f,g) = \int f(x)\overline{g}(x)\nu_b(dx)$. With respect to this inner product, the polynomials $H_{n,b}$ verify the orthogonal conditions $\int H_{m,b}(x)H_{n,b}(x)\nu_b(dx) = \delta_{nm}\, n!/b^n$.

REMARK 3. In fact, the appearance of such constants $\lambda_n = n!/b^n$ was one of the reasons for introducing p-adic Hilbert spaces that are isomorphic to $l^2(p,\lambda)$.

Any $f \in \mathcal{P}(\mathbb{Q}_p)$ can be written in the following way: $f(x) = \sum_{n=0}^N f_n H_{n,b}(x)$, $N = N(f)$, $f_n \in \mathbb{Q}_p(i)$. We introduce the norm $\|f\|^2 = \max_n |f_n|_p^2(|n!|_p/|b|_p^n)$, and we define $L_2^i(\mathbb{Q}_p, \nu_b)$ as the completion of $\mathcal{P}(\mathbb{Q}_p)$ with respect to $\|\cdot\|$. It is evident that the space $L_2^i(\mathbb{Q}_p, \nu_b)$ is the set

$$\left\{ f(x) = \sum_{n=0}^{\infty} f_n H_{n,b}(x),\ f_n \in \mathbb{Q}_p(i) : \text{ the series } \sum_{n=0}^{\infty} f_n \overline{f_n} \frac{n!}{b^n} \text{ converges} \right\}.$$

Let $L_2(\mathbb{Q}_p, \nu_b)$ stand for the subset of $L_2^i(\mathbb{Q}_p, \nu_b)$ consisting of functions that have the Hermite coefficients $f_n \in \mathbb{Q}_p$. This is a Hilbert space over the field $\mathbb{Q}_p$.

For $f(x) \in L_2^i(\mathbb{Q}_p, \nu_b)$ we set

$$\sigma_n^2(f) \equiv \sigma_{n,b}^2(f) = |f_n|_p^2 \left| \frac{n!}{b^n} \right|_p, \tag{6}$$

where

$$f_n = \frac{b^n}{n!} \int f(x) H_{n,b}(x) \nu_{b,p}(dx)$$

are the Hermite coefficients of $f(x)$.

Now we wish to study the relations between $L_2(\mathbb{Q}_p, \nu_b)$-functions and analytic functions. Set $\mathcal{A}_{\mathbb{Q}_p}(\mathcal{U}_r) = \{f \in \mathcal{A}(\mathcal{U}_r) : f : U_r \to \mathbb{Q}_p\}$, i.e., these are functions that have Taylor coefficients belonging to the field $\mathbb{Q}_p$.

THEOREM 4. *Assume $p \neq 2$. Then $L_2(\mathbb{Q}_p, \nu_b) \subset \mathcal{A}_{\mathbb{Q}_p}(\mathcal{U}_{\theta_b})$.*

Now we consider the case $p = 2$. In general, L_2-functions are not analytic on the ball $\mathcal{U}_{\theta_b}$.

THEOREM 5. *Let $s > \theta_b$. Then $\mathcal{A}_{\mathbb{Q}_p}(\mathcal{U}_s) \subset L_2(\mathbb{Q}_p, \nu_b)$.*

Further we construct the L_2-representation of the translation group. If $|b|_p = p^{2k+1}$ we set $s(b) = p^k$, if $|b|_p = p^{2k}$, we set $s(b) = p^{k-1}$. Set $\widehat{T}_\beta(f)(x) = f(x+\beta), \beta \in \mathbb{Q}_p$. We shall prove that these operators are bounded for $\beta \in U_{s(b)}$. Moreover, these operators are isometries of $L_2(\mathbb{Q}_p, \nu_b)$. Using this fact we shall construct a representation of the translation group in the p-adic Hilbert space $L_2(\mathbb{Q}_p, \nu_b)$.

LEMMA 1. *The formula*

$$\widehat{T}_\beta H_{n,b}(x) = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{\beta}{b}\right)^j H_{n-j,b}(x) \tag{7}$$

holds for the translates of Hermite polynomials.

THEOREM 6. *The operator $\widehat{T}_\beta$ belongs to $IS(L_2(\mathbb{Q}_p, \nu_b))$ for every $\beta \in U_{s(b)}$, and the map $T : U_{s(b)} \to IS(L_2(\mathbb{Q}_p, \nu_b))$, $\beta \to \widehat{T}_\beta$, is analytic.*

5. Gaussian representations of position and momentum operators

Just as in ordinary Schrödinger quantum mechanics, let us define the coordinate and momentum operators in $L_2^i(\mathbb{Q}_p, \nu_b)$ by

$$\widehat{\mathbf{q}} f(x) = x f(x),\ \widehat{\mathbf{p}} f(x) = (-i) \left(\frac{d}{dx} - \frac{x}{2b} \right) f(x),$$

where f belongs to the $\mathbb{Q}_p(i)$-linear space $\mathcal{D}$ of linear combinations of Hermite polynomials. The coordinate and momentum operators so defined satisfy on $\mathcal{D}$ the canonical commutation relations

$$[\widehat{\mathbf{q}}, \widehat{\mathbf{p}}] = iI, \tag{8}$$

where I is the unit operator in $L_2^i(\mathbb{Q}_p, \nu_b)$. We shall see that these relations can be extended to the whole of $L_2^i(\mathbb{Q}_p, \nu_b)$.

THEOREM 7 (Albeverio-Khrennikov). *The operators of the coordinate $\widehat{\mathbf{q}}$ and momentum $\widehat{\mathbf{p}}$ are bounded in the Hilbert space $L_2^i(\mathbb{Q}_p, \nu_b)$, with*

$$\|\widehat{\mathbf{q}}\| = \sqrt{|b|_p}, \quad \|\widehat{\mathbf{p}}\| = \frac{1}{\sqrt{|b|_p}}. \tag{9}$$

Moreover $\widehat{\mathbf{q}}$ and $\widehat{\mathbf{p}}$ are symmetric and satisfy (8) on $L_2^i(\mathbb{Q}_p, \nu_b)$.

Proof. Let $f(x) = \sum_{n=0}^{\infty} f_n H_{n,b}(x) \in L_2^i(\mathbb{Q}_p, \nu_b)$. By the recurrence formula

$$H_{n+1,b}(x) = b^{-1}[xH_{n,b}(x) - nH_{n-1,b}(x)], \tag{10}$$

we have

$$\widehat{\mathbf{q}}H_{n,b}(x) = bH_{n+1,b}(x) + nH_{n-1,b}(x), \tag{11}$$

and $\widehat{\mathbf{q}} f(x) = \sum_{n=0}^{\infty} b f_n H_{n+1,b}(x) + \sum_{n=1}^{\infty} n f_n H_{n-1,b}(x)$. Thus, by the strong triangle inequality, we obtain

$$\begin{aligned}
\|\widehat{\mathbf{q}} f\|^2 &\leq \max\left[\max_n |b|_p^2 |f_n|_p^2 \frac{|(n+1)!|_p}{|b|_p^{n+1}},\ \max_n |n|_p^2 |f_n|_p^2 \frac{|(n-1)!|_p}{|b|_p^{n-1}}\right] \\
&= |b|_p \max\left[\max_n |n+1|_p |f_n|_p^2 \frac{|n!|_p}{|b|_p^n},\ \max_n |n|_p |f_n|_p^2 \frac{|n!|_p}{|b|_p^n}\right] \\
&\leq |b|_p \|f\|^2,
\end{aligned}$$

(as $|n|_p \leq 1$ for all $n \in \mathbf{N}$). Therefore, $\|\widehat{\mathbf{q}}\| \leq \sqrt{|b|_p}$. Now we prove that $\|\widehat{\mathbf{q}}\|^2 = |b|_p$. Let $n = p^k$, then

$$D_{k,b} = \|\widehat{\mathbf{q}} H_{p^k,b}\|^2 = \max\left[\frac{|b|_p^2 |(p^k+1)!|_p}{|b|_p^{p^k+1}},\ \frac{|p^k|_p^2 |(p^k-1)!|_p}{|b|_p^{p^k-1}}\right].$$

But $|(p^k+1)!|_p = |p^k!|_p$ and $|p^{2k}(p^k-1)!|_p = p^{-k}|p^k!|_p$. Thus

$$D_{k,b} = |b|_p \frac{|p^k!|_p}{|b|_p^{p^k}} = |b|_p \|H_{p^k,b}\|^2,$$

which proves the first equality in (9).

Further, we have $\frac{d}{dx}H_{n,b}(x)=(x/b)H_{n,b}(x)-H_{n+1,b}(x)=(n/b)H_{n-1,b}(x)$. Set $\widehat{T}_x=(d/dx-(x/2b))$. We have $\widehat{T}_x H_{n,b}(x)=(n/2b)H_{n-1,b}(x)-(1/2)H_{n+1,b}(x)$. To compare this expression with (11), we rewrite it as

$$\widehat{T}_x H_{n,b}(x)=\frac{1}{2b}\big[-bH_{n+1,b}(x)+nH_{n-1,b}(x)\big]. \tag{12}$$

The expression in square brackets is similar to that in (11); the sign does play a role in estimates of max type. Thus we obtain $\|\widehat{T}_x\|=(1/|b|_p)\|\widehat{\mathbf{q}}\|$, which proves the second equality in (9).

Symmetry of the bounded operators $\widehat{\mathbf{q}}$, $\widehat{\mathbf{p}}$ is easily verified. □

Thus, unlike in the Archimedean case (complex Hilbert space), in the p-adic case the canonical commutation relations (8) are valid not only on a dense subspace, but everywhere on the Hilbert space.

6. One parameter groups generated by position and momentum operators

We shall compute numbers $[\gamma(\widehat{\mathbf{q}})]_p^-$ and $[\gamma(\widehat{\mathbf{p}})]_p^-$, see (2), (3) in section 3.

If $|b|_p=p^{2k+1}$ then $\gamma(\widehat{\mathbf{q}})=1/(p^k p^{1/2}p^{1/(p-1)})$. If $p\neq 3$ then $[\gamma(\widehat{\mathbf{q}})]_p^-=1/p^{k+1}$. If $p=3$ then $[\gamma(\widehat{\mathbf{q}})]_p^-=1/p^{k+2}$. If $|b|_p=p^{2k}$ then $\gamma(\widehat{\mathbf{q}})=1/(p^k p^{1/(1-p)})$ and $[\gamma(\widehat{\mathbf{q}})]_p^-=1/p^{k+1}$. Set

$$R(b)=[\gamma(\widehat{\mathbf{q}})]_p^-.$$

If $|b|_p=p^{2k+1}$ then $\gamma(\widehat{\mathbf{p}})=(p^{1/2}/p^{1/(p-1)})p^k$. If $p\neq 3$ then $[\gamma(\widehat{\mathbf{p}})]_p^-=p^k$. If $p=3$ then $[\gamma(\widehat{\mathbf{p}})]_p^-=p^{k-1}$. If $|b|_p=p^{2k}$ then $[\gamma(\widehat{\mathbf{p}})]_p^-=p^{k-1}$. Set

$$r(b)=[\gamma(\widehat{\mathbf{p}})]_p^-.$$

THEOREM 8. *(Albeverio–Khrennikov) The maps* $\alpha\mapsto\widehat{U}(\alpha)=e^{i\alpha\widehat{\mathbf{q}}}$, $\alpha\in U_{R(b)}$, *and* $\beta\mapsto\widehat{V}(\beta)=e^{i\beta\widehat{\mathbf{p}}}$, $\beta\in U_{r(b)}$, *are analytic one-parameter groups of unitary isometric operators acting on* $L_2^i(\mathbb{Q}_p,\nu_b)$. *They satisfy the Weyl commutation relations*

$$\widehat{U}(\alpha)\widehat{V}(\beta)=e^{-i\alpha\beta}\widehat{V}(\beta)\widehat{U}(\alpha). \tag{13}$$

We set

$$\widehat{M}_\beta f(x)=e^{-\beta\widehat{\mathbf{q}}/2b}f(x)=\sum_{n=0}^{\infty}\frac{(-\beta\widehat{\mathbf{q}})^n}{n!(2b)^n}f(x), \tag{14}$$

for $f\in L_2(\mathbb{Q}_p,\nu_b)$. By Theorem 7, we easily obtain

PROPOSITION 2. *The map* $M:U_{r(b)}\mapsto IS(L_2(\mathbb{Q}_p,\nu_b))$, $\beta\to\widehat{M}_\beta$, *is an analytic one-parameter group (indexed by the ball* $U_{r(b)}$*).*

REMARK 4. The function $x\mapsto e^{-\beta x/2b}$ is not defined on the whole of $\mathbb{Q}_p$ and we cannot consider (14) as a pointwise multiplication operator.

7. Operator calculus

It is well known that in the ordinary $L_2(\mathbb{R}, dx)$ space, the unitary group $\widehat{V}(\beta) = e^{i\beta\widehat{\mathbf{p}}}$, with $\beta \in \mathbb{R}$, can be realized as the translation group, with $\widehat{V}(\beta)\psi(x) = \psi(x+\beta)$ for sufficiently well-behaved functions $\psi(x)$. If we consider the equivalent representation in L_2-space with respect to the Gaussian measure $\nu_b(dx) = (e^{-x^2/2b}/\sqrt{2\pi b})dx$ on $\mathbb{R}$, we obtain

$$\widehat{V}(\beta)\psi(x) = e^{-\beta^2/4b}e^{-\beta x/2b}\psi(x+\beta), \tag{15}$$

or

$$\widehat{V}(\beta) = c_\beta \widehat{M}_\beta \widehat{T}_\beta, \tag{16}$$

where $c_\beta = e^{-\beta^2/4b}$. We shall now prove that (16) is also valid in the p-adic case.

Set $\widehat{S}(\beta) = c_\beta \widehat{M}_\beta \widehat{T}_\beta$, $\beta \in U_{r(b)}$, where the operator $\widehat{M}_\beta$ is defined by (14).

THEOREM 9. *The map $\beta \mapsto \widehat{S}_\beta$, $\beta \in U_{r(b)}$, is a one-parameter analytic group of isometric unitary operators acting in $L_2^i(\mathbb{Q}_p, \nu_b)$.*

LEMMA 2. *The groups $\widehat{S}(\beta)$ and $\widehat{V}(\beta)$ have $\widehat{\mathbf{p}}$ as their common generator.*

As a consequence of this lemma, and the analyticity of the one parameter groups $S(\beta)$ and $V(\beta)$, we easily obtain:

THEOREM 10. *The representation (15), (16) holds for the operator group $\widehat{V}(\beta)$.*

By using one-parameter groups $\widehat{U}(\alpha), \widehat{V}(\beta)$, one can formally define pseudo-differential operators. However, a rigorous mathematical theory is still awaiting development.

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