Rend. Sem. Mat. Univ. Pol. Torino
Vol. 67, 2 (2009), 247 – 269
Second Conf. Pseudo-Differential Operators

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A TRANSFORMATION OF ALMOST PERIODIC PSEUDODIFFERENTIAL OPERATORS TO FOURIER MULTIPLIER OPERATORS WITH OPERATOR-VALUED SYMBOLS

Dedicated to Professor Luigi Rodino on the occasion of his 60th birthday

Abstract. We present results for pseudodifferential operators on $\mathbb{R}^d$ whose symbol $a(\cdot,\xi)$ is almost periodic (a.p.) for each $\xi \in \mathbb{R}^d$ and belongs to a Hörmander class $S^m_{\rho,\delta}$. We study a linear transformation $a \mapsto U(a)$ from a symbol $a(x,\xi)$ to a frequency-dependent matrix $U(a)(\xi)_{\lambda,\lambda'}$, indexed by $(\lambda,\lambda') \in \Lambda \times \Lambda$ where Λ is a countable set in $\mathbb{R}^d$. The map $a \mapsto U(a)$ transforms symbols of a.p. pseudodifferential operators to symbols of Fourier multiplier operators acting on vector-valued function spaces. We show that the map preserves operator positivity and identity, respects operator composition and respects adjoints.

1. Introduction

The paper concerns pseudodifferential operators (abbreviated to ΨDO) on $\mathbb{R}^d$ in the Kohn–Nirenberg quantization, where the symbol $a(\cdot,\xi)$ is almost periodic (a.p.) for each $\xi \in \mathbb{R}^d$, and belongs to a Hörmander class $S^m_{\rho,\delta}$. This symbol class is denoted $APS^m_{\rho,\delta}$ and the corresponding operators are called a.p. pseudodifferential operators. We study the symbol transformation $a \mapsto U(a)$ given by

$$U(a)(\xi)_{\lambda,\lambda'} = M_x(a(x,\xi-\lambda')e^{-2\pi i x\cdot(\lambda'-\lambda)})$$

where M_x denotes the mean value functional of a.p. functions. This transformation was introduced, for operator kernels rather than symbols, by E. Gladyshev [4, 5], for the purposes of stochastic processes. The connection between stochastic processes and operator theory originates from the fact that the so-called covariance function of a stochastic process is the kernel of a positive operator. Gladyshev studied a particular class of stochastic processes called *almost periodically correlated*, which means that the symbol of the covariance operator is almost periodic in the first variable.

The element $U(a)(\xi)$ can be considered a matrix indexed by $(\lambda,\lambda') \in \Lambda \times \Lambda$ where $\Lambda \subset \mathbb{R}^d$ is the countable set of frequencies that occur in $\{a(\cdot,\xi)\}_{\xi\in\mathbb{R}^d}$. Thus $U(a)(\xi)$ is an operator that acts between sequence spaces and the function $\xi \mapsto U(a)(\xi)$ may be considered the operator-valued symbol of a Fourier multiplier operator denoted $U(a)(D)$.

Let $a \in APS^m_{\rho,\delta}$ and let l^2_s be the space of sequences $(x_\lambda)_{\lambda\in\Lambda}$ such that the weighted norm

$$\|x\|_{l_s^2} = \left(\sum_{\lambda \in \Lambda} (1+|\lambda|^2)^s |x_\lambda|^2 \right)^{1/2}$$

is finite. Using results by M. A. Shubin, we first observe that the norm of the operator $a(x,D) : H^s(\mathbb{R}_B^d) \mapsto H^{s-m}(\mathbb{R}_B^d)$ is equal to the norm of $a(x,D) : H^s(\mathbb{R}^d) \mapsto H^{s-m}(\mathbb{R}^d)$ for any $s \in \mathbb{R}$. Here $H^s(\mathbb{R}^d)$ denotes the classical Sobolev Hilbert space, and $H^s(\mathbb{R}_B^d)$ denotes the Sobolev–Besicovitch space of a.p. functions, completed from the trigonometric polynomials in the norm

$$\|f\|_{H^s(\mathbb{R}_B^d)} = \left(\sum_{\lambda \in \mathbb{R}^d} (1+|\lambda|^2)^s |f_\lambda|^2 \right)^{1/2},$$

where $f_\lambda = M_x(f(x)e^{-2\pi i x\cdot\lambda})$ is the Bohr–Fourier coefficient of an a.p. function f. Then we prove that the norm of the matrix $U(a)(0) : l_s^2 \mapsto l_{s-m}^2$ is bounded by the norm of the operator $a(x,D) : H^s(\mathbb{R}_B^d) \mapsto H^{s-m}(\mathbb{R}_B^d)$. We also show that $a(x,D)$ is positive on $\mathscr{S}(\mathbb{R}^d)$ if and only if it is positive on the trigonometric polynomials on $\mathbb{R}^d$ and $a(x,D) \geqslant 0$ on $TP(\Lambda)$ if and only if $U(a)(0)$ is a positive definite matrix. Thus much information about the operator $a(x,D)$ can be read off from the evaluation of the matrix symbol $U(a)$ at the origin.

We prove that $U(a)(\xi)$ is a continuous transformation $l_s^2 \mapsto l_{s-m}^2$ for any $\xi \in \mathbb{R}^d$, and the map $\mathbb{R}^d \ni \xi \mapsto U(a)(\xi) \in \mathcal{L}(l_s^2, l_{s-m}^2)$ is continuous. Moreover, $U(a)(D) \geqslant 0$ if $a(x,D) \geqslant 0$. The latter result on preservation of positivity was proved by Gladyshev [5] for uniformly continuous operator kernels. Here $U(a)(D)$ acts on vector-valued function spaces like $\mathscr{S}(\mathbb{R}^d, l_s^2)$. Then we show our main result that the transformation $a \mapsto U(a)$ respects operator composition. More precisely, denote the *symbol product*, corresponding to operator composition, by $a(x,D) \circ b(x,D) = (a\#_0 b)(x,D)$. If $a \in APS_{\rho,\delta}^{m_1}$ and $b \in APS_{\rho,\delta}^{m_2}$, $m_1, m_2 \in \mathbb{R}$, then we have

$$U(a\#_0 b)(\xi) = U(a)(\xi) \cdot U(b)(\xi).$$

Finally, we prove that the requirement that the symbol is almost periodic in the first variable is invariant under a common family of quantizations that is defined using a parameter $t \in \mathbb{R}$. The family includes the Kohn–Nirenberg ($t = 0$) and the Weyl ($t = 1/2$) correspondences.

In conclusion, the transformation $a \mapsto U(a)$ is a linear, injective map that preserves operator identity, positivity, adjoint and composition. In the proofs of our results we use mainly results by Shubin [9, 10, 11, 12].

In scalar-valued function spaces, translation-invariant (or convolution or Fourier multiplier) operators commute, but for vector-valued function spaces, the product in $\mathbb{C}$ is replaced by the matrix product, so translation-invariant operators are not commutative. The transformation $a(x,D) \mapsto a \mapsto U(a)(D)$ transfers the non-commutativity of almost periodic pseudodifferential operators with symbols in $S_{\rho,\delta}^m$ into the non-commutativity of the matrix product.

A brief comment on some parts of the literature on a.p. pseudodifferential operators follows. Coburn, Moyer and Singer [1] developed an index theory for pseudodifferential operators on $\mathbb{R}^d$ with almost periodic principal symbol. Shubin has made many important contributions to the theory of partial differential operators with almost periodic coefficients and a.p. pseudodifferential operators. For example, he introduced the Sobolev–Besicovitch spaces [9] and proved the equality of the spectra for a.p. pseudodifferential operators acting on $L^2(\mathbb{R}^d)$ and the Besicovitch space $B^2(\mathbb{R}^d)$, provided the operator is bounded or elliptic [11, 12].

Lately Turunen, Ruzhansky and Vainikko have worked on pseudodifferential operators with symbols that are *periodic* in the first variable [14, 15, 8]. The operators may be considered to act on functions defined on the torus $\mathbb{T}^d$, and the theory of pseudodifferential operators on manifolds may be used. However, the use of Fourier series representations gives a more elementary and global treatment.

2. Notation and preliminaries

We use $\langle x\rangle = (1+|x|^2)^{1/2}$, $x \in \mathbb{R}^d$, and the Fourier transform is defined by

$$\mathscr{F} f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\xi}\, dx, \quad f \in \mathscr{S}(\mathbb{R}^d).$$

For a multiindex $\alpha = (\alpha_1, \dots, \alpha_d)$, we define the partial differential operator

$$\partial^\alpha f(x) = \partial_x^\alpha f(x) = \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1}\cdots \partial x_d^{\alpha_d}}, \quad x \in \mathbb{R}^d.$$

We use C for a generic positive constant that may vary over equalities and inequalities, we denote by $C^m(\mathbb{R}^d)$ the space of functions such that $\partial^\alpha f$ is continuous for $|\alpha| \leqslant m$ and $C^\infty = \bigcap_m C^m$ is the space of smooth functions. The symbol $C_b(\mathbb{R}^d)$ stands for the space of continuous and supremum bounded functions, and $C_b^\infty(\mathbb{R}^d)$ is the space of functions whose derivatives of all orders are continuous and bounded in supremum norm. The space of compactly supported smooth (test) functions is denoted $C_c^\infty(\mathbb{R}^d)$. The Schwartz space of smooth rapidly decreasing functions is denoted $\mathscr{S}(\mathbb{R}^d)$ and its dual $\mathscr{S}'(\mathbb{R}^d)$ is the space of tempered distributions. A space of trigonometric polynomials is denoted $TP(S)$ and consists of functions of the form

$$f(x) = \sum_{n=1}^{N} a_n e^{2\pi i \xi_n\cdot x}, \quad a_n \in \mathbb{C}, \quad \xi_n \in S \subseteq \mathbb{R}^d.$$

We will consider functions defined on $\mathbb{R}^d$ and taking values in a Hilbert or Banach space X, and then $C(\mathbb{R}^d, X)$ denotes the space of continuous X-valued functions, and likewise for other function spaces. The space of bounded linear transformations between two Hilbert spaces H and H' is denoted $\mathscr{L}(H,H')$, and $\mathscr{L}(H,H) = \mathscr{L}(H)$. The operator norm is denoted $\|\cdot\|_{\mathscr{L}(H,H')}$ or $\|\cdot\|_{\mathscr{L}(H)}$.

A subset Y of a complete metric space X is *precompact* if it is totally bounded, which means that Y can be covered by a finite union of balls of radius ε, for any $\varepsilon > 0$. This definition is equivalent to the property that the closure of Y is compact.

We define a standard family of symbol classes, the so called Hörmander classes. More precisely, the following symbol classes are global versions of Hörmander spaces [3, 6, 13].

DEFINITION 1. *For $m \in \mathbb{R}$ and $0 \leqslant \rho, \delta \leqslant 1$ the space $S^m_{\rho,\delta}$ is defined as the space of all $a \in C^\infty(\mathbb{R}^{2d})$ such that*

$$\sup_{x,\xi \in \mathbb{R}^d} \langle \xi \rangle^{-m+\rho|\alpha|-\delta|\beta|} |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| < \infty, \quad \alpha, \beta \in \mathbb{N}^d. \tag{1}$$

We impose the conditions

$$0 < \rho \leqslant 1, \quad 0 \leqslant \delta < 1, \quad \delta \leqslant \rho.$$

Following convention, we set $S^{-\infty}_{\rho,\delta} = \bigcap_{m\in\mathbb{R}} S^m_{\rho,\delta}$ and $S^{\infty}_{\rho,\delta} = \bigcup_{m\in\mathbb{R}} S^m_{\rho,\delta}$.

The space $S^m_{\rho,\delta}$ is a Fréchet space with seminorms defined by (1).

We consider the Kohn–Nirenberg quantization of pseudodifferential operators. A symbol function a defined on the phase space $\mathbb{R}^{2d}$ gives rise to an operator $a(x,D)$ according to the formula

$$a(x,D)f(x) = \int_{\mathbb{R}^{2d}} e^{2\pi i \xi\cdot(x-y)} a(x,\xi) f(y)\, dy\, d\xi, \quad f \in \mathscr{S}(\mathbb{R}^d). \tag{2}$$

When $a \in S^m_{\rho,\delta}$, the corresponding operator class is denoted $L^m_{\rho,\delta}$. For the symbol classes $S^m_{\rho,\delta}$, the oscillatory integral (2) is generally not absolutely convergent and should be read as the iterated integral

$$a(x,D)f(x) = \int_{\mathbb{R}^d} e^{2\pi i \xi\cdot x} a(x,\xi) \widehat{f}(\xi)\, d\xi. \tag{3}$$

In order to extend the operator to act on other function spaces than $\mathscr{S}(\mathbb{R}^d)$ one modifies the definition (2) into

$$a(x,D)f(x) = \lim_{\varepsilon\to+0} \int_{\mathbb{R}^{2d}} \psi(\varepsilon y)\psi(\varepsilon\xi) e^{2\pi i \xi\cdot(x-y)} a(x,\xi) f(y)\, dy\, d\xi \tag{4}$$

where $\psi \in C_c^\infty(\mathbb{R}^d)$ equals one in a neighborhood of the origin. Integrating by parts we may rewrite (4) as

$$\begin{aligned} a(x,D)f(x) = \int_{\mathbb{R}^{2d}} & e^{2\pi i \xi\cdot(x-y)} (1+|\xi|^2)^{-N} (1-\Delta_\xi)^M a(x,\xi) \\ & \times (1-\Delta_y)^N ((1+|x-y|^2)^{-M} f(y))\, dy\, d\xi, \end{aligned}$$

where Δ denotes the normalized Laplacian $\Delta = (2\pi)^{-2}\sum_1^d \partial_j^2$, which is an absolutely convergent integral for $f \in C_b^\infty(\mathbb{R}^d)$ provided that $2M > d$ and $2N > d+m$. By differentiation under the integral it follows that $a(x,D) : C_b^\infty(\mathbb{R}^d) \mapsto C_b^\infty(\mathbb{R}^d)$ continuously. This procedure is standard and fundamental in pseudo-differential calculus [3, 6, 13].

For an admissible pair of symbols a, b we define the *symbol product* $\#_0$ by

$$c = a\#_0 b \quad \Longleftrightarrow \quad c(x,D) = a(x,D)b(x,D).$$

We have the following well-known result in the theory of pseudodifferential operators [3, 6]. The symbol product is a continuous bilinear map from $S^{m_1}_{\rho,\delta} \times S^{m_2}_{\rho,\delta}$ to $S^{m_1+m_2}_{\rho,\delta}$,

$$S^{m_1}_{\rho,\delta}\#_0 S^{m_2}_{\rho,\delta} \subseteq S^{m_1+m_2}_{\rho,\delta}, \quad m_1, m_2 \in \mathbb{R}. \tag{5}$$

3. Almost periodic functions and pseudodifferential operators

We will work with spaces of almost periodic functions [2, 7, 12]. The basic space of uniform almost periodic functions is denoted $CAP(\mathbb{R}^d)$ and defined as follows. A set $U \subset \mathbb{R}^d$ is called *relatively dense* if there exists a compact set $K \subset \mathbb{R}^d$ such that $(x+K)\cap U \neq \emptyset$ for any $x \in \mathbb{R}^d$. An element $\tau \in \mathbb{R}^d$ is called an ε-almost period of a function $f \in C_b(\mathbb{R}^d)$ if $\sup_x |f(x+\tau) - f(x)| < \varepsilon$. Then $CAP(\mathbb{R}^d)$ is defined as the space of all $f \in C_b(\mathbb{R}^d)$ such that, for any $\varepsilon > 0$, the set of ε-almost periods of f is relatively dense. With the assumption that the uniform almost periodic functions is a subspace of $C_b(\mathbb{R}^d)$, this original definition by H. Bohr is equivalent to the following three [2, 7, 12]:

(i) the set of translations $\{f(\cdot - x)\}_{x\in\mathbb{R}^d}$ is precompact in $C_b(\mathbb{R}^d)$;

(ii) $f = g \circ i_B$ where i_B is the canonical homomorphism from $\mathbb{R}^d$ into the Bohr compactification $\mathbb{R}^d_B$ of $\mathbb{R}^d$ and $g \in C(\mathbb{R}^d_B)$. Hence f can be extended to a continuous function on $\mathbb{R}^d_B$;

(iii) f is the uniform limit of trigonometric polynomials.

The space $CAP(\mathbb{R}^d)$ is a conjugate-invariant complex algebra of uniformly continuous functions. For $f \in CAP(\mathbb{R}^d)$ the mean value functional

$$M(f) = \lim_{T\to+\infty} T^{-d} \int_{s+K_T} f(x)\,dx, \tag{6}$$

where $K_T = \{x \in \mathbb{R}^d :\ 0 \leqslant x_j \leqslant T,\ j = 1,\dots,d\}$, exists uniformly over $s \in \mathbb{R}^d$. By M_x we understand the mean value in the variable x of a function of several variables. The Bohr (–Fourier) transformation [7] is defined by

$$f_\lambda = M_x(f(x)e^{-2\pi i\lambda\cdot x}), \quad \lambda \in \mathbb{R}^d,$$

and $f_\lambda \neq 0$ for at most countably many $\lambda \in \mathbb{R}^d$. The set $\{\lambda \in \mathbb{R}^d : f_\lambda \neq 0\}$ is called the set of frequencies for f.

A function $f \in CAP(\mathbb{R}^d)$ may be reconstructed from its Bohr–Fourier coefficients $(f_\lambda)_{\lambda\in\Lambda}$ using Bochner–Fejér polynomials [7, 12]. We give a brief overview of the results we need. Let $\beta_n \in \mathbb{R}^d$, $n = 1,2,\ldots$, be a *rational basis* for the set of frequencies Λ for f. This means that $(\beta_n)_{n=1}^\infty$ is linearly independent over $\mathbb{Q}$ and each $\lambda \in \Lambda$ can be written

$$\lambda = \sum_{n=1}^{N} q_n \beta_n, \quad q_n \in \mathbb{Q},$$

with unique coefficients $(q_n)_{n=1}^N$. Every countable set $\Lambda \subset \mathbb{R}^d$ has a rational basis contained in Λ [7]. The composite Bochner–Fejér kernel is defined as

$$\begin{aligned} K_{n;\beta_1,\ldots,\beta_n}(x) = &\sum_{|\nu_1|\leqslant(n!)^2,\ldots,|\nu_n|\leqslant(n!)^2} \left(1-\frac{|\nu_1|}{(n!)^2}\right)\cdots\left(1-\frac{|\nu_n|}{(n!)^2}\right) \\ &\times \exp\left(2\pi i\left(\frac{\nu_1}{n!}\beta_1+\cdots+\frac{\nu_n}{n!}\beta_n\right)\cdot x\right). \end{aligned}$$

We denote its coefficients

$$K_{n;\nu_1,\ldots,\nu_n} = \left(1-\frac{|\nu_1|}{(n!)^2}\right)\cdots\left(1-\frac{|\nu_n|}{(n!)^2}\right), \quad |\nu_j| \leqslant (n!)^2, \quad 1 \leqslant j \leqslant n. \tag{7}$$

Since $(\beta_n)_{n=1}^\infty$ is linearly independent over $\mathbb{Q}$, and since $M_x(e^{2\pi i\lambda\cdot x}) = 0$ when $\lambda \neq 0$, we have $M(K_{n;\beta_1,\ldots,\beta_n}) = 1$.

For a given $f \in CAP(\mathbb{R}^d)$ the Bochner–Fejér polynomial of order n is defined by

$$\begin{aligned} P_n(f)(x) &= M_y\left(f(y)K_{n;\beta_1,\ldots,\beta_n}(x-y)\right) \\ &= \sum_{|\nu_1|\leqslant(n!)^2,\ldots,|\nu_n|\leqslant(n!)^2} K_{n;\nu_1,\ldots,\nu_n} f_{\frac{\nu_1}{n!}\beta_1+\cdots+\frac{\nu_n}{n!}\beta_n} \\ &\quad \times \exp\left(2\pi i\left(\frac{\nu_1}{n!}\beta_1+\cdots+\frac{\nu_n}{n!}\beta_n\right)\cdot x\right). \end{aligned} \tag{8}$$

It follows from $M(K_{n;\beta_1,\ldots,\beta_n}) = 1$ and $K_{n;\beta_1,\ldots,\beta_n}(x) \geqslant 0$ [7] that

$$\|P_n(f)\|_{L^\infty} \leqslant \|f\|_{L^\infty}. \tag{9}$$

If we define the function on Λ

$$K_n(\lambda) = \begin{cases} K_{n;\nu_1,\ldots,\nu_n} & \text{if } \lambda = \frac{\nu_1}{n!}\beta_1+\cdots+\frac{\nu_n}{n!}\beta_n, \quad |\nu_j| \leqslant (n!)^2, \quad 1 \leqslant j \leqslant n, \\ 0 & \text{otherwise,} \end{cases}$$

then we may write (8) in shorter form as

$$P_n(f)(x) = \sum_{\lambda\in\Lambda} K_n(\lambda) f_\lambda e^{2\pi i\lambda\cdot x}. \tag{10}$$

We observe that $K_n(\lambda)$ has finite support and $0 \leqslant K_n(\lambda) \leqslant 1$. For an arbitrary $\lambda \in \Lambda$ we may write for some $n > 0$ and $|\nu_j| \leqslant (n!)^2$, $1 \leqslant j \leqslant n$,

$$\begin{aligned}\lambda &= \frac{\nu_1}{n!}\beta_1 + \cdots + \frac{\nu_n}{n!}\beta_n \\ &= \frac{\nu_1(n+m)!/n!}{(n+m)!}\beta_1 + \cdots + \frac{\nu_n(n+m)!/n!}{(n+m)!}\beta_n + 0\cdot\beta_{n+1} + \cdots + 0\cdot\beta_{n+m},\end{aligned}$$

where $m \geqslant 0$ is arbitrary. It follows that

$$K_{n+m}(\lambda) = K_{n+m;\nu_1(n+m)!/n!,\ldots,\nu_n(n+m)!/n!,0,\ldots,0}.$$

For n and $\nu_1,\ldots,\nu_n$ fixed, it follows from (7) that the right hand side approaches 1 as $m \to \infty$, because

$$1 - \frac{|\nu_j|(n+m)!/n!}{((n+m)!)^2} = 1 - \frac{|\nu_j|}{n!(n+m)!} \to 1, \quad m \to \infty, \quad 1 \leqslant j \leqslant n.$$

We may conclude that $K_n(\lambda) \to 1$ as $n \to +\infty$, for any $\lambda \in \Lambda$.

We state the fundamental approximation result for the Bochner–Fejér polynomials [7, 12]. If $f \in CAP(\mathbb{R}^d)$ then we have the uniform limit

$$\sup_{x\in\mathbb{R}^d} |P_n(f)(x) - f(x)| \to 0, \quad n \to \infty. \tag{11}$$

The limit in (11) holds for any $f \in CAP(\mathbb{R}^d)$ whose set of frequencies is contained in Λ.

The next lemma resembles [12, Corollary 2.1]. We give a proof for completeness.

LEMMA 1. *For a precompact set $\mathcal{F} \subset CAP(\mathbb{R}^d)$, the limit*

$$\sup_{x\in\mathbb{R}^d} |P_n(f)(x) - f(x)| \to 0, \quad n \to \infty$$

is uniform over $f \in \mathcal{F}$.

Proof. Denote $\|\cdot\| = \|\cdot\|_{L^\infty}$. Due to the assumption that $\mathcal{F}$ is precompact, there exists for each integer $k > 0$ a finite set $\{f_{k,j}\}_{j=1}^{N_k} \subset \mathcal{F}$ such that $\|f - f_{k,j}\| < 1/k$ holds for each $f \in \mathcal{F}$ for some j, $1 \leqslant j \leqslant N_k$. Let Λ_k be the union of the frequencies that occur in $\{f_{k,j}\}_{j=1}^{N_k}$ and let Λ be the linear hull over $\mathbb{Q}$ of $\bigcup_{k\geqslant 1}\Lambda_k$. Define the Bochner–Fejér kernels $\{K_{n;\beta_1,\ldots,\beta_n}(x)\}_{n\geqslant 1}$ as above from the countable set Λ.

Let $\varepsilon > 0$ and pick an integer $k > \varepsilon^{-1}$. According to limit (11) we have $\|f_{k,j} - P_n(f_{k,j})\| < \varepsilon$ for all $1 \leqslant j \leqslant N_k$ if $n \geqslant N_\varepsilon$ for a sufficiently large integer N_ε. Let $f \in \mathcal{F}$ and pick an $f_{k,j}$ such that $\|f - f_{k,j}\| < 1/k < \varepsilon$. We have, using (9),

$$\begin{aligned}\|f - P_n(f)\| &\leqslant \|f - f_{k,j}\| + \|f_{k,j} - P_n(f_{k,j})\| + \|P_n(f_{k,j} - f)\| \\ &\leqslant \|f - f_{k,j}\| + \|f_{k,j} - P_n(f_{k,j})\| + \|f_{k,j} - f\| < 3\varepsilon, \quad n \geqslant N_\varepsilon.\end{aligned}$$

□

For $m \in \mathbb{N}$, the space $CAP^m(\mathbb{R}^d)$ is defined as all $f \in C^m(\mathbb{R}^d)$ such that $\partial^\alpha f \in CAP(\mathbb{R}^d)$ for $|\alpha| \leqslant m$, and $CAP^\infty(\mathbb{R}^d) = \bigcap_{m\in\mathbb{N}} CAP^m(\mathbb{R}^d)$. Then $CAP^\infty = CAP \cap C_b^\infty$ [12].

The mean value defines an inner product

$$(f,g)_B = M(f\overline{g}), \quad f,g \in CAP(\mathbb{R}^d). \tag{12}$$

The completion of $CAP(\mathbb{R}^d)$ in the norm $\|\cdot\|_B$ is the Hilbert space of Besicovitch a.p. functions $B^2(\mathbb{R}^d)$ [12].

Inspired by the usual Sobolev space norm

$$\|f\|_{H^s(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 \, d\xi \right)^{1/2},$$

Shubin [9] has defined Sobolev–Besicovitch spaces of a.p. functions $H^s(\mathbb{R}^d_B)$ for $s \in \mathbb{R}$, as the completion of $TP(\mathbb{R}^d)$ in the norm corresponding to the inner product

$$(f,g)_{H^s(\mathbb{R}^d_B)} = \sum_{\xi\in\mathbb{R}^d} (1+|\xi|^2)^s f_\xi \overline{g}_\xi, \quad f,g \in TP(\mathbb{R}^d).$$

The spaces $H^s(\mathbb{R}^d_B)$ are Hilbert spaces containing $TP(\mathbb{R}^d)$ as a dense subspace, $H^0(\mathbb{R}^d_B) = B^2(\mathbb{R}^d)$, and one defines

$$H^\infty(\mathbb{R}^d_B) = \bigcap_{s\in\mathbb{R}} H^s(\mathbb{R}^d_B), \quad H^{-\infty}(\mathbb{R}^d_B) = \bigcup_{s\in\mathbb{R}} H^s(\mathbb{R}^d_B).$$

We have the inclusion $CAP^\infty(\mathbb{R}^d) \subset H^\infty(\mathbb{R}^d_B)$, but there is no result corresponding to the Sobolev embedding theorem for the Sobolev–Besicovitch spaces. In fact, $H^\infty(\mathbb{R}^d_B)$ is not embedded in $CAP(\mathbb{R}^d)$ [12]. The reason is that the frequencies may be contained in a bounded set, for example as in

$$f(x) = \sum_{k=1}^\infty \frac{1}{k} e^{2\pi i \xi_k \cdot x}, \quad |\xi_k| = 1.$$

This function is clearly a member of $H^\infty(\mathbb{R}^d_B)$, and if the frequencies $\{\xi_k\}_{k=1}^\infty$ are linearly independent over $\mathbb{Z}$, then $\|f\|_{L^\infty} = \sum_{k=1}^\infty 1/k = \infty$ [12].

Next we define the symbol spaces for almost periodic pseudodifferential operators.

DEFINITION 2. *For $m \in \mathbb{R}$, the space $APS^m_{\rho,\delta}$ is defined as the space of all $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ such that $a(\cdot,\xi) \in CAP(\mathbb{R}^d)$ for all $\xi \in \mathbb{R}^d$. The corresponding operator class in the Kohn–Nirenberg quantization is denoted $APL^m_{\rho,\delta}$, and its members are called almost periodic pseudodifferential operators.*

For fixed $\xi \in \mathbb{R}^d$, we denote the Bohr–Fourier coefficients of $a(\cdot,\xi)$ by

$$a_\lambda(\xi) = (a(\cdot,\xi))_\lambda = M_x(a(x,\xi)e^{-2\pi i \lambda\cdot x}), \quad \xi \in \mathbb{R}^d, \quad \lambda \in \mathbb{R}^d. \tag{13}$$

LEMMA 2. *For $a \in APS^m_{\rho,\delta}$ the set of frequencies*

$$\Lambda = \Lambda(a) = \{\lambda \in \mathbb{R}^d : \exists \xi \in \mathbb{R}^d : a_\lambda(\xi) \neq 0\}$$

is countable.

Proof. As already mentioned $\Lambda_\xi = \{\lambda \in \mathbb{R}^d : a_\lambda(\xi) \neq 0\}$ is countable for each $\xi \in \mathbb{R}^d$. Using $\Lambda = \bigcup_{\xi\in\mathbb{R}^d} \Lambda_\xi$, it suffices to show that $\bigcup_{\xi\in\mathbb{R}^d} \Lambda_\xi \subset \bigcup_{\xi\in\mathbb{Q}^d} \Lambda_\xi$. If $\lambda \in \bigcup_{\xi\in\mathbb{R}^d} \Lambda_\xi$ there exists $\xi \in \mathbb{R}^d$ such that $a_\lambda(\xi) \neq 0$. By the mean value theorem we have

$$a(x,\xi+\eta) - a(x,\xi) = (\nabla_2 \operatorname{Re} a(x,\xi+\theta_1\eta) + i\nabla_2 \operatorname{Im} a(x,\xi+\theta_2\eta))\cdot\eta \tag{14}$$

where ∇_2 denotes the gradient in the second $\mathbb{R}^d$ variable and $0 \leqslant \theta_1, \theta_2 \leqslant 1$. It follows that $|a_\lambda(\xi+\eta) - a_\lambda(\xi)| \leqslant M_x(|a(x,\xi+\eta) - a(x,\xi)|) \leqslant C|\eta|$. Hence there exists $\xi' \in \mathbb{Q}^d$ such that $a_\lambda(\xi') \neq 0$. □

Without loss of generality we may assume that Λ is a linear space over $\mathbb{Q}$. Furthermore it follows from (14) that $\partial_\xi^\alpha a(\cdot,\xi) \in CAP(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}^d$ and $\xi \in \mathbb{R}^d$, since a ξ-derivative is a uniform limit of $CAP(\mathbb{R}^d)$ functions. Thus $\partial_\xi^\alpha \partial_x^\beta a(\cdot,\xi) \in CAP(\mathbb{R}^d)$ for all $\alpha, \beta \in \mathbb{N}^d$ and $\xi \in \mathbb{R}^d$.

LEMMA 3. *Suppose $a \in APS^m_{\rho,\delta}$ and $\lambda \in \Lambda$. Then $a_\lambda \in C^\infty(\mathbb{R}^d)$ and*

$$\partial^\alpha(a_\lambda)(\xi) = (\partial_\xi^\alpha a)_\lambda(\xi), \quad \alpha \in \mathbb{N}^d, \tag{15}$$

$$(\partial_x^\beta a)_\lambda(\xi) = (2\pi i\lambda)^\beta a_\lambda(\xi), \quad \beta \in \mathbb{N}^d. \tag{16}$$

Proof. By differentiation under the mean value we obtain (15). To prove (16), we integrate by parts which gives

$$\begin{aligned}(\partial_x^\beta a)_\lambda(\xi) &= M_x((\partial_x^\beta a)(x,\xi)e^{-2\pi i\lambda\cdot x}) \\ &= M_x(a(x,\xi)(-\partial_x)^\beta(e^{-2\pi i\lambda\cdot x})) \\ &= (2\pi i\lambda)^\beta a_\lambda(\xi).\end{aligned}$$

□

Lemma 3 gives

$$\partial^\alpha(a_\lambda)(\xi) = (\partial_\xi^\alpha a)_\lambda(\xi) = (2\pi i\lambda)^{-\beta}(\partial_\xi^\alpha\partial_x^\beta a)_\lambda(\xi), \quad \lambda \neq 0.$$

From (13) and Definition 1 we thus obtain the estimate

$$|\partial^\alpha(a_\lambda)(\xi)| \leqslant C_{k,\alpha}\langle\lambda\rangle^{-k}\langle\xi\rangle^{m-\rho|\alpha|+\delta k}, \quad k \in \mathbb{N}, \quad \alpha \in \mathbb{N}^d. \tag{17}$$

LEMMA 4. *If* $a \in APS^m_{\rho,\delta}$ *and* $f \in TP(\mathbb{R}^d)$ *then*

$$a(x,D)f(x) = \sum_{\lambda \in \mathbb{R}^d} e^{2\pi i x\cdot\lambda} a(x,\lambda) f_\lambda. \tag{18}$$

Proof. Since $f(x) = \sum_\lambda f_\lambda e^{2\pi i x\cdot\lambda}$ is a finite sum we have by the definition (4)

$$\begin{aligned}
a(x,D)f(x) &= \sum_\lambda f_\lambda \lim_{\varepsilon\to+0} \int_{\mathbb{R}^{2d}} \psi(\varepsilon y)\psi(\varepsilon\xi) e^{2\pi i(\xi\cdot x - y\cdot(\xi-\lambda))} a(x,\xi)\, dy\, d\xi \\
&= \sum_\lambda f_\lambda \lim_{\varepsilon\to+0} \int_{\mathbb{R}^d} a(x,\xi) e^{2\pi i\xi\cdot x}\psi(\varepsilon\xi) \left(\int_{\mathbb{R}^d} \psi(\varepsilon y) e^{-2\pi i y\cdot(\xi-\lambda)} dy \right) d\xi \\
&= \sum_\lambda f_\lambda \lim_{\varepsilon\to+0} \int_{\mathbb{R}^d} a(x,\xi+\lambda) e^{2\pi i x\cdot(\xi+\lambda)} \psi(\varepsilon(\xi+\lambda))\, \varepsilon^{-d} \widehat{\psi}(\xi/\varepsilon)\, d\xi.
\end{aligned} \tag{19}$$

Let us define $g(\xi) = a(x,\xi+\lambda)e^{2\pi i x\cdot(\xi+\lambda)} \in C^\infty(\mathbb{R}^d)$. Using the fact that $\int \varepsilon^{-d}\widehat{\psi}(\xi/\varepsilon) d\xi = \psi(0) = 1$ we obtain

$$\begin{aligned}
&\left| g(0) - \int_{\mathbb{R}^d} g(\xi)\psi(\varepsilon(\xi+\lambda))\varepsilon^{-d}\widehat{\psi}(\xi/\varepsilon)\, d\xi \right| \\
&\leqslant \int_{\mathbb{R}^d} |g(0)-g(\xi)|\varepsilon^{-d}|\widehat{\psi}(\xi/\varepsilon)|\, d\xi + \int_{\mathbb{R}^d} |1-\psi(\varepsilon(\xi+\lambda))||g(\xi)|\varepsilon^{-d}|\widehat{\psi}(\xi/\varepsilon)|\, d\xi \\
&= \int_{\mathbb{R}^d} |g(0)-g(\varepsilon\xi)||\widehat{\psi}(\xi)|\, d\xi + \int_{\mathbb{R}^d} |1-\psi(\varepsilon(\varepsilon\xi+\lambda))||g(\varepsilon\xi)||\widehat{\psi}(\xi)|\, d\xi.
\end{aligned}$$

The integrand of the first term tends to zero as $\varepsilon \to 0$ for each $\xi \in \mathbb{R}^d$. For $0 < \varepsilon < 1$ it is dominated by $C(1+\langle\xi\rangle^{|m|}\langle\lambda\rangle^{|m|})|\widehat{\psi}(\xi)|$ which is integrable, so by Lebesgue's dominated convergence theorem the first integral approaches zero as $\varepsilon \to 0$. Likewise, the second integral approaches zero as $\varepsilon \to 0$, since the integrand approaches zero as $\varepsilon \to 0$ for each $\xi \in \mathbb{R}^d$, and is dominated by $C|\widehat{\psi}(\xi)|\langle\xi\rangle^{|m|}\langle\lambda\rangle^{|m|}$ which is integrable. We conclude that

$$\lim_{\varepsilon\to+0} \int_{\mathbb{R}^d} a(x,\xi+\lambda)e^{2\pi i x\cdot(\xi+\lambda)}\psi(\varepsilon(\xi+\lambda))\varepsilon^{-d}\widehat{\psi}(\xi/\varepsilon) d\xi = a(x,\lambda)e^{2\pi i x\cdot\lambda}$$

which inserted into (19) proves (18). □

As Shubin has shown [9, 12], most of the basic results of pseudodifferential calculus with symbols in $S^m_{\rho,\delta}$, such as asymptotic expansions, the formula for composition of two operators and the formal adjoint of an operator, are true for $APS^m_{\rho,\delta}$, with the conclusion that all involved symbols satisfy $a(\cdot,\xi) \in CAP(\mathbb{R}^d)$ for all $\xi \in \mathbb{R}^d$. In particular we have [12, Theorem 3.1]: If $a \in APS^{m_1}_{\rho,\delta}$ and $b \in APS^{m_2}_{\rho,\delta}$ then $a\#_0 b \in APS^{m_1+m_2}_{\rho,\delta}$.

We will need three more results from Shubin's article [12].

THEOREM 1 (M.A. Shubin). *Let* $A \in APL^m_{\rho,\delta}$.

(i) *If* $u, v \in CAP^\infty(\mathbb{R}^d)$ *then*

$$(Au, v)_B = \lim_{R \to +\infty} |B_R|^{-1} (A(\varphi_R u), \varphi_R v)_{L^2}$$

where $\{\varphi_R\}_{R \geqslant 1} \subset C_c^\infty(\mathbb{R}^d)$ *is a family of functions that satisfy*

$$\varphi_R(x) = \begin{cases} 1 & \text{for } |x| \leqslant R, \\ 0 & \text{for } |x| \geqslant R + R^\kappa, \end{cases}$$
$$|\partial^\alpha \varphi_R(x)| \leqslant C_\alpha R^{-\kappa|\alpha|},$$

where $0 < \kappa < 1$. *Here* $B_R \subset \mathbb{R}^d$ *denotes the ball of radius* R *centered at the origin and* $|B_R|$ *its volume.*

(ii) *If* $u \in \mathscr{S}(\mathbb{R}^d)$ *and* $u_k = u * \psi_k \in CAP^\infty(\mathbb{R}^d)$, *where* $\{\psi_k\}_{k=1}^\infty \subset CAP(\mathbb{R}^d)$ *are chosen in a particular way (see [12, Lemma 4.3]), then*

$$(Au, u)_{L^2} = \lim_{k \to +\infty} (Au_k, u_k)_B.$$

(iii) $\|A\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = \|A\|_{\mathcal{L}(B^2(\mathbb{R}^d))}$.

The result (iii) is an immediate consequence of (i) and (ii).

From Lemma 4 we see that $\langle D \rangle^s$ is a unitary operator from $H^s(\mathbb{R}^d_B)$ to $H^0(\mathbb{R}^d_B) = B^2(\mathbb{R}^d)$, just as in the case of $H^s(\mathbb{R}^d)$. The well-known result that $a \in S^0_{\rho,\delta}$ implies $a(x,D) \in \mathcal{L}(L^2(\mathbb{R}^d))$ [6] has the following consequence.

COROLLARY 1. *If* $a \in APS^m_{\rho,\delta}$ *then for any* $s \in \mathbb{R}$

$$\|a(x,D)\|_{\mathcal{L}(H^s(\mathbb{R}^d), H^{s-m}(\mathbb{R}^d))} = \|a(x,D)\|_{\mathcal{L}(H^s(\mathbb{R}^d_B), H^{s-m}(\mathbb{R}^d_B))} < \infty.$$

Proof. We have

$$\begin{aligned}
\|a(x,D)\|_{\mathcal{L}(H^s(\mathbb{R}^d), H^{s-m}(\mathbb{R}^d))} &= \sup_{\|f\|_{H^s(\mathbb{R}^d)} \leqslant 1} \|a(x,D) f\|_{H^{s-m}(\mathbb{R}^d)} \\
&= \sup_{\|\langle D \rangle^s f\|_{L^2(\mathbb{R}^d)} \leqslant 1} \|\langle D \rangle^{s-m} a(x,D) \langle D \rangle^{-s} \langle D \rangle^s f\|_{L^2(\mathbb{R}^d)} \\
&= \sup_{\|f\|_{L^2(\mathbb{R}^d)} \leqslant 1} \|\langle D \rangle^{s-m} a(x,D) \langle D \rangle^{-s} f\|_{L^2(\mathbb{R}^d)} \\
&= \sup_{\|f\|_{B^2(\mathbb{R}^d)} \leqslant 1} \|\langle D \rangle^{s-m} a(x,D) \langle D \rangle^{-s} f\|_{B^2(\mathbb{R}^d)} \\
&= \sup_{\|f\|_{H^s(\mathbb{R}^d_B)} \leqslant 1} \|a(x,D) f\|_{H^{s-m}(\mathbb{R}^d_B)} \\
&= \|a(x,D)\|_{\mathcal{L}(H^s(\mathbb{R}^d_B), H^{s-m}(\mathbb{R}^d_B))}.
\end{aligned}$$

In fact, the fourth equality is Theorem 1 (iii). The finiteness of the operator norm follows from the observation that the symbol

$$\langle\xi\rangle^{s-m} \#_0 a \#_0 \langle\xi\rangle^{-s} \in S^0_{\rho,\delta},$$

due to (5), and the above mentioned $L^2(\mathbb{R}^d)$-continuity for operators with symbol in $S^0_{\rho,\delta}$. □

4. A transformation of symbols for a.p. pseudodifferential operators

DEFINITION 3. *Let $a \in APS^m_{\rho,\delta}$ and let $\Lambda = \Lambda(a)$ denote the frequencies whose Bohr–Fourier coefficients a_λ are not identically zero. We set*

$$U(a)(\xi)_{\lambda,\lambda'} = a_{\lambda'-\lambda}(\xi - \lambda'), \quad \lambda,\lambda' \in \Lambda, \quad \xi \in \mathbb{R}^d, \tag{20}$$

where $a_\lambda(\xi)$ is the Bohr–Fourier coefficient defined in (13)*.*

We note the property

$$U(a)(\xi)_{\lambda,\lambda'} = U(a)(\xi+\mu)_{\lambda+\mu,\lambda'+\mu}, \quad \mu \in \Lambda.$$

By Lemma 1 the inverse transformation of $a \mapsto U(a)_{\lambda,\lambda'}$ is

$$a(x,\xi) = \lim_{n\to\infty} \sum_{\lambda\in\Lambda} K_n(\lambda) U(a)(\xi)_{-\lambda,0}(\xi) e^{2\pi i\lambda\cdot x}$$

which converges uniformly in x for each ξ. For $a \in S^m_{\rho,\delta}$ the map $a \mapsto U(a)_{\lambda,\lambda'}$ is thus injective.

For fixed $\xi \in \mathbb{R}^d$ we may look upon $U(a)(\xi)$ as a matrix,

$$U(a)(\xi) = [U(a)(\xi)_{\lambda,\lambda'}]_{\lambda,\lambda'\in\Lambda},$$

indexed by $(\lambda,\lambda') \in \Lambda\times\Lambda$. This matrix defines an operator on complex-valued sequences defined on Λ, which are denoted $z = (z_\lambda)_{\lambda\in\Lambda}$, according to

$$(U(a)(\xi)\cdot z)_\lambda = \sum_{\lambda'\in\Lambda} U(a)(\xi)_{\lambda,\lambda'} z_{\lambda'}.$$

It follows from (15) that

$$\partial^\alpha_\xi (U(a))(\xi) = U(\partial^\alpha_\xi a)(\xi). \tag{21}$$

Moreover, denoting translation by $(T_{0,-\eta}a)(x,\xi) = a(x,\xi+\eta)$ we have

$$U(T_{0,-\eta}a)(\xi)_{\lambda,\lambda'} = (T_{0,-\eta}a)_{\lambda'-\lambda}(\xi-\lambda') = U(a)(\xi+\eta)_{\lambda,\lambda'}. \tag{22}$$

Since the operator-valued function $U(a)$ depends on the frequency variable only, it may be used to define a Fourier multiplier operator for vector-valued functions according to

$$U(a)(D)F(x) = \int_{\mathbb{R}^d} e^{2\pi i\xi\cdot x} U(a)(\xi)\cdot \widehat{F}(\xi)\, d\xi, \tag{23}$$

where $F(x) = (F_\lambda(x))_{\lambda\in\Lambda}$ is the vector-valued function

$$\mathbb{R}^d \ni x \mapsto (F_\lambda(x))_{\lambda\in\Lambda}.$$

The inner product for vector-valued functions is

$$\begin{aligned}(F,G)_{L^2(\mathbb{R}^d,l^2)} = (F,G)_{L^2(\mathbb{R}^d,l^2(\Lambda))} &= \int_{\mathbb{R}^d} (F(x),G(x))_{l^2}\,dx \\ &= \int_{\mathbb{R}^d} \sum_{\lambda\in\Lambda} F_\lambda(x)\overline{G_\lambda(x)}\,dx, \quad F,G \in L^2(\mathbb{R}^d,l^2).\end{aligned}$$

If the symbol a does not depend on x, i.e. $a(x,D)$ is a Fourier multiplier (convolution) operator, then $a_\lambda(\xi) = 0$ when $\lambda \neq 0$ follows from (13). Thus $U(a)(\xi)$ is the pointwise multiplier operator

$$(U(a)(\xi)\cdot z)_\lambda = \sum_{\lambda'\in\Lambda} a_{\lambda'-\lambda}(\xi-\lambda')z_{\lambda'} = a_0(\xi-\lambda)z_\lambda = a(\xi-\lambda)z_\lambda,$$

and

$$(U(a)(D)F(x))_\lambda = \int_{\mathbb{R}^d} e^{2\pi i\xi\cdot x} a(\xi-\lambda)\widehat{F_\lambda}(\xi)\,d\xi = (T_\lambda a)(D)F_\lambda(x).$$

Thus $U(a)(D)$ acts pointwise in the λ variable by a convolution in x. If a does not depend on ξ, then $U(a)$ does not depend on ξ either, and $U(a)_{\lambda,\lambda'} = a_{\lambda'-\lambda}$. Thus, in this case we have

$$(U(a)(D)F(x))_\lambda = (U(a)\cdot F(x))_\lambda = \sum_{\lambda'\in\Lambda} a_{\lambda'-\lambda}F_{\lambda'}(x),$$

which is an operator that acts pointwise in x, by a convolution over the index set Λ. In particular we have $U(1)(\xi)_{\lambda,\lambda'} = \delta_{\lambda'-\lambda}$ which denotes the Kronecker delta. This means that $U(1)(D) = I$.

The above discussion is not precise since we have not yet proved in what sense $U(a)(\xi)$ is a continuous operator for fixed $\xi \in \mathbb{R}^d$, and whether the operator-valued function $\xi \mapsto U(a)(\xi)$ is continuous and bounded. Let us therefore address these questions.

We shall first evaluate the operator-valued function $U(a)(\xi)$ in the origin. It will turn out that $U(a)(0)$ contains much information about continuity, positivity and invertibility of $a(x,D)$. We need the sequence spaces

$$l_s^p = l_s^p(\Lambda) = \left\{ (x_\lambda)_{\lambda\in\Lambda} : \|x\|_{l_s^p} = \left(\sum_{\lambda\in\Lambda} \langle\lambda\rangle^{ps}|x_\lambda|^p \right)^{1/p} < \infty \right\}, \tag{24}$$

parametrized by $s \in \mathbb{R}$ and normed by $\|\cdot\|_{l_s^p}$ where $1 \leqslant p \leqslant \infty$. In some places we will use the symbol l_c^2 which denotes the space of square-summable sequences with compact support.

PROPOSITION 1. *For $a \in APS^m_{\rho,\delta}$ we have for any $s \in \mathbb{R}$*

$$\|U(a)(0)\|_{\mathscr{L}(l^2_s, l^2_{s-m})} \leqslant \|a(x,D)\|_{\mathscr{L}(H^s(\mathbb{R}^d_B), H^{s-m}(\mathbb{R}^d_B))} < \infty. \tag{25}$$

Proof. Let $f, g \in TP(\Lambda)$. Lemma 4 gives

$$\begin{aligned}(a(x,D)f,g)_B &= \sum_{\lambda,\lambda'} M_x(a(x,\lambda)e^{2\pi i x\cdot(\lambda-\lambda')}) f_\lambda \overline{g}_{\lambda'} \\ &= \sum_{\lambda,\lambda'} a_{\lambda'-\lambda}(\lambda) f_\lambda \overline{g}_{\lambda'} \\ &= (U(a)(0)\cdot \check{f}, \check{g})_{l^2}\end{aligned} \tag{26}$$

where $\check{f}_\lambda = f_{-\lambda}$. We abbreviate $H^s = H^s(\mathbb{R}^d_B)$. Using the duality $(H^s)' = H^{-s}$ under the form $(\cdot,\cdot)_B$, we obtain

$$\begin{aligned}\|a(x,D)\|_{\mathscr{L}(H^s,H^{s-m})} &= \sup_{\|f\|_{H^s}\leqslant 1} \|a(x,D)f\|_{H^{s-m}} \\ &= \sup_{\|f\|_{H^s}\leqslant 1, \|g\|_{H^{m-s}}\leqslant 1} |(a(x,D)f,g)_B| \\ &\geqslant \sup_{\|f_\cdot\|_{l^2_s}\leqslant 1, \|g_\cdot\|_{l^2_{m-s}}\leqslant 1} |(U(a)(0)\cdot\check{f},\check{g})_{l^2}| \\ &= \|U(a)(0)\|_{\mathscr{L}(l^2_s, l^2_{s-m})},\end{aligned}$$

where we denote $\|f_\cdot\|^2_{l^2_s} = \sum_\lambda \langle\lambda\rangle^{2s}|f_\lambda|^2$. □

As a consequence of (26) and Theorem 1 (i) and (ii) we have the following result on positivity. As customary we say that A is a positive operator on a topological vector space X if $(Af,f)_H \geqslant 0$ for all $f \in X$, where $X \subset H$ and H is a Hilbert space, naturally associated with X. (We avoid the requirement $(Af,f)_H \geqslant 0$ for all $f \in H$ since the expression $(Af,f)_H$ may not be well-defined if A is not a bounded operator on H.) This is denoted $A \geqslant 0$ (where the spaces X and H are understood from the context). We will use the following pairs (X,H): $(\mathscr{S}(\mathbb{R}^d), L^2(\mathbb{R}^d))$, $(TP(\mathbb{R}^d), B^2(\mathbb{R}^d))$, (l^2_c, l^2) and $(\mathscr{S}(\mathbb{R}^d, l^2_c), L^2(\mathbb{R}^d, l^2))$.

COROLLARY 2. *If $a \in APS^m_{\rho,\delta}$ then $a(x,D) \geqslant 0$ on $\mathscr{S}(\mathbb{R}^d)$ if and only if $a(x,D) \geqslant 0$ on $TP(\mathbb{R}^d)$. Moreover, $a(x,D) \geqslant 0$ on $TP(\Lambda)$ if and only if $U(a)(0) \geqslant 0$ on l^2_c.*

The next result gives a continuity statement of the operator-valued map $\xi \mapsto U(a)(\xi)$.

PROPOSITION 2. *If $a \in APS^m_{\rho,\delta}$ then we have*

$$\|U(a)(\xi)\|_{\mathscr{L}(l^1_{|m|}, l^\infty)} \leqslant C\langle\xi\rangle^m, \tag{27}$$

$$U(a) \in C(\mathbb{R}^d, \mathscr{L}(l^1_{|m|}, l^\infty)). \tag{28}$$

Proof. Using the inequality $\langle x+y\rangle^u \leqslant C\langle x\rangle^u\langle y\rangle^{|u|}$, Definition 3 and (17) we obtain

$$|U(a)(\xi)_{\lambda,\lambda'}| \leqslant C\langle \xi-\lambda'\rangle^m \leqslant C\langle\xi\rangle^m\langle\lambda'\rangle^{|m|}.$$

Hence

$$\|U(a)(\xi)\cdot x\|_{l^\infty} \leqslant C\langle\xi\rangle^m\|x\|_{l^1_{|m|}}$$

which proves (27). To prove (28), we note that

$$(U(a)(\xi)-U(a)(\xi+\eta))_{\lambda,\lambda'} = U(a-T_{0,-\eta}a)(\xi)_{\lambda,\lambda'} \tag{29}$$

follows from (22). Thus, by the mean value theorem (14), and again Definition 3 and (17),

$$\begin{aligned}
&\left|(U(a)(\xi)-U(a)(\xi+\eta))_{\lambda,\lambda'}\right| \\
&\leqslant |\eta|\left|(\nabla_2 \operatorname{Re} a)_{\lambda'-\lambda}(\xi-\lambda'+\theta_1\eta)+i(\nabla_2 \operatorname{Im} a)_{\lambda'-\lambda}(\xi-\lambda'+\theta_2\eta)\right| \\
&\leqslant C|\eta|\left(\langle\xi-\lambda'+\theta_1\eta\rangle^{m-\rho}+\langle\xi-\lambda'+\theta_2\eta\rangle^{m-\rho}\right) \\
&\leqslant C|\eta|\langle\lambda'\rangle^{m-\rho}\left(\langle\xi+\theta_1\eta\rangle^{|m-\rho|}+\langle\xi+\theta_2\eta\rangle^{|m-\rho|}\right) \\
&\leqslant C|\eta|\langle\lambda'\rangle^{|m|}\langle\eta\rangle^{|m-\rho|}\langle\xi\rangle^{|m-\rho|},
\end{aligned}$$

and therefore

$$\begin{aligned}
&\|U(a)(\xi)-U(a)(\xi+\eta)\|_{\mathcal{L}(l^1_{|m|},l^\infty)} \\
&= \sup_{\|x\|_{l^1_{|m|}}\leqslant 1}\ \sup_{\lambda\in\Lambda}|((U(a)(\xi)-U(a)(\xi+\eta))\cdot x)_\lambda| \\
&\leqslant C|\eta|\langle\eta\rangle^{|m-\rho|}\langle\xi\rangle^{|m-\rho|} \\
&\to 0, \quad |\eta|\to 0.
\end{aligned}$$

This proves (28). □

The next result gives a sharpening of condition (28), since we have $l^1_{|m|}\subset l^2_{|m|}$ and $l^2_{|m|-m}\subset l^\infty$.

PROPOSITION 3. *If $a\in APS^m_{\rho,\delta}$ then we have for any $s\in\mathbb{R}$*

$$U(a)(\xi)\in\mathcal{L}(l^2_s,l^2_{s-m}),\quad \xi\in\mathbb{R}^d, \tag{30}$$

$$U(a)\in C(\mathbb{R}^d,\mathcal{L}(l^2_s,l^2_{s-m})). \tag{31}$$

Proof. From (22) we see that $U(a)(\xi)=U(T_{0,-\xi}a)(0)$. Since $T_{0,-\xi}a\in APS^m_{\rho,\delta}$ for any $\xi\in\mathbb{R}^d$, (30) follows from Proposition 1.

In order to prove (31), it suffices to prove continuity in the origin, since

$$U(a)(\xi+\eta)-U(a)(\eta)=U(T_{0,-\eta}a)(\xi)-U(T_{0,-\eta}a)(0).$$

We use (29) and again Proposition 1 and Corollary 1, which give

$$\begin{aligned}\|U(a)(\xi)-U(a)(0)\|_{\mathscr{L}(l_s^2,l_{s-m}^2)} &= \|U(T_{0,-\xi}a-a)(0)\|_{\mathscr{L}(l_s^2,l_{s-m}^2)}\\ &\leqslant \|(T_{0,-\xi}a-a)(x,D)\|_{\mathscr{L}(H^s(\mathbb{R}^d),H^{s-m}(\mathbb{R}^d))}.\end{aligned}$$

In the next step we use

$$\|b(x,D)\|_{\mathscr{L}(H^s(\mathbb{R}^d),H^{s-m}(\mathbb{R}^d))} = \|\langle D\rangle^{s-m}b(x,D)\langle D\rangle^{-s}\|_{\mathscr{L}(L^2)}$$

for $b\in S^m_{\rho,\delta}$, and the fact that the $\mathscr{L}(L^2)$-norm of an operator with symbol in $S^0_{\rho,\delta}$ may be estimated by a finite sum of seminorms of the symbol in $S^0_{\rho,\delta}$ (see [6, Theorem 18.1.11] and [3, Theorem 2.80]). By (5) it thus suffices to prove that

$$T_{0,-\xi}a-a\to 0 \quad \text{in} \quad S^m_{\rho,\delta} \quad \text{as} \quad \xi\to 0. \tag{32}$$

The mean value theorem (14) gives

$$a(x,\eta+\xi)-a(x,\eta) = (\nabla_2\,\mathrm{Re}\,a(x,\eta+\theta_1\xi)+i\nabla_2\,\mathrm{Im}\,a(x,\eta+\theta_2\xi))\cdot\xi$$

with $0\leqslant\theta_1,\theta_2\leqslant 1$, so we have

$$\begin{aligned}&\left|\partial_\eta^\alpha\partial_x^\beta(T_{0,-\xi}a-a)(x,\eta)\right|\\ &\leqslant|\xi|\left|\partial_\eta^\alpha\partial_x^\beta\nabla_2\,\mathrm{Re}\,a(x,\eta+\theta_1\xi)+i\partial_\eta^\alpha\partial_x^\beta\nabla_2\,\mathrm{Im}\,a(x,\eta+\theta_2\xi)\right|\\ &\leqslant C|\xi|\left(\langle\eta+\theta_1\xi\rangle^{m-\rho(|\alpha|+1)+\delta|\beta|}+\langle\eta+\theta_2\xi\rangle^{m-\rho(|\alpha|+1)+\delta|\beta|}\right)\\ &\leqslant C|\xi|\langle\xi\rangle^{|m-\rho(|\alpha|+1)+\delta|\beta||}\langle\eta\rangle^{m-\rho(|\alpha|+1)+\delta|\beta|}.\end{aligned}$$

This proves (32), and therefore (31). □

The following result concerns positivity.

PROPOSITION 4. *If $a\in APS^m_{\rho,\delta}$ then we have: $a(x,D)\geqslant 0$ on $\mathscr{S}(\mathbb{R}^d)$ implies $U(a)\,(D)\geqslant 0$ on $\mathscr{S}(\mathbb{R}^d,l_c^2)$. Moreover, $U(a)(D)\geqslant 0$ on $\mathscr{S}(\mathbb{R}^d,l_c^2)$ implies $a(x,D)\geqslant 0$ on $TP(\Lambda)$.*

Proof. Suppose $a(x,D)\geqslant 0$ on $\mathscr{S}(\mathbb{R}^d)$. For $f\in\mathscr{S}(\mathbb{R}^d)$ and $M_\eta f(x)=e^{2\pi i\eta\cdot x}f(x)$ we have, for any $\eta\in\mathbb{R}^d$,

$$\begin{aligned}0&\leqslant(a(x,D)M_\eta f,M_\eta f)_{L^2(\mathbb{R}^d)}\\ &=\iint_{\mathbb{R}^{2d}}e^{2\pi ix\cdot(\xi-\eta)}a(x,\xi)\widehat{f}(\xi-\eta)\overline{f(x)}\,dx\,d\xi\\ &=\iint_{\mathbb{R}^{2d}}e^{2\pi ix\cdot\xi}a(x,\xi+\eta)\widehat{f}(\xi)\overline{f(x)}\,dx\,d\xi\\ &=((T_{0,-\eta}a)(x,D)f,f)_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Thus $(T_{0,-\eta}a)(x,D) \geqslant 0$ on $\mathscr{S}(\mathbb{R}^d)$ for all $\eta \in \mathbb{R}^d$. By Corollary 2 and (22) it follows that $U(a)(\xi) \geqslant 0$ on l_c^2 for all $\xi \in \mathbb{R}^d$. If $F \in \mathscr{S}(\mathbb{R}^d, l_c^2)$ we obtain

$$(U(a)(D)F,F)_{L^2(\mathbb{R}^d,l^2)} = \int_{\mathbb{R}^d} (U(a)(\xi)\cdot \widehat{F}(\xi), \widehat{F}(\xi))_{l^2}\, d\xi \geqslant 0,$$

since the integrand is nonnegative everywhere. Thus $U(a)(D) \geqslant 0$ on $\mathscr{S}(\mathbb{R}^d, l_c^2)$.

Suppose on the other hand that $U(a)(D) \geqslant 0$ on $\mathscr{S}(\mathbb{R}^d, l_c^2)$. Let $z \in l_c^2$ and pick $\varphi \in C_c^\infty(\mathbb{R}^d)$ with support in the unit ball such that $\varphi \geqslant 0$ and $\|\varphi\|_{L^2} = 1$. With $\varphi_\varepsilon(x) = \varepsilon^{-d/2}\varphi(x/\varepsilon)$ and $F_\varepsilon(x)_\lambda = \mathscr{F}^{-1}\varphi_\varepsilon(x) z_\lambda$ we then have

$$\begin{aligned} 0 \leqslant (U(a)(D)F_\varepsilon, F_\varepsilon)_{L^2(\mathbb{R}^d,l^2)} &= \int_{\mathbb{R}^d} (U(a)(\xi)\cdot z, z)_{l^2} \varphi_\varepsilon(\xi)^2\, d\xi \\ &\to (U(a)(0)\cdot z, z)_{l^2}, \quad \varepsilon \to 0, \end{aligned}$$

where we have used (31) and the shrinking support of φ_ε. Therefore $U(a)(0) \geqslant 0$ on l_c^2 which implies that $a(x,D) \geqslant 0$ on $TP(\Lambda)$ according to Corollary 2. □

The previous result is similar to Gladyshev's results [4, 5], which were formulated in the framework of almost periodically correlated (or cyclostationary) stochastic processes and vector-valued weakly stationary stochastic processes. The so-called covariance operator of a second-order stochastic process is a positive operator, and an almost periodically correlated stochastic process has a covariance operator whose symbol is almost periodic in the first variable. Weakly stationary stochastic processes have translation invariant covariance operators, that is, they are convolution (or Fourier multiplier) operators. Gladyshev showed that the transformation (20), $a \mapsto U(a)$, which he formulated in terms of operator kernels, transforms a uniformly continuous kernel corresponding to a positive a.p. pseudodifferential operator to the kernel of a positive translation-invariant operator acting on vector-valued function spaces. The kernel of the operator (2) is

$$k_a(x,y) = \int_{\mathbb{R}^d} e^{2\pi i \xi\cdot(x-y)} a(x,\xi)\, d\xi = (\mathscr{F}_2^{-1}a)(x, x-y),$$

understood as an oscillatory integral. Here $\mathscr{F}_2$ denotes partial Fourier transform in the second $\mathbb{R}^d$ variable. The study of almost periodically correlated stochastic processes is in many respects rather similar to the theory of positive a.p. pseudodifferential operators. The symbol classes $S^m_{\rho,\delta}$ are however rarely used for stochastic processes. One usually restricts to operators whose kernels are continuous functions.

The next result concerns composition.

THEOREM 2. *If $a \in APS^{m_1}_{\rho,\delta}$ and $b \in APS^{m_2}_{\rho,\delta}$, $m_1, m_2 \in \mathbb{R}$, then*

$$U(a\#_0 b)(\xi) = U(a)(\xi)\cdot U(b)(\xi), \quad \xi \in \mathbb{R}^d. \tag{33}$$

Proof. Let Λ denote the linear hull over $\mathbb{Q}$ of $\Lambda(a)\cup\Lambda(b)$. According to (30) in Proposition 3, $U(a)(\xi) \in \mathscr{L}(l_s^2, l_{s-m_1}^2)$ and $U(b)(\xi) \in \mathscr{L}(l_s^2, l_{s-m_2}^2)$ for any $s \in \mathbb{R}$. Therefore the sum

$$(U(a)(\xi)\cdot U(b)(\xi))_{\lambda,\lambda'} = \sum_{\mu\in\Lambda} U(a)(\xi)_{\lambda,\mu} U(b)(\xi)_{\mu,\lambda'} = \sum_{\mu\in\Lambda} a_{\mu-\lambda}(\xi-\mu) b_{\lambda'-\mu}(\xi-\lambda') \tag{34}$$

is absolutely convergent for all $(\lambda,\lambda')\in\Lambda\times\Lambda$, and the matrix $U(a)(\xi)\cdot U(b)(\xi)$ maps l_s^2 to $l_{s-m_1-m_2}^2$ continuously for any $s\in\mathbb{R}$ and any $\xi\in\mathbb{R}^d$.

We study the left hand side of (33) by regularizing the symbol b in two steps. First we pick a test function $\varphi\in C_c^\infty(\mathbb{R}^d)$ which equals one in a neighborhood of the origin, set $\varphi_\varepsilon(\xi)=\varphi(\varepsilon\xi)$ and define

$$b_\varepsilon(x,\xi) = b(x,\xi)\varphi_\varepsilon(\xi)\in S_{\rho,\delta}^{-\infty}, \quad 0\leqslant\varepsilon\leqslant 1.$$

By [6, Proposition 18.1.2] $\varphi_\varepsilon\to 1$ in $S_{1,0}^\theta$ as $\varepsilon\to 0$ for any $\theta>0$. Since convergence in $S_{1,0}^\theta$ implies convergence in $S_{\rho,\delta}^\theta$ and $b_\varepsilon = b\#_0\varphi_\varepsilon$, it follows from (5) that $b_\varepsilon\to b$ in $S_{\rho,\delta}^{m_2+\theta}$ as $\varepsilon\to 0$, and

$$a\#_0 b = \lim_{\varepsilon\to 0} a\#_0 b_\varepsilon \quad \text{in} \quad S_{\rho,\delta}^{m_1+m_2+\theta}, \quad \theta>0.$$

Convergence in $S_{\rho,\delta}^m$ for any $m\in\mathbb{R}$ implies the uniform convergence

$$\sup_{x\in\mathbb{R}^d} |a\#_0 b(x,\xi) - a\#_0 b_\varepsilon(x,\xi)| \to 0, \quad \varepsilon\to 0,$$

for any $\xi\in\mathbb{R}^d$, and therefore we have for the Bohr–Fourier coefficients

$$(a\#_0 b)_\mu(\xi) = \lim_{\varepsilon\to 0}(a\#_0 b_\varepsilon)_\mu(\xi), \quad \mu\in\mathbb{R}^d, \quad \xi\in\mathbb{R}^d. \tag{35}$$

In the second step we regularize the symbol b_ε. Fix $\alpha,\beta\in\mathbb{N}^d$ and define the family of functions $\mathcal{F} = \{\partial_\xi^\alpha\partial_x^\beta b_\varepsilon(\cdot,\xi)\}_{\xi\in\mathbb{R}^d}\subset CAP(\mathbb{R}^d)$. The family $\mathcal{F}$ depends continuously in the $CAP(\mathbb{R}^d)$ norm on ξ by (14), and has compact support with respect to ξ. Thus $\mathcal{F}$ is precompact, and by Lemma 1 the Fourier series reconstruction with the Bochner–Fejér polynomials

$$\partial_\xi^\alpha\partial_x^\beta b_\varepsilon(x,\xi) = \lim_{n\to\infty} P_n(\partial_\xi^\alpha\partial_x^\beta b_\varepsilon(\cdot,\xi))(x) = \lim_{n\to\infty}\sum_{\lambda\in\Lambda} K_n(\lambda)(\partial_\xi^\alpha\partial_x^\beta b_\varepsilon)_\lambda(\xi)e^{2\pi i\lambda\cdot x} \tag{36}$$

is uniformly convergent in both variables, i.e. in $\mathbb{R}^{2d}$. By Lemma 3 we have

$$(\partial_\xi^\alpha\partial_x^\beta b_\varepsilon)_\lambda(\xi) = \partial_\xi^\alpha(\partial_x^\beta b_\varepsilon)_\lambda(\xi) = (2\pi i\lambda)^\beta\partial_\xi^\alpha(b_\varepsilon)_\lambda(\xi),$$

which means that we can rewrite (36) as the uniform limit over $\mathbb{R}^{2d}$

$$\partial_\xi^\alpha\partial_x^\beta b_\varepsilon(x,\xi) = \lim_{n\to\infty}\partial_\xi^\alpha\partial_x^\beta\left(\sum_{\lambda\in\Lambda} K_n(\lambda)(b_\varepsilon)_\lambda(\xi)e^{2\pi i\lambda\cdot x}\right). \tag{37}$$

Let us denote, observing that $(b_\varepsilon)_\lambda(\xi) = b_\lambda(\xi)\varphi(\varepsilon\xi)$,

$$b_{\varepsilon,n}(x,\xi) = \varphi(\varepsilon\xi)\sum_{\lambda\in\Lambda} K_n(\lambda) b_\lambda(\xi) e^{2\pi i\lambda\cdot x}.$$

The fact that $b_\varepsilon(x,\cdot)$ and $b_{\varepsilon,n}(x,\cdot)$ have support in a compact set, common for all $x\in\mathbb{R}^d$, in combination with the uniform limit (37), implies that

$$\sup_{x,\xi\in\mathbb{R}^d} \langle\xi\rangle^{-m+\rho|\alpha|-\delta|\beta|} \left|\partial_\xi^\alpha\partial_x^\beta\left(b_{\varepsilon,n}(x,\xi) - b_\varepsilon(x,\xi)\right)\right| \to 0, \quad n\to\infty,$$

for any $m\in\mathbb{R}$. This holds for any $\alpha,\beta\in\mathbb{N}^d$, and hence $b_{\varepsilon,n}\to b_\varepsilon$ in $S^m_{\rho,\delta}$ as $n\to\infty$ for any $m\in\mathbb{R}$. This means by (5) that $a\#_0 b_{\varepsilon,n}\to a\#_0 b_\varepsilon$ in $S^m_{\rho,\delta}$ as $n\to\infty$ for any $m\in\mathbb{R}$. As above we thus obtain

$$(a\#_0 b)_\mu(\xi) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}(a\#_0 b_{\varepsilon,n})_\mu(\xi), \quad \mu\in\mathbb{R}^d, \quad \xi\in\mathbb{R}^d, \tag{38}$$

using (35).

Since the symbol $c_\lambda(x,\xi) = e^{2\pi i\lambda\cdot x} b_\lambda(\xi)\varphi(\varepsilon\xi)$ gives the pseudodifferential operator

$$c_\lambda(x,D)g(x) = \int_{\mathbb{R}^d} e^{2\pi i\xi\cdot x} b_\lambda(\xi-\lambda)\varphi(\varepsilon(\xi-\lambda))\widehat{g}(\xi-\lambda)\,d\xi, \quad g\in\mathscr{S}(\mathbb{R}^d), \tag{39}$$

it follows that

$$\begin{aligned} a(x,D)(c_\lambda(x,D)g)(x) &= \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} a(x,\xi)\mathscr{F}(c_\lambda(x,D)g)(\xi)\,d\xi \\ &= \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} a(x,\xi) b_\lambda(\xi-\lambda)\varphi(\varepsilon(\xi-\lambda))\widehat{g}(\xi-\lambda)\,d\xi \\ &= \int_{\mathbb{R}^d} e^{2\pi i x\cdot(\xi+\lambda)} a(x,\xi+\lambda) b_\lambda(\xi)\varphi(\varepsilon\xi)\widehat{g}(\xi)\,d\xi, \end{aligned}$$

and thus

$$a\#_0 c_\lambda(x,\xi) = a(x,\xi+\lambda) b_\lambda(\xi)\varphi(\varepsilon\xi) e^{2\pi i\lambda\cdot x}.$$

This gives

$$(a\#_0 b_{\varepsilon,n})(x,\xi) = \sum_{\lambda\in\Lambda} K_n(\lambda) a(x,\xi+\lambda) b_\lambda(\xi)\varphi(\varepsilon\xi) e^{2\pi i\lambda\cdot x}.$$

Hence

$$\begin{aligned} \lim_{n\to\infty}(a\#_0 b_{\varepsilon,n})_\mu(\xi) &= \varphi(\varepsilon\xi)\lim_{n\to\infty}\sum_{\lambda\in\Lambda} K_n(\lambda) a_{\mu-\lambda}(\xi+\lambda) b_\lambda(\xi) \\ &= \varphi(\varepsilon\xi)\sum_{\lambda\in\Lambda} a_{\mu-\lambda}(\xi+\lambda) b_\lambda(\xi), \end{aligned} \tag{40}$$

due to $0\leqslant K_n\leqslant 1$, $K_n(\lambda)\to 1$ as $n\to\infty$ for all $\lambda\in\Lambda$, the absolutely convergent sum (34), and the dominated convergence theorem. Now (38) and (40) yield

$$(a\#_0 b)_\mu(\xi) = \sum_{\lambda\in\Lambda} a_{\mu-\lambda}(\xi+\lambda) b_\lambda(\xi), \quad \mu\in\Lambda, \quad \xi\in\mathbb{R}^d.$$

Finally we have

$$\begin{aligned} U(a\#_0 b)(\xi)_{\lambda,\lambda'} &= (a\#_0 b)_{\lambda'-\lambda}(\xi-\lambda') \\ &= \sum_{\mu\in\Lambda} a_{\lambda'-\lambda-\mu}(\xi-\lambda'+\mu)b_\mu(\xi-\lambda') \\ &= \sum_{\mu\in\Lambda} a_{\mu-\lambda}(\xi-\mu)b_{\lambda'-\mu}(\xi-\lambda'). \end{aligned}$$

A comparison with (34) completes the proof. □

To summarize our findings hitherto, the transformation $a \mapsto U(a)$ maps a symbol $a \in S^m_{\rho,\delta}$ defined on the phase space $\mathbb{R}^d \times \mathbb{R}^d$ to an operator-valued symbol $U(a)$ that depends on the frequency variable $\xi \in \mathbb{R}^d$ only. The operator corresponding to the symbol $U(a)$ acts on sequence-space valued function spaces, e.g. $\mathscr{S}(\mathbb{R}^d, l^2_c)$. The operator corresponding to the symbol $U(a)$ is thus a convolution (Fourier multiplier) operator. The map $a(x,D) \mapsto U(a)(D)$ is linear, injective, preserves identity and positivity, and respects operator composition,

$$a(x,D)b(x,D) \mapsto U(a\#_0 b)(D) = U(a)(D)\cdot U(b)(D).$$

Convolution operators do not commute when function spaces are vector-valued as they do for scalar-valued function spaces. The transformation $a \mapsto U(a)$ encodes the non-commutativity of $a(x,D)$ and $b(x,D)$ in the matrix product of the symbols $U(a)$ and $U(b)$. That is, with the notation for the commutator $[A,B] = AB - BA$, we have

$$[a(x,D), b(x,D)] \mapsto U(a)(D)\cdot U(b)(D) - U(b)(D)\cdot U(a)(D),$$

where the right hand side operator acts by

$$\begin{aligned} &[U(a)(D), U(b)(D)]F(x) \\ &= \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}\left(U(a)(\xi)\cdot U(b)(\xi) - U(b)(\xi)\cdot U(a)(\xi)\right)\cdot \widehat{F}(\xi)\,d\xi. \end{aligned}$$

In our final result we show that the basic assumption of this paper, i.e. that symbols are almost periodic in the first variable, is invariant under the quantization. More precisely, let us introduce the family of quantizations

$$a_t(x,D)f(x) = \int_{\mathbb{R}^{2d}} e^{2\pi i\xi\cdot(x-y)} a((1-t)x+ty,\xi)\,f(y)\,dy\,d\xi \tag{41}$$

parametrized by $t \in \mathbb{R}$. The Kohn–Nirenberg quantization is obtained for $t = 0$ and the Weyl quantization has $t = 1/2$. The following result says that if an operator is expressed in two different quantizations, then if its symbol is almost periodic in the first variable in one quantization, it will have the same property in any other quantization. In other words, the fact that we have worked in the Kohn–Nirenberg quantization is not essential.

PROPOSITION 5. *If* $a \in APS^m_{\rho,\delta}$*,* $s,t \in \mathbb{R}$*,* $s \neq t$*, and* $a_t(x,D) = b_s(x,D)$*, then* $b \in APS^m_{\rho,\delta}$.

Proof. We use a technique that is similar to the proof of Theorem 2. If $a,b \in \mathscr{S}(\mathbb{R}^{2d})$ and $f \in \mathscr{S}(\mathbb{R}^d)$ then the integral over ξ in (41) is a partial Fourier transform, so we get

$$\begin{aligned} a_t(x,D)f(x) &= \int_{\mathbb{R}^d} \mathscr{F}_2 a((1-t)x+ty, y-x) f(y)\,dy \\ &= \int_{\mathbb{R}^d} \mathscr{F}_2 a(x+ty,y) f(y+x)\,dy \\ &= \iint_{\mathbb{R}^{2d}} \widehat{a}(z,y) e^{2\pi i t z\cdot y} e^{2\pi i z\cdot x} f(y+x)\,dy\,dz. \end{aligned}$$

Thus if $a_t(x,D) = b_s(x,D)$ we have

$$\widehat{b}(x,\xi) = e^{-2\pi i(s-t)x\cdot\xi}\widehat{a}(x,\xi),$$

which extends by continuity to $a,b \in \mathscr{S}'(\mathbb{R}^{2d})$ [3]. This transformation is often denoted [6]

$$b(x,\xi) = e^{-2\pi i(s-t)D_x\cdot D_\xi} a(x,\xi) := (Ta)(x,\xi). \tag{42}$$

According to [3, Theorem 2.37], we have

$$e^{-2\pi i(s-t)D_x\cdot D_\xi} : S^m_{\rho,\delta} \mapsto S^m_{\rho,\delta} \quad \text{continuously}, \quad m \in \mathbb{R}. \tag{43}$$

Therefore it suffices to prove that $(Ta)(\cdot,\xi) \in CAP(\mathbb{R}^d)$ for all $\xi \in \mathbb{R}^d$.

We proceed with a regularization of the symbol a as in the proof of Theorem 2. Thus let $\varphi \in C_c^\infty(\mathbb{R}^d)$ equal one in a neighborhood of the origin, set $\varphi_\varepsilon(\xi) = \varphi(\varepsilon\xi)$ and define $a_\varepsilon(x,\xi) = a(x,\xi)\varphi_\varepsilon(\xi)$. Then $a_\varepsilon \to a$ in $S^{m+\theta}_{\rho,\delta}$ as $\varepsilon \to 0$ for any $\theta > 0$. By the continuity (43) we have $Ta_\varepsilon \to Ta$ in $S^{m+\theta}_{\rho,\delta}$ as $\varepsilon \to 0$. Moreover, if we define

$$a_{\varepsilon,n}(x,\xi) = \varphi(\varepsilon\xi) \sum_{\lambda\in\Lambda} K_n(\lambda) a_\lambda(\xi) e^{2\pi i\lambda\cdot x}$$

then we obtain $a_{\varepsilon,n} \to a_\varepsilon$ in $S^{m'}_{\rho,\delta}$ as $n \to \infty$ for any $m' \in \mathbb{R}$, as in the proof of Theorem 2. Again by the continuity (43) it follows that $Ta_{\varepsilon,n} \to Ta_\varepsilon$ in $S^{m'}_{\rho,\delta}$ as $n \to \infty$. It follows that for each fixed $\xi \in \mathbb{R}^d$ we have the uniform limits

$$(Ta)(\cdot,\xi) = \lim_{\varepsilon\to 0}\lim_{n\to\infty} (Ta_{\varepsilon,n})(\cdot,\xi).$$

Since $CAP(\mathbb{R}^d)$ is closed under uniform convergence [7], the proof is complete if we show that $(Ta_{\varepsilon,n})(\cdot,\xi) \in CAP(\mathbb{R}^d)$ for any $\xi \in \mathbb{R}^d$, $\varepsilon > 0$ and $n \in \mathbb{N}$.

We have, since $(a_\varepsilon)_\lambda(\xi) = a_\lambda(\xi)\varphi(\varepsilon\xi)$,

$$\mathscr{F}(a_{\varepsilon,n})(\eta,z) = \sum_{\lambda\in\Lambda} K_n(\lambda)\delta_\lambda(\eta)\mathscr{F}(a_\varepsilon)_\lambda(z),$$

where $\delta_\lambda = \delta_0(\cdot - \lambda)$ denotes a translated Dirac distribution. Hence we have

$$e^{-2\pi i(s-t)\eta\cdot z}\mathscr{F}(a_{\varepsilon,n})(\eta,z) = \sum_{\lambda\in\Lambda} K_n(\lambda)e^{-2\pi i(s-t)\lambda\cdot z}\delta_\lambda(\eta)\mathscr{F}(a_\varepsilon)_\lambda(z)$$
$$= \sum_{\lambda\in\Lambda} K_n(\lambda)\delta_\lambda(\eta)\mathscr{F}(T_{(s-t)\lambda}(a_\varepsilon)_\lambda)(z)$$

and, since $Ta = \mathscr{F}^{-1}M\mathscr{F}$ where $(Mf)(\eta,z) = e^{-2\pi i(s-t)\eta\cdot z}f(\eta,z)$,

$$(Ta_{\varepsilon,n})(x,\xi) = \sum_{\lambda\in\Lambda} K_n(\lambda)(T_{(s-t)\lambda}(a_\varepsilon)_\lambda)(\xi)e^{2\pi i\lambda\cdot x}$$
$$= \sum_{\lambda\in\Lambda} K_n(\lambda)(a_\varepsilon)_\lambda(\xi-(s-t)\lambda)e^{2\pi i\lambda\cdot x}.$$

Hence $(Ta_{\varepsilon,n})(\cdot,\xi)$ is a trigonometric polynomial, because the sum is finite, so we may conclude that $(Ta_{\varepsilon,n})(\cdot,\xi) \in CAP(\mathbb{R}^d)$ for any $\xi \in \mathbb{R}^d$, $\varepsilon > 0$ and $n \in \mathbb{N}$. □

REMARK 1. We have worked in the Kohn–Nirenberg quantization and the transformation $a \mapsto U(a)$. For the Weyl quantization, the corresponding transformation is $a \mapsto V(a)$ where

$$V(a)(\xi)_{\lambda,\lambda'} = a_{\lambda'-\lambda}\left(\xi - \frac{\lambda+\lambda'}{2}\right).$$

With the Weyl product defined by $a_{1/2}(x,D)b_{1/2}(x,D) = (a\#b)_{1/2}(x,D)$, we then have $V(a\#b)(\xi) = V(a)(\xi)\cdot V(b)(\xi)$, corresponding to Theorem 2. Moreover, $V(\overline{a})(\xi)_{\lambda,\lambda'} = \overline{V(a)(\xi)_{\lambda',\lambda}}$, i.e. $V(\overline{a})(\xi) = V(a)(\xi)^*$ where A^* denotes the Hermitian (conjugate transpose) matrix, which gives $V(\overline{a})(D) = V(a)(D)^*$. Since $\overline{a}_{1/2}(x,D) = a_{1/2}(x,D)^*$, we obtain as a consequence that the transformation $a_{1/2}(x,D) \mapsto V(a)(D)$, as well as $a(x,D) \mapsto U(a)(D)$, respects adjoints.

Acknowledgement

This paper originates from a talk that the author gave at the Second Conference on Pseudo-Differential Operators and Related Topics, Växjö University, Sweden, 23–27 June 2008, organized by ISAAC to honour Professor Luigi Rodino on his 60th birthday. I am grateful to the organizers, Professors G. Garello, J. Toft and M.-W. Wong, for the invitation to the very interesting and stimulating conference and for the hospitality.

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AMS Subject Classification: 47G30, 43A60

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Lavoro pervenuto in redazione il 07.05.2009 e, in forma definitiva, il 02.07.2009