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PSEUDO-SPECTRAL DERIVATIVE OF QUADRATIC QUASI-INTERPOLANT SPLINES

Abstract. In this paper we propose a local spline method for the approximation of the derivative of a function f. It is based on an optimal spline quasi-interpolant operator Q_2, introduced in [12]. Differentiating $Q_2 f$, we construct the pseudo-spectral derivative at the quasi-interpolation knots and the corresponding differentiation matrix. An error analysis is proposed. Some numerical results and comparisons with other known methods are given.

1. Introduction

In the approximation of the derivative of a function f in a certain interval $[a,b]$, the choice of the method is not a secondary problem.

The differentiation of interpolation polynomials leads to classical finite differences for the approximate computation of derivatives, but this approximation is not stable for increasing values of polynomial degree in case of uniform knot partition. In order to overcome this problem the Gauss-Lobatto Chebyshev knots can be considered [10].

Another approach can consist in approximating the derivative of f by the derivative of a spline, generally expressed by a local basis of B-splines and defined by either a quasi-interpolating operator or an interpolating one. However, while quasi-interpolant (q-i) splines [9, 11, 12, 13] have a direct construction, the interpolant ones need the solution of a linear system of equations. This is one of the reasons why, in the literature, local q-i spline operators represent very useful tools in many applications [14]. In particular, in [9], a general method to construct quasi-interpolant spline operators, that can be also applied to approximate the derivative of a function f, is presented, and some convergence properties are proved. Recently, in [8, 13], the authors have proposed some uniform discrete quasi-interpolant spline operators, giving a simple explicit formula, and have presented a method for numerical differentiation based on these operators. In particular they have constructed the differentiation matrices for uniform quadratic and cubic splines, that are very useful in applications.

When we have to approximate a function exibiting varied features and abrupt transitions, a possible approach is to 'adapt' the knot partition by thickening the points where the function has more irregularities. For this reason, in the literature, non uniform quasi-interpolant spline operators have been introduced (see e.g. [1, 9, 11, 12]). In particular, in [12], a non uniform optimal local quasi-interpolant spline operator, the discrete quadratic C^1 Q_2, is presented, providing an explicit formula for the coefficient functionals.

In this paper we propose a method, based on the above operator Q_2, for the numerical evaluation of the first derivative of a function f. Such method generalizes some results, obtained in [13], for the uniform case to the non uniform one. In Section 3 we construct the derivative of $Q_2 f$, called the pseudo-spectral derivative and the corresponding differentiation matrix D_2. In Section 4 we propose the error analysis. Finally, in Section 5 we present some numerical results, giving comparisons with other known methods.

We remark that the same technique could be used to estimate higher-order derivatives of f considering non uniform quasi-interpolant splines of higher degree, if the functionals defining them are known.

2. On a local discrete quadratic C^1 spline quasi-interpolant

Let $I = [a,b]$ be a bounded interval endowed with some partition

$$\Delta_k = \{a = x_0 < x_1 < \ldots < x_k < x_{k+1} = b\},$$

we define

$$\mathcal{S}_2^1(\Delta_k) = \{s \in C^1([a,b]) : s_{|[x_i,x_{i+1})} \in \mathbb{P}_2, i = 0,1,\ldots,k\},$$

where $\mathbb{P}_2$ denotes the space of polynomials in x of degree 2.

We denote by $\mathcal{N} = \{N_j^2(x)\}_{j=1}^{k+3}$ the basis of normalized quadratic B-splines with knots:

$$x_{-2} = x_{-1} = x_0 < x_1 < \ldots < x_k < x_{k+1} = x_{k+2} = x_{k+3}, \tag{1}$$

spanning $\mathcal{S}_2^1(\Delta_k)$ [2]. With our notations the support of the B-spline N_j^2 is $[x_{j-3}, x_j]$.

Considering the knot partition (1), we define $h_j = x_{j-1} - x_{j-2}$ for $0 \le j \le k+4$ and the set of points $\mathcal{T}_k = \{t_i, i = 1,\ldots,k+3\}$, where

$$\begin{aligned} &t_1 = a, \\ &t_i = \tfrac{1}{2}(x_{i-2} + x_{i-1}), \quad \text{for } i = 2,\ldots,k+2, \\ &t_{k+3} = b. \end{aligned}$$

Now we consider a linear quasi-interpolant operator exact on $\mathbb{P}_2$ (i.e. $Q_2 p = p$, $p \in \mathbb{P}_l$, $l \le 2$) of the form:

$$Q_2 : C(I) \to \mathcal{S}_2^1(\Delta_k),$$

introduced in [11] and defined by

$$Q_2 f = \sum_{j=1}^{k+3} m_j(f) N_j^2 \tag{2}$$

with knot partition (1). The coefficients m_j are local linear functional which are combination of discrete values of f at points in $\mathcal{T}_k$, laying in the support of N_j^2:

$$\begin{aligned} &m_1(f) = f_1,\; m_{k+3}(f) = f_{k+3}, \\ &m_j(f) = a_j f_{j-1} + b_j f_j + c_j f_{j+1}, \quad \text{for } j = 2,\ldots,k+2, \end{aligned}$$

where $f_j = f(t_j)$ and

$$(3) \qquad \begin{aligned} a_j &= -\frac{h_j^2}{(h_{j-1}+h_j)(h_{j-1}+2h_j+h_{j+1})}, \\ b_j &= 1+\frac{h_j^2}{(h_{j-1}+h_j)(h_j+h_{j+1})}, \\ c_j &= -\frac{h_j^2}{(h_j+h_{j+1})(h_{j-1}+2h_j+h_{j+1})}. \end{aligned}$$

We can express (2) in the following form

$$(4) \qquad Q_2 f = \sum_{j=1}^{k+3} f_j \tilde{N}_j^2,$$

where $\{\tilde{N}_j^2\}_{j=1}^{k+3}$ are the fundamental functions of Q_2, defined in [11] by

$$(5) \qquad \begin{aligned} \tilde{N}_1^2 &= N_1^2 + a_2 N_2^2, \\ \tilde{N}_j^2 &= c_{j-1}N_{j-1}^2 + b_j N_j^2 + a_{j+1}N_{j+1}^2, \quad j = 2, \ldots k+2, \\ \tilde{N}_{k+3}^2 &= c_{k+2}N_{k+2}^2 + N_{k+3}^2. \end{aligned}$$

3. Pseudo-spectral derivative and differentiation matrix

We define the pseudo-spectral derivative of f as the derivative of the quadratic quasi-interpolant spline $Q_2 f$. We denote it by $Q_2' f$ and we remark that it belongs to $\mathcal{S}_1^0(\Delta_k)$, space of linear splines defined on Δ_k.

Therefore, taking into account (2) and (4), we have:

$$(6) \qquad Q_2' f = \left(\sum_{j=1}^{k+3} m_j(f) N_j^2\right)' = \sum_{j=1}^{k+3} f_j (\tilde{N}_j^2)'.$$

Since the derivative of the spline N_j^2 [2] is

$$(7) \qquad (N_j^2)'(x) = 2\left(\frac{N_{j-1}^1(x)}{h_j+h_{j-1}} - \frac{N_j^1(x)}{h_{j+1}+h_j}\right), \quad j = 1, \ldots, k+3,$$

with $N_0^1(x) \equiv 0$, $N_{k+3}^1(x) \equiv 0$ and $\{N_j^1(x)\}_{j=1}^{k+2}$ set of normalized linear B-splines spanning $\mathcal{S}_1^0(\Delta_k)$, from (5) and (7) we obtain

$$
\begin{aligned}
(\tilde{N}_1^2)' &= 2\left(\frac{a_2-1}{h_2}N_1^1 - \frac{a_2}{h_3+h_2}N_2^1\right),\\
(\tilde{N}_j^2)' &= 2\left(\frac{c_{j-1}}{h_{j-1}+h_{j-2}}N_{j-2}^1 + \frac{b_j-c_{j-1}}{h_j+h_{j-1}}N_{j-1}^1\right.\\
&\quad \left. + \frac{a_{j+1}-b_j}{h_{j+1}+h_j}N_j^1 - \frac{a_{j+1}}{h_{j+2}+h_{j+1}}N_{j+1}^1\right), \quad j=2,\dots k+2,\\
(\tilde{N}_{k+3}^2)' &= 2\left(\frac{c_{k+2}}{h_{k+2}+h_{k+1}}N_{k+1}^1 + \frac{1-c_{k+2}}{h_{k+2}}N_{k+2}^1\right).
\end{aligned}
$$

Evaluating (6) at the points of $\mathcal{T}_k$, we have

$$Q_2' f(t_i) = \sum_{j=1}^{k+3} f_j (\tilde{N}_j^2)'(t_i), \quad i=1,\dots,k+3.$$

Therefore the pseudo-spectral derivative at the quasi-interpolation knots can be computed only using the values of f and $(\tilde{N}_j^2)'$ at $\mathcal{T}_k$.

The values of $(\tilde{N}_j^2)'$ at $\mathcal{T}_k$ can be stored in a matrix $D_2 \in \mathbb{R}^{(k+3)\times(k+3)}$: $d_{ij} = (\tilde{N}_j^2)'(t_i)$, for $i,j=1,\dots,k+3$, called differentiation matrix.

Setting y for the vector with components $y_j = f(t_j)$, $j=1,\dots,k+3$ and y' for the vector with components $y_j' = Q_2' f(t_j)$, $j=1,\dots,k+3$, we simply write:

$$y' = D_2 y. \tag{8}$$

The differentiation matrix D_2 has a structure as follows:

$$
D_2 = \left(\begin{array}{cccccccc}
\times & \times & \times & & & & & \\
\times & \times & \times & \times & & & \mathbf{0} & \\
\hline
\times & \times & \times & \times & \times & & & \\
 & \times & \times & \times & \times & \times & \mathbf{0} & \\
 & & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & \mathbf{0} & & \times & \times & \times & \times & \times \\
 & & & & \times & \times & \times & \times \quad \times \\
\hline
 & \mathbf{0} & & & & \times & \times & \times \quad \times \\
 & & & & & & \times & \times \quad \times
\end{array}\right)
= \left(\begin{array}{c}
D_2^{(1)} \\
\hline
D_2^{(2)} \\
\hline
D_2^{(3)}
\end{array}\right),
$$

where $D_2^{(1)}$, $D_2^{(3)} \in \mathbb{R}^{2\times(k+3)}$, $D_2^{(2)} \in \mathbb{R}^{(k-1)\times(k+3)}$ is a banded matrix, with bandwidth 2 (i.e. 2 bands above and below the diagonal) and the elements different from zero are:

- for $D_2^{(1)}$:

$$d_{11}=2(a_2-1)/h_2,\ \ d_{12}=2b_2/h_2,\ \ d_{13}=2c_2/h_2,$$

$$\begin{aligned}&d_{21}=(a_2-1)/h_2-a_2/(h_2+h_3),\ \ d_{22}=b_2/h_2+(a_3-b_2)/(h_2+h_3),\\&d_{23}=c_2/h_2+(b_3-c_2)/(h_2+h_3),\ \ d_{24}=c_3/(h_2+h_3),\end{aligned}$$

- for $D_2^{(2)}$:

$$\begin{aligned}&d_{i,i-2}=-a_{i-1}/(h_{i-1}+h_i),\\&d_{i,i-1}=(a_i-b_{i-1})/(h_{i-1}+h_i)-a_i/(h_i+h_{i+1}),\\&d_{i,i}=(b_i-c_{i-1})/(h_{i-1}+h_i)+(a_{i+1}-b_i)/(h_i+h_{i+1}),\\&d_{i,i+1}=c_i/(h_{i-1}+h_i)+(b_{i+1}-c_i)/(h_i+h_{i+1}),\\&d_{i,i+2}=c_{i+1}/(h_i+h_{i+1}),\\&\qquad i=3,\dots,k+1\end{aligned}$$

- for $D_2^{(3)}$:

$$\begin{aligned}&d_{k+2,k}=-a_{k+1}/(h_{k+1}+h_{k+2}),\\&d_{k+2,k+1}=(a_{k+2}-b_{k+1})/(h_{k+1}+h_{k+2})-a_{k+2}/h_{k+2},\\&d_{k+2,k+2}=(b_{k+2}-c_{k+1})/(h_{k+1}+h_{k+2})-b_{k+2}/h_{k+2},\\&d_{k+2,k+3}=c_{k+2}/(h_{k+1}+h_{k+2})+(1-c_{k+2})/h_{k+2},\end{aligned}$$

$$\begin{aligned}&d_{k+3,k+1}=-2a_{k+2}/h_{k+2},\ \ d_{k+3,k+2}=-2b_{k+2}/h_{k+2},\\&d_{k+3,k+3}=2(1-c_{k+2})/h_{k+2}.\end{aligned}$$

4. Error analysis

In this section we analyse the error $E_1=(f-Q_2f)'$ when $f\in C^r(I)$, $1\le r\le 3$.

In order to do it, we recall that a sequence of partitions $\{\Delta_k\}$ is locally uniform if there exists a constant $A\ge 1$ such that

$$\frac{x_{i+1}-x_i}{x_{j+1}-x_j}\le A,\quad \text{for all } i \text{ and } j=i\pm 1.$$

We introduce, for $1\le p\le k+1$, the following notations

$$\begin{aligned}&I_p=[x_{p-1},x_p],\\&J_p=[x_{p-3},x_{p+2}],\\&\overline{h}_p=\max_{p-1\le j\le p+3}h_j,\\&\overline{h}=\max_{1\le j\le k+3}h_j,\\&\delta_p=\min_{p-2\le j\le p-1}x_{j+2}-x_j.\\&\delta=\min_{1\le j\le k+1}\delta_j.\end{aligned}\tag{9}$$

To estimate $\|E_1\|_\infty$ we proceed as in [7] and [9].

Firstly, we want to obtain a local bound for $|E_1(t)|$, with $t \in I_p$, $1 \le p \le k+1$, and then we extend the results to the interval I.

We define for any $x \in J_p$ and $f \in C^r(J_p)$

$$R(x) = f(x) - \sum_{i=0}^{r} \frac{f^{(i)}(t)}{i!}(x-t)^i,$$

and to give a bound to $|E_1(t)|$ it is only necessary to estimate $|Q_2' R(t)|$, as shown in [9].

Since $t \in I_p$ we have

$$\left|Q_2' R(t)\right| = \left|\sum_{j=p-1}^{p+3} R(t_j)(\tilde{N}_j^2)'(t)\right| \le \sum_{j=p-1}^{p+3} \left|R(t_j)\right| \left|(\tilde{N}_j^2)'(t)\right|, \tag{10}$$

with $\tilde{N}_0^2 = \tilde{N}_{k+4}^2 = 0$ and $R(t_0) = R(t_{k+4}) = 0$.

In Lemma 1 and 2 we estimate $\left|R(t_j)\right|$ and $\left|(\tilde{N}_j^2)'(t)\right|$ respectively.

LEMMA 1. *Let $f \in C^r(J_p)$, $r = 1,2$. Then for $i = p-1,\dots,p+3$*

$$\left|R(t_j)\right| \le \frac{(5/2)^{r+1}}{r!} \overline{h}_p^r \omega(f^{(r)}, \overline{h}_p, J_p).$$

The proof of Lemma 1 is similar to that given in Lemma 3.3 of [7] and then here we omit it.

LEMMA 2. *Suppose $t \in I_p$. Then, for $j = 3,\dots,k+1$*

$$\left|(\tilde{N}_j^2)'(t)\right| \le \frac{6}{\delta_p}, \tag{11}$$

and

$$\left|(\tilde{N}_1^2)'(t)\right| \le \frac{3}{\delta_p}, \quad \left|(\tilde{N}_{k+3}^2)'(t)\right| \le \frac{3}{\delta_p}, \tag{12}$$

$$\left|(\tilde{N}_2^2)'(t)\right| \le \frac{5}{\delta_p}, \quad \left|(\tilde{N}_{k+2}^2)'(t)\right| \le \frac{5}{\delta_p}. \tag{13}$$

Proof. From (5), for $j = 3,\dots,k+1$, we have

$$\left|(\tilde{N}_j^2)'(t)\right| \le |c_{j-1}||N_{j-1}^2| + |b_j||N_j^2| + |a_{j+1}||N_{j+1}^2|.$$

Taking into account Lemma 2.1 of [9], we can bound the first derivative of the normalized B-splines $\{N_j^2\}$, obtaining

$$\left|(\tilde{N}_j^2)'(t)\right| \le \frac{2}{\delta_p}\left(|c_{j-1}| + |b_j| + |a_{j+1}|\right) \tag{14}$$

and, since from Theorem 1.2 of [12] we have $|a_j| \le \frac{1}{2}$, $|b_j| \le 2$ and $|c_j| \le \frac{1}{2}$, we get

$$|c_{j-1}| + |b_j| + |a_{j+1}| \le 3. \tag{15}$$

Therefore (14) and (15) yield (11).

For $j = 1$ and $j = k+3$, from (5), we have

$$\left|(\tilde{N}_1^2)'(t)\right| \le |N_1^2| + |a_2||N_2^2|, \quad \left|(\tilde{N}_{k+3}^2)'(t)\right| \le |c_{k+2}||N_{k+2}^2| + |N_{k+3}^2|$$

and, from Theorem 1.2 of [12]

$$(1 + |a_2|) \le \frac{3}{2}, \quad (1 + |c_{k+2}|) \le \frac{3}{2}.$$

Therefore we obtain (12).

For $j = 2$ and $j = k+2$, from (5), we have

$$\left|(\tilde{N}_2^2)'(t)\right| \le |c_1||N_1^2| + |b_2||N_2^2| + |a_3||N_3^2|,$$

$$\left|(\tilde{N}_{k+2}^2)'(t)\right| \le |c_{k+1}||N_{k+1}^2| + |b_{k+2}||N_{k+2}^2| + |a_{k+3}||N_{k+3}^2|.$$

Since, from (3), c_1 and a_{k+3} are equal to zero, taking into account Theorem 1.2 of [12], we get

$$(|b_2| + |a_3|) \le \frac{5}{2}, \quad (|c_{k+1}| + |b_{k+2}|) \le \frac{5}{2},$$

and we obtain (13). □

Now we can give a local estimate for $|E_1(t)|$.

THEOREM 1. *Suppose $t \in I_p$ and let $f \in C^r(J_p)$, $r = 1, 2$. Then*

$$\left|(f - Q_2 f)'(t)\right| \le C_{r,p} \overline{h}_p^{\,r-1} \omega(f^{(r)}, \overline{h}_p, J_p), \tag{16}$$

where

$$C_{r,p} = \frac{\Gamma_p (5/2)^{r+1}}{r!} \frac{\overline{h}_p}{\delta_p}, \tag{17}$$

and $\Gamma_p \le 30$.

If in addition $\{\Delta_k\}$ is locally uniform with constant A, then

$$C_{r,p} = \frac{\Gamma_p (5/2)^{r+1}}{r!} A^2. \tag{18}$$

Proof. The inequality (16) and the constant values (17) follow immediately from (10), Lemma 1 and Lemma 2.

If $\{\Delta_k\}$ is locally uniform with constant A, then [5]

$$\overline{h}_p = \max_{p-3 \le i \le p+1} (x_{i+1} - x_i) \le A^2 (x_p - x_{p-1}),$$
$$\delta_p = \min_{p-2 \le i \le p-1} (x_{i+2} - x_i) \ge (x_p - x_{p-1}).$$

Therefore

$$\frac{\overline{h}_p}{\delta_p} \leq A^2. \tag{19}$$

From (19) and (17), we get (18). □

Theorem 1 leads immediately to the following global results.

THEOREM 2. *Let $f \in C^r(I)$, $r \leq 2$. Then*

$$\left\|(f - Q_2 f)'\right\|_\infty \leq \overline{C}_r \overline{h}^{r-1} \omega(f^{(r)}, \overline{h}, I),$$

where

$$\overline{C}_r = \frac{30(5/2)^{r+1}}{r!} \frac{\overline{h}}{\delta}.$$

If in addition $f \in C^3(I)$, then

$$\left\|(f - Q_2 f)'\right\|_\infty \leq \overline{C}_2 \overline{h}^2 \left\|f^{(3)}\right\|_\infty.$$

If $\{\Delta_k\}$ is locally uniform with constant A, then the constants $\overline{C}_r$ are independent on k and

$$\overline{C}_r = \frac{30(5/2)^{r+1}}{r!} A^2.$$

Now we analyse the error E_1 at the points t_i, $i = 1, \ldots, k+3$.

The logical scheme here proposed is similar to that one presented in [14] for the error of spline derivative in the uniform case.

THEOREM 3. *If $f \in C^3(I)$, for $i = 3, \ldots, k+1$, we obtain*

$$y_i' - f_i' = -\frac{h_{i+1}^2 H_{i+1} + h_{i-1}^2 H_{i-1} - 2h_i^2 H_i}{48(h_{i-1} + h_i)(h_i + h_{i+1})} f_i^{(3)} + O(\overline{h}_{i-1}^3), \tag{20}$$

where $\overline{h}_{i-1}$ is defined in (9) and

$$\begin{aligned}
H_{i+1} &= 2h_{i+1}(h_{i-1} + h_i) + h_{i+2}(h_{i-1} + h_i) - h_i h_{i-1}, \\
H_{i-1} &= 2h_{i-1}(h_{i+1} + h_i) + h_{i-2}(h_{i+1} + h_i) - h_i h_{i+1}, \\
H_i &= 2h_i(h_{i-1} + h_{i+1}) + h_i^2 + 4h_{i+1} h_{i-1}.
\end{aligned}$$

For $i = 1, 2$ we have

$$y_1' - f_1' = -\frac{h_2(2h_2 + h_3)}{24} f_1^{(3)} + O(\overline{h}_0^3), \tag{21}$$

where we define $\overline{h}_0 = \max_{0 \leq i \leq 3} h_i$ and

$$y_2' - f_2' = -\frac{h_3^2(h_4 + 2h_3) - 2h_2^2(h_2 + 2h_3)}{48(h_2 + h_3)} f_2^{(3)} + O(\overline{h}_1^3), \tag{22}$$

and analogous results hold for $i = k+2, k+3$.

Proof. From (8), for $3 \le i \le k+1$, we have

$$\begin{aligned} y_i' = \quad & -\frac{a_{i-1}}{h_{i-1}+h_i} f(t_{i-2}) + \left(\frac{a_i - b_{i-1}}{h_{i-1}+h_i} - \frac{a_i}{h_i + h_{i+1}} \right) f(t_{i-1}) \\ & + \left(\frac{b_i - c_{i-1}}{h_{i-1}+h_i} + \frac{a_{i+1} - b_i}{h_i + h_{i+1}} \right) f(t_i) \\ & + \left(\frac{c_i}{h_{i-1}+h_i} - \frac{b_{i+1} - c_i}{h_i + h_{i+1}} \right) f(t_{i+1}) + \frac{c_{i+1}}{h_i + h_{i+1}} f(t_{i+2}). \end{aligned} \tag{23}$$

Now we apply Taylor formula to the function f in a neighbourhood of t_i, and we evaluate the corresponding polynomial at the points t_{i-2}, t_{i-1}, t_{i+1} and t_{i+2}. For example at the point t_{i+1} we have

$$\begin{aligned} f(t_{i+1}) = \quad & f(t_i) + (t_{i+1} - t_i) f'(t_i) + \frac{1}{2}(t_{i+1} - t_i)^2 f^{(2)}(t_i) \\ & + \frac{1}{6}(t_{i+1} - t_i)^3 f^{(3)}(t_i) + O\left(\left(\frac{h_i + h_{i+1}}{2} \right)^4 \right). \end{aligned}$$

Then, in (23), we replace $f(t_{i-2})$, $f(t_{i-1})$, $f(t_{i+1})$, $f(t_{i+2})$ with their Taylor expansion and, taking into account (3), after some algebra, we obtain (20).

Using the same technique for y_1' and y_2', we obtain the expressions (21), (22) and analogous results hold for t_{k+2} and t_{k+3}. □

5. Numerical results

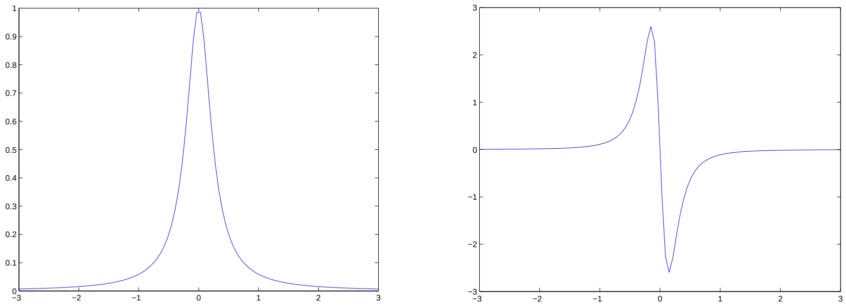
In this section we propose some numerical results, obtained by a computational procedure that we have developed in Matlab environment [3].

We use the following test functions f_p, $p = 1, 2$:

$$f_1(x) = \frac{1}{1+16x^2},$$
$$f_2(x) = \frac{1}{1+16x^2} \sin(3\pi x),$$

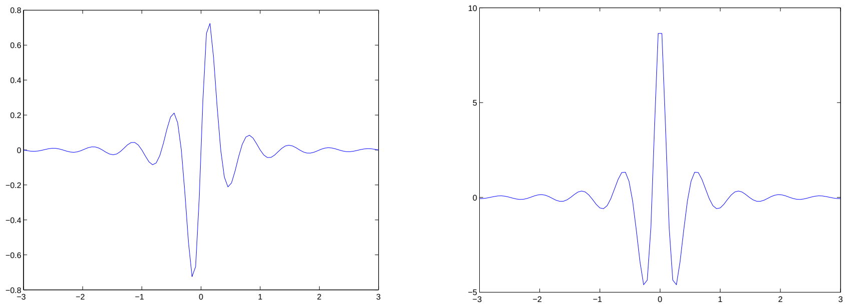
on the interval $I = [-3, 3]$, shown in (*a*) in Figures 1 and 2, with their first derivatives shown in (*b*).

In particular we compare our method, based on the spline operator Q_2, using both a uniform and a non uniform knot partition, with the classical centered-difference formula of order 2 (see e.g. [10]) of f_p at the point t_i and with the derivative of the quadratic spline interpolating the data $\{(t_i, f_p(t_i)\}_{i=1}^{k+3}$. More precisely, given these data, we construct the parabolic spline interpolating the function f at the data sites $\{t_i\}_{i=1}^{k+3}$, denoted by $S_2 f_p$, and then we compute its derivative, $S_2' f_p$. For the construction of the



(a) (b)

Figure 1: The function f_1 (a) and its first derivative (b)



(a) (b)

Figure 2: The function f_2 (a) and its first derivative (b)

quadratic interpolant spline we can refer to [2] and [3] for theoretical and computational considerations, respectively.

For the non uniform cases we use the partition $\Delta_k = \{x_j\}_{j=0}^{k+1}$ defined by

$$x_j = a + (b-a)\left(\frac{2j}{k+2}\right)^2, \qquad j = 0, \ldots, \left[\tfrac{k+1}{2}\right],$$
$$x_j = b - (b-a)\left(\frac{2j}{k+2}\right)^2, \qquad j = \left[\tfrac{k+1}{2}\right] + 1, \ldots, k+1,$$

with knots thickening around the origin. The sequence of partitions $\{\Delta_k\}$, above defined, is locally uniform, with constant $A = 3$ [6]. This kind of partition is a particular case of symmetrically graded meshes proposed in [4]. We remark that choosing a non uniform partition Δ_k, we can control the behaviour of the first derivative of f using a greater number of knots where it has strong variations.

We set, for $p = 1,2$,

$$\varepsilon_p^{u.} = \max_{v\in\mathcal{T}_k} \left| f_p'(v) - (Q_2^{u.})' f_p(v) \right|$$
$$\varepsilon_p^{n.u.} = \max_{v\in\mathcal{T}_k} \left| f_p'(v) - (Q_2^{n.u.})' f_p(v) \right|$$
$$\bar{\varepsilon}_p^{n.u.} = \max_{v\in\mathcal{T}_k} \left| f_p'(v) - S_2' f_p(v) \right|$$
$$\hat{\varepsilon}_p^{u.} = \max_{v\in\mathcal{T}_k} \left| f_p'(v) - \delta f_p(v) \right|$$

where $(Q_2^{u.})' f_p$ and $(Q_2^{n.u.})' f_p$ are the approximations given by the spline operator Q_2 using a uniform and a non uniform partition respectively, δf_p is the approximation given by the centered-difference formula and $S_2' f_p$ is the derivative of the quadratic interpolant spline. The label $n.u.$ denotes the use of a non uniform knot partition in the construction of the corresponding approximation, while the label $u.$ denotes the use of a uniform one.

The results obtained by using the different methods above introduced are reported in Table 4.1, for increasing values of k.

We can notice that, using a non uniform partition, we obtain better results, and that the behaviour of the quasi-interpolant spline and the interpolant one is almost the same, but, as remarked above, in the q-i case we do not solve any system of equations.

k	$\varepsilon_1^{u.}$	$\varepsilon_1^{n.u.}$	$\bar{\varepsilon}_1^{n.u.}$	$\hat{\varepsilon}_1^{u.}$
64	1.9(-1)	1.5(-2)	1.3(-2)	3.8(-1)
128	3.3(-2)	3.6(-3)	3.4(-3)	1.0(-1)
256	7.3(-3)	8.9(-4)	8.8(-4)	2.6(-2)
512	1.7(-3)	2.2(-4)	2.2(-4)	6.8(-3)
1024	4.3(-4)	5.6(-5)	5.6(-5)	1.7(-3)
k	$\varepsilon_2^{u.}$	$\varepsilon_2^{n.u.}$	$\bar{\varepsilon}_2^{n.u.}$	$\hat{\varepsilon}_2^{u.}$
64	1.2(0)	7.5(-2)	6.9(-2)	2.1(0)
128	2.1(-1)	1.8(-2)	1.8(-2)	6.0(-1)
256	4.4(-2)	5.0(-3)	4.5(-3)	1.6(-1)
512	1.0(-2)	1.3(-3)	1.1(-3)	4.0(-2)
1024	2.5(-3)	3.3(-4)	2.9(-4)	9.9(-3)

Table 4.1: Maximum absolute errors by different methods

6. Conclusion

In this paper we have proposed a local spline method, based on the optimal quadratic spline quasi-interpolant operator Q_2, introduced in [12], for the approximation of the derivative of a function f in a certain interval $[a,b]$. Differentiating $Q_2 f$, we have constructed the pseudo-spectral derivative at the quasi-interpolation knots and the corresponding differentiation matrix D_2. Furthermore we have presented an error analysis and we have given some numerical results, comparing our method with other known methods.

We can remark that a generalization of the obtained results to quasi-interpolant splines of higher degree could be an interesting future topic.

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