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In Memoriam Aristide Sanini

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GRASSMANN BUNDLES AND HARMONIC MAPS*

Introduction

The "classical" Gauss map Υ associates to any point x of an oriented surface M, immersed in $\mathbb{R}^3$, the unit normal vector N_x applied at a point O in $\mathbb{R}^3$, and so determines a mapping from M to the unit sphere S^2. It is therefore called the *spherical representation* of M.

This representation enables one to obtain plenty of information on various geometrical aspects of the surface. In particular, it provides an "extrinsic" interpretation of the Gaussian curvature of M. A classical result is expressed by the following:

THEOREM 1. *The Gauss map $\Upsilon : M \to S^2$ is conformal if and only if*
1. *M is a minimal surface, or*
2. *M is contained in a sphere.*

In both cases, Υ is a harmonic map. The analysis of Gauss maps, extended in a suitable way, is the focal point of many interesting research topics. Particular attention is dedicated to the conditions under which those maps are conformal or harmonic.

In his article [19], R. Osserman provides an excellent overview on the evolution of the concept of Gauss map and the information regarding the geometry of submanifolds that can be deduced from it.

Building on the classical concept of Gauss map, it is possibile to define the "generalized" one, which associates to each point x of an m-dimensional manifold isometrically immersed in $\mathbb{R}^n$, the subspace of $\mathbb{R}^n$ parallel to T_xM, i.e.,

$$\Upsilon : M \to G_m(n),$$

where $G_m(n)$ is the Grassmannian of m-planes in $\mathbb{R}^n$, having a well-known structure of a homogeneous (indeed, symmetric) Riemannian space.

Subsequently, M. Obata constructed in [18] a Gauss map for an m-dimensional Riemannian manifold M isometrically immersed in a simply connected space N with constant sectional curvature. Such a construction of a Gauss map is based on the mapping of $x \in M$ to the m-dimensional totally geodesic submanifold of N tangent to M at x and leads to several particularly significant results regarding conformality conditions.

The more recent theory of harmonic maps (see for example the extensive reports of J. Eells–L. Lemaire [8, 9]) immediately reveals remarkable points in contact with the theory of Gauss maps. We mainly refer to those results (see [7]) generalizing the theorem cited above (see [4]):

*Translation of a *Rapporto Interno, Politecnico di Torino, 1988.*

THEOREM 2 (Chern [2]). *Consider an isometric immersion f of an orientable surface M inside $\mathbb{R}^n$. Then f is harmonic (i.e., minimal) if and only if the Gauss map from M to $\tilde{G}_2(n)$ is anti-holomorphic, where $\tilde{G}_2(n)$ is the Grassmannian of oriented 2-planes in $\mathbb{R}^n$ which can be identified with the complex quadric Q_{n-2} in $\mathbb{CP}^{n-1}$.*

THEOREM 3 (Ruh–Vilms [21]). *A submanifold M of $\mathbb{R}^n$ has parallel mean curvature vector if and only if the Gauss map $\Upsilon : M \to G_m(n)$ is harmonic.*

More recently, C.M. Wood [24] and G.R. Jensen–M. Rigoli [11], considering a submanifold M of a generic Riemannian manifold N, define the Gauss map as the map from M to the Grassmann bundle $G_m(TN)$ of m-planes tangent to N endowed with a suitable metric. They analyse several aspects of the harmonic and conformal conditions, extending the previous results.

From another point of view, S.S. Chern and R.K. Lashof [6], considering a submanifold M isometrically immersed in $\mathbb{R}^n$, define the "spherical" Gauss map (another extension of the classical concept) as the correspondence $\nu : M \to S^{n-1}$ that associates to each unit vector v, normal to M in a point $x \in M$, a point in S^{n-1} obtained by parallel transport of v to the origin of $\mathbb{R}^n$.

In the article cited above, Jensen and Rigoli study the analogous problem in the case of a manifold M isometrically immersed in a generic Riemannian manifold N, associating to any unit vector normal to M the same element in the unit tangent bundle T_1N of N. They analyse also several problems related to the harmonicity of the map.

The present report aims also to expose some recent proper achievements regarding the subject and is divided in three parts.

The first part, entitled "Grassmann bundles and distributions", can be summarized as follows. Section 1 describes the construction of the Riemannian metric on the Grassmann bundle $G_p(TM)$ of p-planes tangent to a a manifold M. This is due to Jensen–Rigoli, and has been already applied by E. Musso–F. Tricerri [17] in the case of unit tangent bundles. The fibres of the Riemannian submersion $G_p(TM) \to M$ are totally geodesic and isometric to the Grassmannian $G_p(m)$ endowed with the standard metric.

Section 2 analyses some aspects related to the curvature of $G_p(TM)$.

Any given p-dimensional distribution over M singles out a section ϕ of $G_p(TM)$, and in Section 3 we determine the conditions under which ϕ is harmonic. Section 4 contains some examples of such a situation in the case in which M is the sphere S^3, the Heisenberg group or another Lie group admitting a left-invariant metric.

In the second part "Isometric immersions and maps between Grassman bundles" we analyse (starting in Section 5) the map F from $G_p(TM)$ to $G_p(TN)$ induced by an isometric immersion f of M inside N. If $p = \dim M$ then $G_p(TM)$ can be identified with M and F coincides with the Gauss map Υ.

Then in Section 6 we define the tension field of F and the conditions under which it is harmonic. We exhibit a significant example of a minimal surface M of the Heisenberg group H for which the Gauss map is conformal but not harmonic.

Section 7 develops a detailed analysis of the harmonic properties of F under the hypothesis that N has constant sectional curvature. The results are completely analogous to those obtained by the author (see [22]) in the case of the map induced between the unit tangent bundles by a Riemannian immersion of M to N. Furthermore, when F coincides with the Gauss map, the results we achieve are compared to those of E. Ruh–J. Vilms and T. Ishihara described in [10].

The third part is dedicated to the "Spherical Gauss map". In Section 8 we introduce the Riemannian metric on the unit normal bundle $T_1^{\perp}M$ of a manifold M isometrically immersed in N. Later, in Section 9, we study the harmonicity of the spherical Gauss map $\nu : T_1^{\perp}M \to T_1^{\perp}N$ applying a technique analogous to the one adopted in the Second part (ν has already been analysed by Jensen–Rigoli with another method). In Section 10, we add some remarks and examples in the case in which N has a constant sectional curvature.

The present report contains two appendices:

– Appendix A, in which we recall several facts regarding the bundle of Darboux frames and the classical conditions (Gauss, Codazzi, Ricci) on the curvature tensors on a manifold.

– Appendix B, which describes the computation of the tension field of a map between Riemannian manifolds in terms of orthonormal coframes, following the method adopted by S.S. Chern–S.I. Goldberg [5].

I. GRASSMANN BUNDLES AND DISTRIBUTIONS

1. The Grassmann bundle of a Riemannian manifold

Let (M,g) be a Riemannian manifold of dimension m. The bundle of orthogonal frames of M, which has the orthogonal group $O(m)$ as a structure group, is characterized by the $\mathbb{R}^m$-valued canonical form $\theta = (\theta^i)$ and the $\mathfrak{o}(m)$-valued 1-form $\omega = (\omega^i_j)$ determined by the Levi-Civita connection.

Denoting by R_a right translation on $O(m)$ determined by an element a of $O(m)$, we have

$$(1)\qquad (R_a^*\theta)^i = (a^{-1})^i_h \theta^h,$$

$$(2)\qquad (R_a^*\omega)^i_j = (a^{-1})^i_h \omega^h_k a^k_j.$$

Furthermore

$$(3)\qquad d\theta^i = -\omega^i_j \wedge \theta^j \qquad (\omega^i_j + \omega^j_i = 0),$$

$$(4)\qquad d\omega^i_j = -\omega^i_k \wedge \omega^k_j + \frac{1}{2} R^M_{ijhk} \theta^h \wedge \theta^k,$$

where R^M_{ijhk} are the curvature functions on $O(M)$ associated to the Riemannian curvature tensor R^M of g; i.e.,

$$R^M_{ijhk}(u) = R^M(u_i, u_j, u_h, u_k) = ((\nabla_{[u_i,u_j]} - \nabla_{u_i}\nabla_{u_j} + \nabla_{u_j}\nabla_{u_i})u_h, u_k), \tag{5}$$

$u = (x, u_1, \ldots u_m)$ is an element of $O(M)$.

DEFINITION 1. *The Grassmann bundle of p-planes in the tangent spaces of M is the bundle on M associated to $O(M)$ with fibre the Grassmannian of p-planes in $\mathbb{R}^m$:*

$$G_p(m) = \frac{O(m)}{O(p) \times O(m-p)}.$$

In other words,

$$G_p(TM) = O(M) \times_{O(M)} G_p(m). \tag{6}$$

The bundle $G_p(TM)$ can be defined in the following equivalent way (we refer the reader to [13, vol. I, Prop. 5.5, p. 57]):

$$G_p(TM) = \frac{O(M)}{O(p) \times O(m-p)}, \tag{7}$$

where $O(M)$ is a principal bundle over $G_p(TM)$ with structure group $O(p) \times O(m-p)$ identified with a subgroup of $O(m)$ as follows:

$$(a_1, a_2) \in O(p) \times O(m-p) \ \mapsto \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \in \ O(m).$$

From now on, we shall exploit the representation of $G_p(TM)$ defined by (6).

The canonical projection $\psi : O(M) \to G_p(TM)$ is given by

$$\psi(u) = [u_1, \ldots, u_p]_x$$

where $u = (x, u_1, \ldots, u_p, u_{p+1}, \ldots, u_m) \in O(M)$ and $[u_1, \ldots, u_p]_x$ denotes the subspace of T_xM generated by the orthonormal vectors $u_1, \ldots, u_p$.

Consider on $O(M)$ the quadratic semidefinite positive form

$$Q = \sum(\theta^i)^2 + \lambda^2 \sum(\omega^a_r)^2 \tag{8}$$

with $r = 1, \ldots, p$, $a = p+1, \ldots, m$, and λ an arbitrary real positive constant. The following facts are well known:

(i) The quadratic form Q is $O(p) \times O(m-p)$-invariant: this follows directly from (1) and (2) with $a = (a_1, a_2) \in O(p) \times O(m-p)$.

(ii) The bilinear form on M associated to Q, i.e.,

$$Q(X,Y) = \sum \theta^i(X)\theta^i(Y) + \lambda^2 \sum \omega^a_r(X)\omega^a_r(Y),$$

vanishes if and only if X or Y are tangent to the fibres of the submersion $\psi : O(M) \to G_p(TM)$.

For this reason (see also [17]), as the rank of the form Q is $m+p(m-p)$ and equals to the dimension of $G_p(TM)$, there exists a unique Riemannian metric ds^2_λ on $G_p(TM)$ such that:

$$\psi^* ds^2_\lambda = Q.$$

In a sequel to this article, we shall consider $G_p(TM)$ endowed with the Riemannian matric ds^2_λ defined by Jensen–Rigoli in [11].

Observe that if we consider on $O(M)$ the Riemannian metric

$$\tilde{g} = \sum(\theta^i)^2 + \frac{1}{2}\lambda^2 \sum(\omega^i_j)^2,$$

one has that ψ is a Riemannian submersion with totally geodesic fibres of $(O(M),\tilde{g})$ over $(G_p(TM), ds^2_\lambda)$.

Let U denote an open set of $G_p(TM)$ and $\sigma : U \to O(M)$ a local section of the bundle $O(M) \xrightarrow{\psi} G_p(TM)$. Thus σ associates to each p-dimensional subspace $[\pi] \subset T_xM$ an orthonormal basis in $x \in M$ such that its first p vectors belong to $[\pi]$.

The $m+p(m-p)$ 1-forms

$$\rho^i = \sigma^*\theta^i, \qquad \rho^{ar} = \lambda\sigma^*\omega^a_r \tag{9}$$

yield an orthonormal coframe on U with respect to the metric ds^2_λ. The forms associated to the Levi-Civita connection with respect to the frame in question are determined by the conditions:

$$\begin{aligned} d\rho^i &= -\rho^i_j \wedge \rho^j - \rho^i_{ar} \wedge \rho^{ar}, \\ d\rho^{ar} &= -\rho^{ar}_j \wedge \rho^j - \rho^{ar}_{bs} \wedge \rho^{bs}, \end{aligned}$$

imposing also skew-symmetry.

A standard computation using (3) and (4), leads to

$$\begin{cases} \rho^i_j = -\rho^j_i = \sigma^*\{\omega^i_j + \frac{1}{2}\lambda^2 R^M_{arji}\omega^a_r\} \\ \rho^i_{ar} = -\rho^{ar}_i = \sigma^*\{\frac{1}{2}\lambda R^M_{arji}\theta^j\} \\ \rho^{ar}_{bs} = -\rho^{bs}_{ar} = \sigma^*\{\delta^a_b\omega^r_s + \delta^r_s\omega^a_b\}. \end{cases} \tag{10}$$

Equation (8) implies also that the natural projection $\Gamma : (G_p(TM), ds^2_\lambda) \to (M,g)$ is a *Riemannian submersion* with *totally geodesic fibres*.

This property can be verified directly using (10). Indeed, let us denote by $\{E_i, E_{ar}\}$ the dual basis of the orthonormal coframe (9), so $\{E_{ar}\}$ is the basis of the vertical distribution V tangent to the fibres and $\{E_i\}$ the basis of the horizontal one H. We have:

$$(\nabla_{E_{bs}} E_{ar}, E_i) = \rho^i_{ar}(E_{bs}) = 0.$$

In the next sections the horizontal and the vertical component of a vector field X, tangent to $G_p(TM)$, will be denoted respectively by X^H and X^V, so

$$X = X^H + X^V.$$

Finally, with a suitable choice of the constants, each fibre of $G_p(TM)$ is isometric to $G_p(m)$ endowed with the canonical metric (we refer the reader to [13, vol. II, p. 272]). Indeed, if we consider

$$G_p(m) = \frac{O(m)}{H}, \qquad H = O(p) \times O(m-p),$$

and the decomposition of the Lie algebra

$$\mathfrak{o} = \mathfrak{h} + \mathfrak{m}$$

with

$$\mathfrak{m} = \begin{pmatrix} 0 & -X^T \\ X & 0 \end{pmatrix} \qquad X \in M(m-p, p, \mathbb{R}),$$

one has that $Ad(H)\mathfrak{m} \subset \mathfrak{m}$. The scalar product on $\mathfrak{m}$ obtained by restriction to $\mathfrak{m}$ of the inner product

$$(A,B) = -\frac{1}{2}\lambda^2 Tr(AB) \tag{11}$$

on $\mathfrak{o}(m)$ defines a metric $d\bar{s}^2_\lambda$ on $G_p(m)$, invariant under the left action of $O(m)$ on $G_p(m)$. The choice of the same arbitrary positive constant λ in (8) and (11) implies that the isometry between $\mathbb{R}^m$ and T_xM, determined by an orthonormal frame in $x \in M$, extends to an isometry from $(G_p(m), d\bar{s}^2_\lambda)$ to $G_p(T_xM)$ (the fibre of $(G_p(TM), ds^2_\lambda)$ corresponding to x).

In particular, we have a *Riemannian product*:

$$(G_p(T\mathbb{R}^m), ds^2_\lambda) \cong \mathbb{R}^m \times (G_p(m), d\bar{s}^2_\lambda). \tag{12}$$

Recall that by *Vilms' Theorem* [1, (9.59), p. 249], ds^2_λ is the unique Riemannian metric on $G_p(TM)$ for which the projection $\Gamma : (G_p(TM), ds^2_\lambda) \to (M, g)$ is a Riemannian submersion with completely geodesic fibres isometric to $(G_p(m), d\bar{s}^2_\lambda)$ and a horizontal distribution associated to the Levi-Civita connection.

REMARKS 1. The canonical map of $G_p(TM)$ to $G_{m-p}(TM)$ which associates to each p-plane in T_xM the orthogonal $(m-p)$-plane is an isometry (with the same choice of the constant λ). This follows from (8) exchanging the indices a and r.

2. The unit tangent bundle T_1M of M can be identified (see [17]) with

$$T_1M = \frac{O(M)}{O(m-1)}.$$

Its metric is determined by (8) with $p = 1$ and coincides with the Sasaki metric if we assume $\lambda = 1$. Let us denote by $G_1(TM)$ the quotient of T_1M with respect to the equivalence relation identifying opposite unit vectors.

3. Obviously M can be identified with $G_m(TM)$, and from (8) it follows that this identification is an isometry.

2. The curvature of a Grassmann bundle

We denote by $\rho^X = (\rho^i, \rho^{ar})$ the forms belonging to the orthonormal coframe (9) of $G_p(TM)$, with $E_X = (E_i, E_{ar})$ the dual basis and with ρ^X_Y the forms associated to the Levi-Civita connection of $G_p(TM)$ determined by (10).

Starting from the structure equations

$$d\rho^X_Y + \rho^X_Z \wedge \rho^Z_Y = \frac{1}{2} R^G_{XYZT} \rho^Z \wedge \rho^T \tag{13}$$

by an elementary computation one can determine the components of the curvature tensor R^G of $G_p(TM)$:

$$R^G_{ijhk}(\pi) = \{R^M_{ijhk} + \tfrac{1}{2}\lambda^2 R^M_{arji} R^M_{arhk} - \tfrac{1}{4}\lambda^2 R^M_{arhi} R^M_{arkj} + \tfrac{1}{4}\lambda^2 R^M_{arki} R^M_{arhj}\}(\sigma[\pi]),$$

$$R^G_{ijh(ar)}(\pi) = \tfrac{1}{2}\lambda\{\nabla_h R^M_{arji}\}(\sigma[\pi]),$$

$$R^G_{ij(ar)(bs)}(\pi) = \{R^M_{rsji}\delta_{ab} - R^M_{abji}\delta_{rs} + \tfrac{1}{4}\lambda^2 R^M_{arki} R^M_{bsjk} - \tfrac{1}{4}\lambda^2 R^M_{bski} R^M_{arjk}\}(\sigma[\pi]),$$

$$R^G_{i(ar)h(bs)}(\pi) = \{\tfrac{1}{2}\lambda R^M_{srhi}\delta_{ab} - \tfrac{1}{2}\lambda R^M_{abhi}\delta_{rs} - \tfrac{1}{4}\lambda^2 R^M_{bsji} R^M_{arhj}\}(\sigma[\pi]),$$

$$R^G_{i(ar)(bs)(ct)}(\pi) = 0,$$

$$R^G_{(ar)(bs)(ct)(du)}(\pi) = \tfrac{1}{\lambda^2}\{\delta_{ab}\delta_{cd}(\delta_{rt}\delta_{su} - \delta_{ru}\delta_{st}) + \delta_{rs}\delta_{tu}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})\}(\sigma[\pi]).$$

From these expressions we obtain the components of the Ricci tensor Ric^G of $G_p(TM)$:

$$\mathrm{Ric}^G_{ih}(\pi) = \left\{\mathrm{Ric}^M_{ih} + \frac{1}{2}\lambda^2 R^M_{arji} R^M_{arhj}\right\}(\sigma[\pi]), \tag{14}$$

$$\mathrm{Ric}^G_{i(ar)}(\pi) = -\frac{1}{2}\lambda \nabla_j R^M_{jiar}(\sigma[\pi]), \tag{15}$$

$$\mathrm{Ric}^G_{(ar)(bs)}(\pi) = \left\{\frac{m-2}{\lambda^2}\delta_{ab}\delta_{rs} + \frac{1}{4}\lambda^2 R^M_{arji} R^M_{bsji}\right\}(\sigma[\pi]), \tag{16}$$

where the notation (ar) etc. is used only to separate the indices.

From (15) follows that the horizontal and the vertical distribution are orthogonal with respect to Ric^G and thus H is a Yang–Mills distribution (see [1, p. 243–244]) if M has harmonic curvature.

In the sequel, we shall assume that M has constant sectional curvature c, i.e.,

$$R^M_{ijhk} = c(\delta_{ih}\delta_{jk} - \delta_{ik}\delta_{jh}),$$

and we will examine the conditions under which $G_p(TM)$ is Einstein. From (14) and

(16) follows that the non-zero components of the tensor Ric^G are:

$$\mathrm{Ric}^G_{st} = \{c(m-1) - \tfrac{1}{2}\lambda^2 c^2 (m-p)\}\delta_{st},$$

$$\mathrm{Ric}^G_{ab} = \{c(m-1) - \tfrac{1}{2}\lambda^2 c^2 p\}\delta_{ab},$$

$$\mathrm{Ric}^G_{(ar)(bs)} = \{\tfrac{m-2}{\lambda^2} + \tfrac{1}{2}\lambda^2 c^2\}\delta_{ab}\delta_{rs}.$$

These last equations imply immediately that $G_p(TM)$ is Einstein if and only if

$$m = 2p, \tag{17}$$

$$c^2\lambda^4(p+1) - 2c\lambda^2(2p-1) + 4p - 4 = 0. \tag{18}$$

Equation (18) is consistent if and only if $p = 1$ and either $c\lambda^2 = 0$ or $c\lambda^2 = 1$. So we have:

PROPOSITION 1. *If M is a Riemannian manifold with constant sectional curvature, its Grassmann bundle $G_p(TM)$ is Einstein if and only if M is a quotient of the plane or the sphere S^2 and $p = 1$.*

On the other hand, it is well-known (see [12]) that $T_1(S^2)$, of which $G^1(S^2)$ is obviously a quotient, is isometric to $\mathbb{RP}^3$.

3. Sections of the Grassmann bundle

A distribution D of rank p on M determines a section ϕ of the Grassmann bundle

$$G_p(TM) \xrightarrow{\Gamma} M$$

in a natural way. It therefore appears reasonable to seek a relationship between geometrical properties of the distribution D, and those of the map ϕ between the Riemannian manifolds (M,g) and $(G_p(TM), ds^2_\lambda)$.

Afterwards, we will determine the conditions under which ϕ is harmonic.

Let us consider, as in Section 1, a section σ of the bundle $O(M) \xrightarrow{\Psi} G_p(TM)$. The distribution D determines a section

$$\sigma\cdot\phi : M \to O(M),$$

which means

$$(\sigma\cdot\phi)(x) = (x, \overline{e}_1, \ldots, \overline{e}_m)$$

where $(\overline{e}_1, \ldots, \overline{e}_m)$ is an orthonormal frame of T_xM in which the first p elements belong to $D_x \subset T_xM$.

In relation to the orthonormal coframe (9) of $G_p(TM)$, we have

$$\phi^*\rho^i = \phi^*\sigma^*\theta^i = \overline{\omega}^i, \tag{19}$$

$$\phi^*\rho^{ar} = \lambda\phi^*\sigma^*\omega^a_r = \lambda\overline{\Gamma}^a_{jr}\overline{\omega}^j. \tag{20}$$

($i = 1,\dots,m$, $r = 1,\dots,p$, $a = p+1,\dots,m$), where $(\overline{\omega}^i)$ is the coframe dual to $(\overline{e_i})$, and $\overline{\Gamma}^i_{jk}$ are the components of the Levi-Civita connection with respect to $(\overline{e_i})$, i.e.,

$$\overline{\Gamma}^i_{jk} = (\nabla^M_{\overline{e}_j}\overline{e}_k, \overline{e}_i).$$

From (19) and (20), it immediately follows that

$$\phi^*(ds^2_\lambda) = \sum(\overline{\omega}^i)^2 + \lambda^2\sum(\overline{\Gamma}^a_{jr}\overline{\omega}^j)^2.$$

Thus ϕ is an isometric immersion if and only if

$$\overline{\Gamma}^a_{jr} = 0,$$

or in other words ∇^M_X maps D into D for all $X \in TM$. Then we set

$$\rho^X = (\rho^i, \rho^{ar}), \qquad \phi^*(\rho^X) = a^X_j\overline{\omega}^j$$

and (19) and (20) directly imply that

$$a^i_j = \delta^i_j, \qquad a^{ar}_j = \lambda\overline{\Gamma}^a_{jr}.$$

We indicate the tension field of ϕ by

$$\tau(\phi) = \tau^i(\phi)E_i + \tau^{ar}(\phi)E_{ar},$$

with

$$\tau^H(\phi) = \tau^i(\phi)E_i, \qquad \tau^V = \tau^{ar}(\phi)E_{ar};$$

its components are determined following the method described in Appendix B, exploiting in particular (10). A simple computation leads to

$$\tau^i(\phi) = \lambda^2 R^M_{arji}\overline{\Gamma}^a_{jr}, \tag{21}$$

$$\tau^{ar}(\phi) = \lambda\{\overline{e}_j(\overline{\Gamma}^a_{jr}) - \overline{\Gamma}^a_{hr}\overline{\Gamma}^h_{jj} - \overline{\Gamma}^a_{js}\overline{\Gamma}^s_{jr} + \overline{\Gamma}^a_{jb}\overline{\Gamma}^b_{jr}\}. \tag{22}$$

From these relations we observe that if $\overline{\Gamma}^a_{jr} = 0$, i.e., $\nabla^M_X D \subseteq D$. The map ϕ, being isometric, is also harmonic and thus minimal.

The following section will give several examples of harmonic maps from M into $G_p(TM)$ which are non-trivial in the sense that they correspond to distributions that are not parallel. It is important to keep in mind:

PROPOSITION 2. *If the map $\phi : M \to G_p(TM)$ determines a harmonic distribution D, then the map $\phi^\perp : M \to G_{m-p}(TM)$ determined by the distribution $D^\perp$ is also harmonic.*

This result follows directly from (21) and (22) exchanging the role of the indices $a, b = p+1,\dots,m$ with $r, s = 1,\dots,p$.

The condition $\tau^{ar}(\phi)=0$ on its own characterizes the *vertically harmonic distributions* studied by C.M. Wood [25]. With a simple computation one can prove that the vanishing of (22) is equivalent (see [25], Theorem 1.11) to

$$\bar{\nabla}^*\bar{\nabla}d^{\perp}|D=0 \quad (=\bar{\nabla}^*\bar{\nabla}d|D^{\perp}),$$

where d and $d^{\perp}$ are respectively the projections on D and $D^{\perp}$, and $\bar{\nabla}$ is the connection determined over the vector bundles D and $D^{\perp}$ by the Levi-Civita connection on M, i.e.,

$$\bar{\nabla}_X v=\begin{cases} d(\nabla^M_X v) & \text{if } v\in D,\\ d^{\perp}(\nabla^M_X v) & \text{if } v\in D^{\perp}.\end{cases}$$

4. Examples of distributions with harmonic map into the Grassmann bundle

Example 1. The sphere S^3.

Consider the sphere S^3 in $\mathbb{R}^4$ given by

$$x_1^2+x_2^2+x_3^2+x_4^2=1,$$

and the orthonormal basis of S^3 formed by the vectors

$$e_1=(-x_2,x_1,x_4,-x_3),\quad e_2=(-x_3,-x_4,x_1,x_2),\quad e_3=(-x_4,x_3,-x_2,x_1).$$

Denoting by $(\omega^1,\omega^2,\omega^3)$ the dual basis, and by (ω^i_j) the matrix of the Levi-Civita connection, we easily obtain

$$\omega^2_1=\omega^3,\quad \omega^3_1=-\omega^2,\quad \omega^3_2=\omega^1,$$

exploiting mainly the fact that

$$[e_1,e_2]=2e_3,\quad [e_3,e_1]=2e_2,\quad [e_2,e_3]=2e_1.$$

Referring to (21) and (22), we have:

– the one-dimensional distributions determined by e_1,e_2,e_3 respectively are (non-trivial) harmonic sections of $G_1(TS^3)$;

– the two-dimensional distributions $\{e_1,e_2\}$, $\{e_1,e_3\}$, $\{e_2,e_3\}$ determine harmonic sections of $G_2(TS^3)$, in accordance with Proposition 2.

As S^3 can be identified with the group $Sp(1)$ of unit quaternions, it is easy to prove that e_1,e_2,e_3 form a basis of left-invariant vector fields. The metric of S^3 is bi-invariant and the Levi-Civita connection is given by

$$\nabla_X Y=\frac{1}{2}[X,Y],$$

where X and Y are left-invariant vector fields. Furthermore, it is easy to prove that every unit left-invariant vector field u, and so the two-dimensional distribution orthogonal to u, determines a harmonic section of $G_1(TS^3)$ and one of $G_2(TS^3)$.

Example 2. The three-dimensional Heisenberg group.

Let us consider the Heisenberg group (we refer the reader to [23, p. 72–74] for example), i.e., the subgroup of $GL(3,\mathbb{R})$ formed by the matrices

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, \tag{23}$$

endowed with the left-invariant metric

$$g = dx^2 + dz^2 + (dy - xdz)^2. \tag{24}$$

Considering the orthonormal coframe

$$\omega^1 = dx, \qquad \omega^2 = dz, \qquad \omega^3 = dy - xdz \tag{25}$$

with dual frame

$$e_1 = \frac{\partial}{\partial X}, \qquad e_2 = \frac{\partial}{\partial Z} + X\frac{\partial}{\partial Y}, \qquad e_3 = \frac{\partial}{\partial Y} \tag{26}$$

we easily obtain:

$$[e_1, e_2] = e_3, \qquad [e_2, e_3] = [e_3, e_1] = 0.$$

The connection forms of the Levi-Civita connection are

$$\omega_1^2 = -\frac{1}{2}\omega^3, \quad \omega_1^3 = -\frac{1}{2}\omega^2, \quad \omega_2^3 = \frac{1}{2}\omega^1$$

and the non-vanishing components of the curvature tensor are

$$R_{1212} = -\frac{3}{4}, \quad R_{1313} = R_{2323} = \frac{1}{4}. \tag{27}$$

Comparing with (21) and (22) we can prove that

– the one-dimensional distributions determined by e_1, e_2, e_3 and their orthogonal complements induce harmonic maps from H to $G_1(TH)$ and $G_2(TH)$;

– (with some computations) the only left-invariant unit vector fields that determine harmonic sections of $G_1(TH)$ are $\pm e_3$ and all the unit vectors of the plane $\{e_1, e_2\}$ (a situation quite different from the case of S^3).

Observe that $\{e_1, e_2\}$ describes a contact distribution on H that has been extensively studied for its remarkable geometric properties (we refer the reader to [15] and [20]).

Example 3. Three-dimensional unimodular Lie groups.

The groups S^3 and H are examples of three-dimensional unimodular Lie groups. The classification of these groups has been provided by J. Milnor [16]. For such a group G there exists a basis of left-invariant vector fields $\{e_1, e_2, e_3\}$ such that:

$$[e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad [e_1, e_2] = \lambda_3 e_3.$$

Considering on G a left-invariant metric with respect to which $\{e_1, e_2, e_3\}$ is an orthonormal basis and denoting by $\{\omega^1, \omega^3, \omega^3\}$ the dual basis, it is easy to prove that the Levi-Civita connection forms are

$$\omega_1^2 = \tfrac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3)\omega^3,$$
$$\omega_1^3 = \tfrac{1}{2}(-\lambda_1 + \lambda_2 - \lambda_3)\omega^2,$$
$$\omega_2^3 = \tfrac{1}{2}(\lambda_2 + \lambda_3 - \lambda_1)\omega^1\,.$$

Some computations determine the non-zero components of the curvature tensor

$$R_{ijij} = \frac{1}{4}\{\lambda_i^2 + \lambda_j^2 - 3\lambda_k^2 + 2\lambda_i\lambda_k + 2\lambda_j\lambda_k - 2\lambda_i\lambda_j\},$$

where $i \neq j \neq k$ assume the values $1, 2, 3$.

Referring to (21) and (22), it is easy to prove that e_1, e_2, e_3 (and the corresponding orthogonal distributions) determine harmonic maps from G to $G_1(TG)$ and $G_2(TG)$.

Example 4. Certainly the three-dimensional unimodular Lie groups do not exhaust the examples of groups with left-invariant metrics admitting harmonic distributions.

This is the case of a four-dimensional group with orthonormal left-invariant basis e_1, e_3, e_3, e_4 such that

$$[e_1, e_3] = e_4, \qquad [e_1, e_4] = -e_3$$

and all other commutators vanishing. It is easy to prove that this group is flat ($R = 0$) and the unique non-zero connection form is $\omega_3^4 = \omega^1$. We can verify for example that the two-dimensional distribution determined by $\{e_1, e_3\}$ is a non-parallel harmonic section of $G_2(TG)$.

II. ISOMETRIC IMMERSIONS, MAPS BETWEEN GRASSMANN BUNDLES

5. The map induced by an isometry between Grassmann bundles

A Riemannian immersion $f : M \to N$ induces in a natural way an immersion

$$F : G_p(TM) \to G_p(TN)$$

which associates to each p-plane tangent to M in a point x its image in $T_{f(x)}N$ via the differential of f. In the special case in which $p = m = \dim M$, F coincides with the Gauss map $\Upsilon : M \to G_m(TN)$.

We can define on $G_p(TN)$ (with $p \leq \dim M$) a metric $d\tilde{s}_\lambda^2$ in a way that is completely analogous to the one described in Section 1 in the case of $G_p(TM)$. The choice for the metric on $G_p(TN)$ with the same constant λ as on $G_p(TM)$ corresponds to rendering an isometry the inclusion of the fibre of $G_p(TM)$ into the fibre of $G_p(TN)$ relative to the same point $x \in M$. Recalling the discussion at the end of Section 1, this means that the immersion of $G_p(m)$ into $G_p(n)$ induced by the natural immersion of $\mathbb{R}^m$ in $\mathbb{R}^n$, namely $\mathbb{R}^m \to (\mathbb{R}^m, O) \subset \mathbb{R}^n$, is isometric.

Let us denote by $\tilde{\theta} = (\tilde{\theta}^A)$ and $\tilde{\omega} = (\tilde{\omega}^A_B)$ the $\mathbb{R}^n$-valued canonical form and the $\mathfrak{o}(n)$-valued form associated to the Levi-Civita connection defined on $O(N)$. Let $\tilde{\sigma}$ denote a section of the bundle

$$O(N) \xrightarrow{\tilde{\psi}} G_p(TN).$$

The Riemannian metric $d\tilde{s}^2_\lambda$ on $G_p(TN)$ is determined by the orthonormal coframe

$$\tilde{\rho}^A = \tilde{\sigma}^*\tilde{\theta}^A, \qquad \tilde{\rho}^{ar} = \lambda\tilde{\sigma}^*\tilde{\omega}^a_r, \qquad \tilde{\rho}^{\alpha r} = \lambda\tilde{\sigma}^*\tilde{\omega}^\alpha_r. \tag{28}$$

The indices involved in the previous equations for the entire second part of the present report will vary as follows:

$$\begin{aligned} &A, B, \ldots = 1, \ldots, n, \qquad i, j, \ldots = 1, \ldots, m, \qquad r, s, \ldots = 1, \ldots, p, \\ &a, b, \ldots = p+1, \ldots, m, \quad \alpha, \beta, \ldots = m+1, \ldots, n. \end{aligned}$$

In analogy to equation (10), the Levi-Civita connection forms for $(G_p(TN), d\tilde{s}^2_\lambda)$ are given by:

$$\begin{cases} \tilde{\rho}^A_B = -\tilde{\rho}^B_A = \tilde{\sigma}^*(\tilde{\omega}^A_B + \frac{1}{2}\lambda^2 R^N_{arBA}\tilde{\omega}^a_r + \frac{1}{2}\lambda^2 R^N_{\alpha rBA}\tilde{\omega}^\alpha_r) \\ \tilde{\rho}^A_{ar} = -\tilde{\rho}^{ar}_A = \tilde{\sigma}^*(\frac{1}{2}\lambda R^N_{arBA}\tilde{\theta}^B) \\ \tilde{\rho}^A_{\alpha r} = -\tilde{\rho}^{\alpha r}_A = \tilde{\sigma}^*(\frac{1}{2}\lambda R^N_{\alpha rBA}\tilde{\theta}^B) \\ \tilde{\rho}^{ar}_{bs} = -\tilde{\rho}^{bs}_{ar} = \tilde{\sigma}^*(\delta^a_b\tilde{\omega}^r_s + \delta^r_s\tilde{\omega}^a_b) \\ \tilde{\rho}^{ar}_{\beta s} = -\tilde{\rho}^{\beta s}_{ar} = \tilde{\sigma}^*(\delta^r_s\tilde{\omega}^a_\beta) \\ \tilde{\rho}^{\alpha r}_{\beta s} = -\tilde{\rho}^{\beta s}_{\alpha r} = \tilde{\sigma}^*(\delta^\alpha_\beta\tilde{\omega}^r_s + \delta^r_s\tilde{\omega}^\alpha_\beta). \end{cases} \tag{29}$$

Since we wish to explore the geometrical implications of an Riemannian immersion $f : M \to N$, we need to exploit the bundle of Darboux frames $O(N,M)$ along f (for more details see Appendix A).

It will be helpful to keep in mind the following diagram:

$$
\begin{array}{ccccc}
O(M) & \xleftarrow{\ s\ } & O(N,M) & \xrightarrow{\ k\ } & O(N) \\
\psi \downarrow\uparrow \sigma & \overset{\chi}{\nearrow}\ \underset{\eta}{\swarrow} & & & \tilde{\sigma} \uparrow\downarrow \tilde{\psi} \\
G_pTM & & \xrightarrow{\qquad F \qquad} & & G_p(TN) \\
\downarrow & & & & \downarrow \\
M & & \xrightarrow{\qquad f \qquad} & & N
\end{array}
$$

Here, $\eta = \psi \circ s$ is the submersion which associates to each adapted orthonormal frame $u = (x, u_1, \dots, u_p, u_{p+1}, \dots, u_m, u_{m+1}, \dots, u_n)$ the subspace $[u_1, \dots, u_p] \subset T_xM$.

Let χ be a local section of the bundle

$$O(N,M) \xrightarrow{\eta} G_p(TM)$$

defined on an open subset U of $G_p(TM)$; it determines a local section σ of

$$O(M) \xrightarrow{\psi} G_p(TM)$$

such that

$$\sigma = s \circ \chi; \tag{30}$$

let then $\tilde{\sigma}$ denote a local section of $O(N) \xrightarrow{\tilde{\psi}} G_p(TN)$ such that

$$\tilde{\sigma} \circ F = k \circ \chi. \tag{31}$$

Consider

$$\tilde{\rho}^{\Sigma} = (\tilde{\rho}^A, \tilde{\rho}^{ar}, \tilde{\rho}^{\alpha r}), \qquad \rho^X = (\rho^i, \rho^{ar}),$$

the 1-forms corresponding to the orthonormal coframes of $G_p(TN)$ and $G_p(TM)$ as in (28) and (9), and set

$$F^* \tilde{\rho}^{\Sigma} = a^{\Sigma}_X \rho_X. \tag{32}$$

In the sequel we will denote by $\overline{\theta}^A$ and $\overline{\omega}^A_B$ the forms induced on $O(N,M)$ by the forms $\tilde{\theta}^A$ and $\tilde{\omega}^A_B$ defined on $O(N)$ via the injection k, i.e.,

$$\overline{\theta}^A = k^* \tilde{\theta}^A, \qquad \overline{\omega}^A_B = k^* \tilde{\omega}^A_B. \tag{33}$$

Using equation (31) we obtain for example

$$\chi^* \overline{\theta}^i = \chi^* k^* \tilde{\theta}^i = F^* \tilde{\sigma}^* \tilde{\theta}^i = F^* \tilde{\rho}^i,$$

and from (30) we obtain

$$\chi^*\overline{\theta^i} = \chi^* s^* \theta^i = \sigma^* \theta^i = \rho^i,$$

which implies

$$F^*\tilde{\rho}^i = \rho^i.$$

Analogously

$$F^*\tilde{\rho}^\alpha = 0, \qquad F^*\tilde{\rho}^{ar} = \rho^{ar}, \qquad F^*\tilde{\rho}^{\alpha r} = \lambda\chi^*(h^\alpha_{rj}\overline{\theta}^j) = \lambda(\chi^* h^\alpha_{rj}\rho^j),$$

where the functions

$$\chi^* h^\alpha_{rj} = h^\alpha_{rj} \cdot \chi \tag{34}$$

evaluated on an element $[\pi] \in G_p(TM)$, are the components of the second fundamental form of the immersion f with respect to the adapted frame $\chi([\pi])$.

It follows that the coefficients a^Σ_X in (32) are given by

$$\begin{cases} a^i_j = \delta^i_j, & a^i_{ar} = 0 \\ a^\alpha_j = 0, & a^\alpha_{ar} = 0 \\ a^{ar}_j, & a^{ar}_{bs} = \delta^a_b \delta^r_s \\ a^{\alpha r}_j = \lambda\chi^*(h^\alpha_{rj}), & a^{\alpha r}_{bs} = 0. \end{cases} \tag{35}$$

Equations (35) imply that

$$F^* d\tilde{s}^2_\lambda = \sum(\rho^i)^2 + \sum(\rho^{ar})^2 + \lambda^2\left(\sum h^\alpha_{ri} h^\alpha_{rj} \cdot \chi\right)\rho^i\rho^j. \tag{36}$$

Since the metric on $G_p(TM)$ is given by

$$ds^2_\lambda = \sum(\rho^i)^2 + \sum(\rho^{ar})^2, \tag{37}$$

the map F is an *isometric immersion* of $(G_p(TM), ds^2_\lambda)$ in $(G_p(TN), d\tilde{s}^2_\lambda)$ *if and only if* f is *totally geodesic* (i.e., $h = 0$).

Furthermore, if $p < m$, the forms (36) and (37) are proportional if and only if they coincide and this occurs only in the case $h = 0$.

If $p = m$, from (36) and (37) (in which the forms ρ^{ar} do not appear any more) occurs that that $F = \Upsilon$ is *conformal* if and only if there exists a function ℓ on M such that

$$\sum_{k=1}^m h^\alpha_{ik} k^\alpha_{jk} = \ell^2 \delta_{ij}$$

so, setting

$$L(X,Y) = \sum_{k=1}^m (h(u_k, X), h(u_k, Y)), \quad X, Y \in TM, \tag{38}$$

we have

$$L(X,Y) = \ell^2 g(X,Y). \tag{39}$$

Observe that equation (39) is independent from the frame and equivalent to:

$$L(X,Y)=0, \qquad X\perp Y. \tag{40}$$

Keeping in mind the Gauss equations (see (100)) we obtain

$$R^N(u_k,u_i,u_k,u_j)=\mathrm{Ric}^M(u_i,u_j)+L(u_i,u_j)-mH\cdot h(u_i,u_j). \tag{41}$$

The bilinear form $\tilde{Q}_\lambda$ associated to $\Upsilon^* d\tilde{s}^2_\lambda$ is therefore given by:

$$\tilde{Q}_\lambda(X,Y)=(X,Y)+\lambda^2\{mH\cdot h(X,Y)-\mathrm{Ric}^M(X,Y)+R^N(u_k,X,u_k,Y)\}, \tag{42}$$

with $X,Y\in TM$.

Equation (42), and the conditions implying that the Gauss map Υ is conformal, are developed in the article of Jensen–Rigoli that we have already cited and also in [24]. Hence we get an extension of the results obtained by Obata in [18] expressed by

THEOREM 4 (Obata). *Assume that N has constant sectional curvature and*

1. *Υ is conformal,*
2. *M is Einstein,*
3. *M is pseudo-umbilical, i.e., $h(X,Y)\cdot H=h(X,Y)|H|^2$;*

then two of the above conditions imply the third.

In the case in which M is a surface in $\mathbb{R}^3$ (and thus $\mathrm{Ric}^M(X,Y)=Kg(X,Y)$ where K is the Gaussian curvature) we obtain the classical result, as observed in the Introduction of this report:

The Gauss map $M\to S^2$ is conformal if either M is a minimal surface, or M is contained in a sphere. In fact, these conditions are equivalent to being pseudo-umbilical in the case of surfaces in $\mathbb{R}^3$.

6. Tension field of the map induced between Grassmann bundles

For the computation of the tension field of F, we exploit the method described in Appendix B, and we set

$$Da^\Sigma_X\equiv da^\Sigma_X-a^\Sigma_Y\rho^Y_X+a^\Omega_X F^*\tilde{\rho}^\Sigma_\Omega=a^\Sigma_{XY}\rho^Y,$$

where a^Σ_X, ρ^Y_X and $\tilde{\rho}^\Sigma_\Omega$ are given by (35), (10) and (29), respectively. Recalling (10), (30), (33), we obtain

$$\begin{cases} \rho^i_j=\chi^*\left\{\overline{\omega}^i_j+\frac{1}{2}\lambda^2 R^M_{arji}\overline{\omega}^a_r\right\} \\[2ex] \rho^i_{ar}=\chi^*\left\{\frac{1}{2}\lambda R^M_{arji}\overline{\theta}^j\right\} \\[2ex] \rho^{ar}_{bs}=\chi^*\left\{\delta^a_b\overline{\omega}^r_s+\delta^r_s\overline{\omega}^a_b\right\}. \end{cases} \tag{43}$$

Then (29), (31), (33) imply

$$(44)\qquad \begin{cases} F^*\tilde{\rho}^A_B = \chi^*\{\overline{\omega}^A_B + \frac{1}{2}\lambda^2 R^N_{arBA}\overline{\omega}^a_r + \frac{1}{2}\lambda^2 R^N_{\alpha rBA}\overline{\omega}^\alpha_r\} \\ F^*\tilde{\rho}^A_{ar} = \chi^*\{\frac{1}{2}\lambda R^N_{arjA}\overline{\theta}^j\} \\ F^*\tilde{\rho}^A_{\alpha r} = \chi^*\{\frac{1}{2}\lambda R^N_{\alpha rjA}\overline{\theta}^j\} \\ F^*\tilde{\rho}^{ar}_{bs} = \chi^*\{\delta^a_b\overline{\omega}^r_s + \delta^r_s\overline{\omega}^a_b\} \\ F^*\tilde{\rho}^{ar}_{\beta s} = \chi^*\{\delta^r_s\overline{\omega}^\alpha_\beta\} \\ F^*\tilde{\rho}^{\alpha r}_{\beta s} = \chi^*\{\delta^r_s\overline{\omega}^\alpha_\beta + \delta^\alpha_\beta\overline{\omega}^r_s\}. \end{cases}$$

The components of the tension field $\tau(F)$ with respect to the orthonormal basis $\tilde{E}_\Sigma = \{\tilde{E}_i, \tilde{E}_\alpha, \tilde{E}_{ar}\tilde{E}_{\alpha r}\}$ dual of the basis (28) of $G_p(TN)$ are given by

$$\tau^\Sigma(F) = a^\Sigma_{XX}$$

and so, using equations (35), (43) and (44) we get

$$(45)\qquad \tau^i(F) = \lambda^2\chi^*(R^N_{\alpha rji}h^\alpha_{rj})$$

$$(46)\qquad \tau^\alpha(F) = \chi^*(h^\alpha_{jj} + \lambda^2 R^N_{\beta rj\alpha}h^\beta_{rj})$$

$$(47)\qquad \tau^{ar}(F) = -\lambda\chi^*(h^\alpha_{rj}h^\alpha_{aj})$$

$$(48)\qquad \tau^{\alpha r}(F) = (h^\alpha_{rjj}).$$

The computations leading to the previous equations are simple except for case (48), which we display explicitly:

$$(49)\qquad \begin{aligned} Da^{\alpha r}_j = {} & da^{\alpha r}_j - a^{\alpha r}_i\rho^i_j + a^i_j F^*\tilde{\rho}^{\alpha r}_i + a^{\beta s}_j F^*\tilde{\rho}^{\alpha r}_{\beta s} \\ = {} & \lambda\chi^*(dh^\alpha_{rj}) - \lambda\chi^*(h^\alpha_{ri})\chi^*(\overline{\omega}^i_j + \tfrac{1}{2}\lambda^2 R^M_{arji}\overline{\omega}^a_r) - \tfrac{1}{2}\lambda\chi^*(R^N_{\alpha rkj}\overline{\theta}^k) \\ & +\lambda\chi^*(h^\beta_{sj})\chi^*(\delta^r_s\overline{\omega}^\alpha_\beta + \delta^\alpha_\beta\overline{\omega}^r_s) \\ = {} & a^{\alpha r}_{jk}\rho^k + a^{\alpha r}_{j(bs)\rho^{bs}}. \end{aligned}$$

$$(50)\qquad \begin{aligned} Da^{\alpha r}_{bs} = {} & da^{\alpha r}_{bs} - a^{\alpha r}_j\rho^j_{bs} + a^{ct}_{bs}F^*\tilde{\rho}^{\alpha r}_{ct} \\ = {} & -\lambda\chi^*(h^\alpha_{rj})\chi^*(\tfrac{1}{2}\lambda R^M_{bskj}\overline{\theta}^k) + \chi^*(\delta^r_s h^\alpha_{bk}\overline{\theta}^k) \\ = {} & a^{\alpha r}_{(bs)k}\rho^k + a^{\alpha r}_{(bs)(ct)}\rho^{ct}. \end{aligned}$$

From (50), we obtain

$$a^{\alpha r}_{(bs)(ct)} = 0, \qquad a^{\alpha r}_{(bs)k} = \chi^*(h^\alpha_{bk}\delta^r_s - \frac{1}{2}\lambda^2 R^M_{bskj}h^\alpha_{rj}).$$

Since

$$a^{\alpha r}_{(bs)k} = a^{\alpha r}_{k(bs)},$$

substituting in (49) yields

$$\lambda\chi^*(dh^\alpha_{rj}) - \lambda\chi^*(h^\alpha_{ri}\overline{\omega}^i_j) - \tfrac{1}{2}\lambda\chi^*(R^N_{\alpha rkj}\overline{\theta}^k) + \lambda\chi^*(h^\beta_{rj}\overline{\omega}^\alpha_\beta - h^\alpha_{sj}\overline{\omega}^s_r)$$
$$= a^{\alpha r}_{jk}\chi^*(\overline{\theta}^k) + \chi^*(h^\alpha_{\beta j})\delta^r_s\chi^*(\lambda\overline{\omega}^b_s),$$

from which we get

$$a^{\alpha r}_{jk}\chi^*(\overline{\theta}^k) = \lambda\chi^*\left\{dh^\alpha_{rj} - h^\alpha_{ri}\overline{\omega}^i_j - h^\alpha_{ij}\overline{\omega}^i_r + h^\beta_{rj}\overline{\omega}^\alpha_\beta - \frac{1}{2}R^N_{\alpha rkj}\overline{\theta}^k\right\}.$$

From equation (103), it follows that

$$a^{\alpha r}_{jk} = \lambda\chi^*\left\{h^\alpha_{rjk} - \frac{1}{2}R^N_{\alpha rkj}\overline{\theta}^k\right\},$$

taking the trace of which we obtain equation (48).

With particular attention to (109), we also have

$$\tau^{\alpha r}(F) = \lambda\chi^*(m\nabla^\perp_r H^\alpha - R^N_{jrj\alpha}). \tag{51}$$

The vanishing of all the components of $\tau(F)$ corresponds to the fact that F is harmonic. On the other hand, the vanishing of $\tau^{ar}(F)$ and $\tau^{\alpha r}(F)$ is equivalent to being F *vertically harmonic*.

We postpone the analysis of these conditions in the relevant case in which N has constant sectional curvature to the following section. Here we provide an example of a Gauss map ($p = m$) which highlights the role of the curvature of N.

From equations (44) – (48) it follows naturally that, if M is a totally geodesic submanifold of N, the map F is harmonic irrespective of the curvature of N. For this reason, we will not consider this trivial case in the sequel.

Example.

In the three-dimensional Heisenberg group G (recall Section 4, **2**), we consider the isometrically immersed surface defined by the equation $y = 0$. In other words, this is the submanifold consisting of matrices of the type

$$\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is therefore generated by the product of two one-parameter subgroups of H.

We will show that S is a *minimal surface* in H whose *Gauss map* Υ *is conformal but neither harmonic nor vertically harmonic*.

Adopting the same notation as in Section 4, we consider the orthonormal basis $u = \{u_1, u_2, u_3\}$ of H defined as

$$u_1 = e_1, \qquad u_2 = \frac{1}{\sqrt{1+x^2}}(e_2 - xe_3), \qquad u_3 = \frac{1}{\sqrt{1+x^2}}(xe_2 + e_3).$$

It is easy to prove that the restriction $u|_S$ yields a Darboux frame with u_3 unit normal vector.

Denoting by h the second quadratic form of S in H, we get

$$h(u_1,u_1)\cdot u_3 = 0, \qquad h(u_1,u_2)\cdot u_3 = \frac{x^2-1}{2(x^2+1)}, \qquad h(u_2,u_2)\cdot u_3 = 0,$$

which implies that S is a minimal surface *not* totally geodesic of H.

The components of the curvature tensor R with respect to the frame u are

$$R_{1212} = \frac{x^2-3}{4(x^2+1)}, \quad R_{1213} = -\frac{x}{1+x^2}, \quad R_{1223} = 0,$$
$$R_{1313} = \frac{1-3x^2}{4(1+x^2)}, \qquad R_{1323} = 0, \qquad R_{2323} = \frac{1}{4},$$

where $R_{1212} = R(u_1,u_2,u_1,u_2)$ etc. Bearing in mind (45), (46) and (48), for the tension field of the Gauss map of S in $G_2(TH)$ we have

$$\tau^1(\Upsilon) = \lambda^2 \frac{x(1-x^2)}{2(1+x^2)^2}, \quad \tau^2(\Upsilon) = 0, \quad \tau^3(\Upsilon) = 0,$$
$$\tau^{3,1}(\Upsilon) = 0, \qquad \tau^{3,2}(\Upsilon) = \lambda \frac{x}{1+x^2},$$

which implies that Υ is neither harmonic nor vertically harmonic.

7. Harmonicity of the map between Grassmann bundles

We examine the different cases that can occur for the harmonicity of the map

$$F : (G_p(TM), ds^2_\lambda) \to G_p(TN), d\bar{s}^2_\lambda)$$

induced by a Riemannian immersion f of M in N. We will distinguish the case $p < m$ from $p = m$, and we will discuss with particular attention the case of constant sectional curvature on N.

Case 1. $p < m = \dim M$.

The vanishing of the components $\tau^{ar}(F)$ given in (47) corresponds to the fact that, for each couple of orthogonal vectors X, Y tangent to M, we have

(52) $$L(X,Y) = \sum h(u_j, X)\cdot h(u_j, Y) = 0, \qquad (X \perp Y).$$

This condition is equivalent (via (40)) to the fact that the Gauss map $\Upsilon : M \to G_p(TN)$ is harmonic.

Suppose that N has constant sectional curvature c, and let us distinguish further the case $c = 0$ from $c \neq 0$.

Subcase 1.1. $p < m$, $R^N = 0$.

We still obtain $\tau^i(F) = 0$; the condition $\tau^\alpha(F) = 0$ is equivalent to $H = 0$, i.e., that f is a minimal immersion. Under such hypothesis we have also $\tau^{ar} = 0$. Observe that if $H = 0$ and $R^N = 0$, equation (41) implies that

$$\mathrm{Ric}^M = -L \tag{53}$$

and for this reason the condition $\tau^{ar} = 0$ can be expressed by one of the following equivalent conditions:

– M is Einstein;

– the Gauss map $\Upsilon : M \to G_m(TN)$ is conformal (and thus recalling (53), homothetic).

This fact motivates the following:

PROPOSITION 3. *If N is a flat space, the map $F : G_p(TM) \to G_p(TN)$ with $p < \dim M$ is harmonic if and only if the following conditions are satisfied:*

– f is a minimal immersion;

– M is Einstein or (equivalently) the Gauss map $\Upsilon : M \to G_m(TN)$ is conformal (homothetic if $\dim M > 2$).

Under the same hypothesis, F is vertically conformal if and only if

– the mean curvature vector is parallel;

– the Gauss map is conformal.

Subcase 1.2. $p < m$, N has constant sectional curvature $c \neq 0$.

In this case the condition $\tau^i(F) = 0$ is still identically satisfied, but we have

$$\tau^\alpha(F) = mH^\alpha - c\lambda^2 \sum_{r=1}^{p} h^\alpha_{rr}.$$

For this reason, the condition $\tau^\alpha = 0$ is satisfied (independently of the choice of frame) if and only if for each vector X tangent to M we have that

$$h(X,X) = \frac{m}{cp\lambda^2} H.$$

Thus

$$\lambda^2 = \frac{m}{pc} \tag{54}$$

$$h(X,X) = H, \quad \forall X, \ |X| = 1, \tag{55}$$

conditions leading to $c > 0$ and then

$$h(X,Y) = g(X,Y)H, \qquad X,Y \in TM. \tag{56}$$

Equation (56) means that M should be a *totally umbilical submanifold* of N. The conditions $\tau^{ar}(F) = 0$ and $\tau^{\alpha r}(F) = 0$ are then identically satisfied (indeed M has constant curvature and $\nabla^{\perp}_X H = 0$, see for example [3, p. 50–51]).

Obviously a choice of λ different from (54) implies that F is harmonic only if $h = 0$, which means that M is a totally geodesic submanifold of N.

In conclusion,

PROPOSITION 4. *If N is a manifold with constant positive sectional curvature, the map $F : G_p(TM) \to G_p(TN)$ (with $p < \dim M$) is harmonic if either M is a totally geodesic submanifold of N, or the following conditions hold:*

– *M is a totally umbilical submanifold of N;*

– *the choice of the constant λ for the metric of $G_p(TN)$ is the same as in* (54)*.*

As in the case in which N is flat, *the map F is vertically conformal if and only if:*

– *the Gauss map is conformal;*

– *the mean curvature vector H is parallel.*

REMARK 1. The results we have obtained are substantially independent of the rank $p < \dim M$ of the Grassmannian bundles (except for the choice of the constant λ according to equation (54)). In this way we obtain results completely analogous to those proved in [22] for the case of unit tangent bundles.

Case 2. $p = m$ and thus $F = \Upsilon$.

In such a case, there is no component $\tau^{ar}(F)$. Assuming that N has constant sectional curvature, we distinguish the following subcases:

Subcase 2.1. $p = m$, $R^N = 0$.

We get always $\tau^i(F) = 0$ and $\tau^{\alpha}(F) = 0$ only if M is minimal, from which follows also $\tau^{\alpha r}(F) = 0$. From these considerations we deduce

PROPOSITION 5. *If N is flat, the Gauss map $\Upsilon : M \to G_m(TN)$ is harmonic if and only if M is a minimal submanifold of N.*

REMARK 2. If $N = \mathbb{R}^n$ it turns out that

$$(G_m(TN, d\tilde{s}^2_\lambda)) \cong \mathbb{R}^n \times (G_m(n), d\bar{s}^2_\lambda),$$

(recall from Section 1), where $F = \Upsilon = (f, \overline{\Upsilon})$ and $\overline{\Upsilon} : M \to G_m(n)$ is the generalized Gauss map. The *vertical* harmonicity of Υ coincides with the harmonicity of $\overline{\Upsilon}$ and may be expressed (recall (48)) by the condition $\nabla^{\perp} H = 0$. In such a way we recover the result of Ruh–Vilms presented in [21].

Subcase 2.2. $p = m$, N has constant sectional curvature $c \neq 0$.

The condition $\tau^i(F) = 0$ is always satisfied, and (46) implies that

$$\tau^\alpha(F) = m(1 - \lambda^2 c)H^\alpha$$

so either $H = 0$ and consequently $\tau^{\alpha r} = 0$, or

$$\lambda^2 = \frac{1}{c}, \qquad \nabla^\perp H = 0. \tag{57}$$

In conclusion,

PROPOSITION 6. *If N has non-zero constant sectional curvature, the Gauss map $\Upsilon : M \to G_m(TN)$ is harmonic if one of the following conditions holds:*

– M is a minimal submanifold of N;

– M has parallel mean curvature vector, N has positive curvature c and the metric of $G_m(TN)$ is obtained by setting $\lambda^2 = 1/c$.

Furthermore Υ is vertically harmonic if the mean curvature vector is parallel.

REMARK 3. When N is a sphere $S^n(r)$ (and so $c = 1/r^2$), besides the Gauss map $\Upsilon : M \to G_m(TS^n(r))$, T. Ishihara in [10] and M. Obata in [18] analyse other types of Gauss mappings related to the immersion of of $S^n(r)$ in $\mathbb{R}^{n+1}$. We can therefore consider the following mappings:

(i) $\Upsilon_1 : M \to G_m(n+1)$,
which associates to each point $x \in M$ the m-dimensional subspace of $\mathbb{R}^n$ parallel to the tangent space of T_xM;

(ii) $\Upsilon_2 : M \to G_{m+1}(n+1)$,
which associates to $x \in M$ the subspace of $\mathbb{R}^{n+1}$ singled out by the space tangent to M and the unit vector X/r; this is exactly the Gauss map introduced by Obata.

Regarding the harmonicity of these maps we have
(i)$'$ Υ_1 is harmonic if and only if the mean curvature vector H^* of M in $\mathbb{R}^{n+1}$ is parallel, in accordance with the Ruh–Vilms Theorem. Since the mean curvature H of M in $S^n(r)$ is determined by:

$$H = H^* + \frac{x}{r^2},$$

H^* parallel is equivalent to H parallel and thus Υ_1 *harmonic is equivalent to* Υ *vertically harmonic*.

(ii)$'$ The map Υ_2 is harmonic if and only if M is a minimal submanifold of $S^n(r)$ (see Theorem 4.8 in [10] where the Υ_1 is denoted by g_3). In conclusion, Υ_2 harmonic implies that Υ is harmonic.

III. THE SPHERICAL GAUSS MAP

A Riemannian immersion $f : M \to N$ induces a map $\nu : T_1^{\perp}M \to T_1 N$ between the unit normal bundle of M and the unit tangent bundle of N, defined by

$$\nu(x,v) = (f(x),v).$$

Jensen and Rigoli (see [11]) examine in particular the conditions under which ν is harmonic, exploiting a method analogous to the one adopted in [22] for the map induced by f on the unit tangent bundles.

In this part of the report, the analysis of the harmonic properties of ν will be developed using the *repère mobile* (moving frame) method that we have already adopted in the previous sections.

If $N = \mathbb{R}^n$, then $T_1 N \cong \mathbb{R}^n \times S^{n-1}$ and the map from $T_1^{\perp}M$ to S^{n-1} coincides with the map defined by Chern–Lashof in [6] for the study of the total curvature.

8. Riemannian structure of the normal unit bundle of a submanifold

Our aim is to examine the spherical Gauss map $\nu : T_1^{\perp}M \to T_1 N$,

$$\nu(x,v) = (f(x),v), \tag{58}$$

induced by the isometric immersion f of an m-dimensional submanifold M in an n-dimensional manifold N, so first of all we specify the metrics on $T_1^{\perp}M$ and $T_1 N$.
Since

$$T_1 N = \frac{O(N)}{O(n-1)},$$

its metric is the one already introduced in Section 5 for $G_p(TN)$ with $p = 1$, which we re-propose with a slight variation of the notation.

Let $\psi_n : O(N) \to T_1 N$ denote the canonical submersion

$$\psi_n(y,u_1,\ldots,u_n) = (y,u_n), \tag{59}$$

and define on $T_1 N$ the metric $d\tilde{s}^2_{\lambda}$ so that

$$\psi_n^* \tilde{s}^2_{\lambda} = \sum (\tilde{\theta}^A)^2 + \lambda^2 \sum (\tilde{\omega}^a_n)^2, \tag{60}$$

where $A = 1,\ldots,n$, $a = 1,\ldots,n-1$, and $(\tilde{\theta}^A)$, $(\tilde{\omega}^A_B)$ denote as usual the canonical form and the Levi-Civita connection form on $O(N)$.

Given a local section $\tilde{\sigma}$ of the bundle $O(N) \xrightarrow{\psi_n} T_1 N$, the 1-forms

$$\rho^A = \tilde{\sigma}^* \tilde{\theta}^A, \qquad \tilde{\rho}^{an} = \lambda \tilde{\sigma}^* \tilde{\omega}^a_n \tag{61}$$

give an orthonormal coframe of T_1N. The forms associated to the Levi-Civita connection of $(T_1N, d\tilde{s}^2_\lambda)$ computed with respect to this coframe are:

$$(62)\qquad \begin{cases} \tilde{\rho}^A_B = -\tilde{\rho}^B_A = \tilde{\sigma}^*\{\tilde{\omega}^A_B + \frac{1}{2}\lambda^2 R^N_{anBA}\tilde{\omega}^a_n\} \\ \tilde{\rho}^A_{an} = -\tilde{\rho}^{an}_A = \tilde{\sigma}^*\{\frac{1}{2}\lambda R^N_{anBA}\tilde{\theta}^B\} \\ \tilde{\rho}^{an}_{bn} = -\tilde{\rho}^{bn}_{an} = \tilde{\sigma}^*(\tilde{\omega}^a_b)\,. \end{cases}$$

To determine the metric on $T_1^\perp M$, we consider the submersion π_n of the bundle of Darboux frames $O(N,M)$ on $T_1^\perp M$ defined by

$$(63)\qquad \pi_n(x, u_1, \ldots, u_m, u_{m+1}, \ldots, u_n) = (x, u_n)\,.$$

Observe that the fibres of π_n are diffeomorphic to $O(m)\times O(n-m-1)$ immersed in $O(n)$ as follows:

$$(64)\quad (a', a'') \mapsto a = \begin{pmatrix} a' & 0 & 0 \\ 0 & a'' & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \ a' = (a^i_k) \in O(m),\ \ a'' = (a^\alpha_\beta) \in O(n-m-1),$$

with $i,k = 1,\ldots,m$ and $\alpha,\beta = m+1,\ldots,n-1$.

Therefore

$$T_1^\perp M = \frac{O(N,M)}{O(m)\times O(n-m-1)}.$$

Denoting by κ the canonical immersion $O(N,M)\to O(N)$ and setting

$$\overline{\theta}^A = \kappa^*\tilde{\theta}^A, \qquad \overline{\omega}^A_B = \kappa^*\tilde{\omega}^A_B,$$

we consider on $O(N,M)$ the quadratic form

$$(65)\qquad \overline{Q} = \sum(\overline{\theta}^i)^2 + \lambda^2\sum(\overline{\omega}^\alpha_n)^2.$$

For each $a \in O(m)\times O(n-m)$ of type (64), we have

$$(66)\qquad R_a\cdot\kappa = \kappa\cdot R_a.$$

If a is of the form specified by (64) then

$$R^*_a\overline{\theta}^i = (a^{-1})^i_k\overline{\theta}^k, \qquad R^*_a\overline{\omega}^\alpha_n = (a^{-1})^\alpha_\beta\overline{\omega}^\beta_n,$$

and hence

(i) $\overline{Q}$ is invariant under the right action of $O(m)\times O(n-m-1)$ on $O(N,M)$.
(ii) $\overline{Q}$ is a semidefinite positive form on $O(N,M)$ of rank equal to $n-1$, the dimension of $T_1^\perp M$.
(iii) The bilinear form associated to $\overline{Q}$ annihilates the vertical vector fields of the submersion π_n.

For this reason, there exists unique Riemannian metric ds^2_λ on $T_1^\perp M$ such that

$$\pi_n^* ds^2_\lambda = \overline{Q},$$

and this is the metric on $T_1^\perp M$ that we will refer to in the sequel of the section. Given a local section χ of $O(N,M) \xrightarrow{\pi_n} T_1 M$, i.e.,

$$\chi(x,v) = (x, u_1, \ldots, u_m, u_{m+1}, \ldots, u_{n-1}, v),$$

the 1-forms

$$\rho^i = \chi^* \overline{\theta}^i, \qquad \rho^{\alpha n} = \lambda \chi^* \overline{\omega}^\alpha_n \tag{67}$$

constitute an orthonormal basis of $T_1^\perp M$. Since we have

$$\overline{\omega}^\alpha_i = \kappa^* \tilde{\omega}^\alpha_i = h^\alpha_{ij} \overline{\theta}^j, \qquad \overline{\omega}^n_i = h^n_{ij} \overline{\theta}^j, \tag{68}$$

a standard computation leads to the following expression of the Levi-Civita connection forms on $(T^\perp M, ds^2_\lambda)$ with respect to the orthonormal frame (67):

$$\begin{cases} \rho^i_j = -\rho^j_i = \chi^* \{ \overline{\omega}^i_j + \frac{1}{2}\lambda^2 (R^N_{\alpha n j i} + h^\alpha_{kj} h^n_{ki} - h^\alpha_{ki} h^n_{kj}) \overline{\omega}^\alpha_n \} \\ \rho^i_{\alpha n} = -\rho^{\alpha n}_i = \frac{1}{2}\lambda \chi^* \{ (R^N_{\alpha n j i} + h^\alpha_{kj} h^n_{ki} - h^\alpha_{ki} h^n_{kj}) \overline{\theta}^j \} \\ \rho^{\alpha n}_{\beta n} = -\rho^{\beta n}_{\alpha n} = \chi^* (\overline{\omega}^\alpha_\beta). \end{cases} \tag{69}$$

Also, considering the normal curvature tensor $R^\perp$, and referring to (101),

$$\begin{cases} \rho^i_j = -\rho^j_i = \chi^* \{ \overline{\omega}^i_j + \frac{1}{2}\lambda^2 R^\perp_{\alpha n j i} \overline{\omega}^\alpha_n \} \\ \rho^i_{\alpha n} = -\rho^{\alpha n}_i = \chi^* \{ \frac{1}{2}\lambda R^\perp_{\alpha n j i} \overline{\theta}^j \} \\ \rho^{\alpha n}_{\beta n} = -\rho^{\beta n}_{\alpha n} = \chi^* (\overline{\omega}^\alpha_\beta). \end{cases} \tag{70}$$

REMARK 4. If M is a hypersurface of N, equations (67) imply directly that $T_1 M$ is isometric to M.

9. The tension field of the spherical Gauss map

Assume that the section $\tilde{\sigma}$ of $O(N) \xrightarrow{\psi_n} T_1 N$ is chosen in a way that

$$\tilde{\sigma} \circ \nu = \kappa \circ \nu \tag{71}$$

and so we are in the situation described by the following diagram:

$$
\begin{array}{ccc}
O(N,M) & \xrightarrow{\kappa} & O(N) \\
\pi_n \big\downarrow \big\uparrow \chi & & \tilde{\sigma} \big\uparrow \big\downarrow \psi_n \\
T_1^{\perp}M & \xrightarrow{\nu} & T_1 N \\
\big\downarrow & & \big\downarrow \\
M & \xrightarrow{f} & N.
\end{array}
$$

Considering (71), the orthonormal coframes of T_1N and $T^{\perp}M$ defined by (61) and (67) satisfy

$$
\begin{aligned}
&\nu^*\tilde{\rho}^i = \rho^i, \qquad \nu^*\tilde{\rho}^\alpha = 0, \qquad \nu^*\tilde{\rho}^n = 0,\\
&\nu^*\tilde{\rho}^{in} = -\lambda\chi^*\overline{\omega}_i^n = \lambda\chi^*(h_{ij}^n)\rho^j, \qquad (72)\\
&\nu^*\tilde{\rho}^i = \lambda\chi^*\overline{\omega}_n^\alpha = \rho^{\alpha n}.
\end{aligned}
$$

In other words, denoting the coframes of T_1N and $T^{\perp}M$ by $\tilde{\rho}^\Sigma = (\tilde{\rho}^i, \tilde{\rho}^\alpha, \tilde{\rho}^n, \tilde{\rho}^{in}, \tilde{\rho}^{\alpha n})$ and $\rho^X = (\rho^i, \rho^{\alpha n})$, we set

$$
\nu^*\tilde{\rho}^\Sigma = a_X^\Sigma \rho^X. \qquad (73)
$$

It follows that

$$
\left\{
\begin{array}{ll}
a_j^i = \delta_j^i, & a_{\alpha n}^i = 0\\
a_j^\alpha = 0, & a_{\beta n}^\alpha = 0\\
a_j^n = 0, & a_{\alpha n}^n = 0\\
a_j^{in} = -\lambda\chi^*(h_{ij}^n), & a_{\alpha n}^{in} = 0\\
a_j^{\alpha n} = 0 & a_{\alpha n}^{in} = \delta_\beta^\alpha.
\end{array}
\right. \qquad (74)
$$

From the above relations, we deduce that

$$
\nu^* d\tilde{s}_\lambda^2 = \sum(\rho^i)^2 + \sum(\rho^{\alpha n})^2 + \lambda^2 \sum(h_{ij}^n h_{ik}^n \cdot \chi)\rho^j \rho_k. \qquad (75)
$$

Since $ds_\lambda^2 = \Sigma(\rho^i)^2 + \Sigma(\rho^{\alpha n})^2$ is the metric on $T_1^{\perp}M$, it follows that:

(i) ν is an isometry only if $h = 0$, i.e., M is a totally geodesic submanifold of N;

(ii) if $\dim N - \dim M \geq 2$ and if $h \neq 0$ the metric $\nu^* d\tilde{s}_\lambda^2$ cannot be conformal or in particular homothetic to ds_λ^2;

(iii) if M is a hypersurface (in such a case we should not consider the forms $\rho^{\alpha n}$) the metrics $\nu^* d\tilde{s}_\lambda^2$ and ds_λ^2 are mutually conformal if and only if on M there exists a function ℓ such that

$$
h_{ij}^n h_{ik}^n = \ell^2 \delta_{jk}, \qquad (76)
$$

which is the same condition as (39).

Equation (76) is equivalent to asserting that the absolute values of the principal curvatures of M are equal. This can be easily proved using an orthonormal frame on M diagonalising the matrix (h^n_{ij}).

The tension field of the map ν is determined as usual following the method described in Appendix B, by setting

$$Da^{\Sigma}_X \equiv da^{\Sigma}_X - a^{\Sigma}_Y \rho^Y_X + a^{\Omega}_X \nu^*(\tilde{\rho}^{\Sigma}_{\Omega}) = a^{\Sigma}_{XY}\rho^Y. \tag{77}$$

Thus

$$\tau^{\Sigma}(\nu) = a^{\Sigma}_{XX}, \tag{78}$$

where the coefficients a^{Σ}_X are given by (74), ρ^Y_X are the coefficients of the Levi-Civita connection on $T^{\perp}M$ (see (69)) and the forms $\nu^*(\tilde{\rho}^{\Sigma}_{\Omega})$ can be computed starting from (62) and using (71), (72).

Simple computations lead to

$$\tau^i(\nu) = -\lambda^2\chi^*(R^N_{knji}h^n_{kj}) \tag{79}$$

$$\tau^{\alpha}(\nu) = \chi^*(h^{\alpha}_{jj} - \lambda^2 R^N_{inj\alpha}h^n_{ij}) \tag{80}$$

$$\tau^n(\nu) = \chi^*(h^n_{jj} - \lambda^2 R^N_{injn}h^n_{ij}) \tag{81}$$

$$\tau^{in}(\nu) = -\lambda\chi^*(h^n_{ijj}) = -\lambda\chi^*(m\nabla^{\perp}_i H^n - R^N_{jijn}) \tag{82}$$

$$\tau^{\alpha n}(\nu) = -\lambda\chi^*(h^n_{ij}h^{\alpha}_{ij}). \tag{83}$$

The computation of the components of $\tau(\nu)$ is simple. The only slightly more complicated case is (82), which is treated in a manner analogous to (48).

10. Harmonicity of the spherical Gauss map

We now examine the conditions under which the spherical Gauss map ν is harmonic, devoting particular attention to the case in which N has constant sectional curvature.

With the intention of interpreting the vanishing of the components $\tau^{\alpha n}(\nu)$ given by (83) (which makes sense only if $n-m \geq 2$), we consider for each element $v \in T^{\perp}M$ the symmetric 2-form

$$h_v = h \cdot v,$$

or better

$$h_v(X,Y) = h(X,Y)\cdot v. \tag{84}$$

We then obtain $\tau^{\alpha n}(\nu) = 0$ if and only if

$$h_v \cdot h_w = \sum h_v(u_i,u_j)\cdot h_w(u_i,u_j) = 0, \qquad \forall v,w \in T^{\perp}_1 M,\ \ v \perp w, \tag{85}$$

for any orthonormal frame $\{u_i\}$ on M. There should therefore exist a function μ on M such that

$$h_v \cdot h_w = \mu^2 (v,w), \qquad \forall v,w \in T_1^{\perp} M, \tag{86}$$

which implies in particular that

$$\|h\|^2 = \sum h_{v_\alpha} \cdot h_{v_\alpha} = (n-m)\mu^2,$$

where $\{v_\alpha\}$ is an orthonormal basis of $T_1^{\perp}M$. In the language introduced in [11], equation (86) means that *the second fundamental form is conformal*.

Assuming that N has constant sectional curvature c, then the tangential component $\tau^i(\nu)$ always vanishes, while

$$\tau^\alpha(\nu) = mH^\alpha, \qquad \tau^n(\nu) = m(1-\lambda^2 c)H^n, \tag{87}$$

so both these expressions should be zero in order for ν to be harmonic. For this reason if $n-m \geq 2$, then $H = 0$, from which it follows that $\tau^{\alpha n}(\nu) = 0$. In conclusion:

PROPOSITION 7. *Let N be a manifold with constant sectional curvature. The spherical Gauss map $\nu : T_1^{\perp}M \to T_1 N$ induced by the isometric immersion f of M in N, with* $\dim N - \dim M \geq 2$*, is harmonic if and only the following conditions hold:*

1. *f is minimal;*
2. *the second fundamental form is conformal.*

Example.

If M is a minimal surface in N, with $\dim N \geq 4$, an orthonormal frame of M can be chosen in a way that $h^\alpha_{ij} = 0$ for $\alpha > 4$. The minimality conditions reduce to

$$h^3_{11} + h^3_{22} = 0, \qquad h^4_{11} + h^4_{22} = 0,$$

and with a suitable choice of an orthonormal basis of M we can assume $h^3_{12} = 0$. The condition that the second fundamental form is conformal leads to

$$h^4_{11} = h^4_{22} = 0, \qquad (h^3_{11})^2 = (h^4_{12})^2.$$

This implies that M should be a *minimal isotropic surface*, i.e., $|h(X,X)| =$ constant for $|X| = 1$.

Otherwise, if M is a *hypersurface* inside the manifold N with constant curvature c, the components of $\tau(\nu)$ are determined by (79), (81) and (82). The components not identically zero are:

$$\tau(\nu) = m(1-\lambda^2 c)H^n, \qquad \tau^{in}(\nu) = -\lambda m \nabla_i^{\perp} H^n,$$

and so the spherical Gauss map ν is harmonic if and only if

$$(1-\lambda^2 c)H = 0, \qquad \nabla_i^{\perp} H = 0. \tag{88}$$

We conclude that:

1. either N is flat and M is a minimal hypersurface; or
2. N is not flat,

$$c > 0, \qquad \lambda^2 = \frac{1}{c}, \qquad \nabla_i^{\perp} H = 0,$$

and so M is a hypersurface with mean curvature vector with constant norm, a condition equivalent in such a case to $\nabla_i^{\perp} H = 0$.

REMARK 5. The spherical Gauss map ν associated to the Riemannian submersion $T_1 N \to N$, is called *vertically harmonic* if the vertical component of $\tau(\nu)$ vanishes, i.e.,

$$\tau^{in}(\nu) = 0, \qquad \tau^{\alpha n}(\nu) = 0.$$

Equations (82) and (83) imply therefore that under the hypothesis that N has constant sectional curvature, ν is vertically harmonic if the following conditions are satisfied:

1. the mean curvature vector is parallel;
2. the second fundamental form is conformal.

If M is a hypersurface, the only condition is $|H| = \text{constant}$.

Appendix A. Darboux frames

Consider a Riemannian immersion f of an m-dimensional manifold M in an n-dimensional manifold N. We denote by $O(M)$ and $O(N)$ the principal bundles of orthonormal frames on M and N, with structure groups $O(m)$, $O(n)$.

We denote by $\theta = (\theta^i)$ and $\omega = (\omega^i_j)$ $(i, j = 1, \ldots, m)$ the canonical $\mathbb{R}^m$-valued form on $O(M)$ and the $\mathfrak{o}(m)$-valued form associated to the Levi-Civita connection on M respectively. Then $\tilde{\theta} = (\tilde{\theta}^i)$ and $\tilde{\omega} = (\tilde{\omega}^i_j)$, $(A, B = 1, \ldots, n)$ denote the analogous forms on $O(N)$. Hence,

$$d\theta^i = -\omega^i_j \wedge \theta^j \qquad (\omega^i_j + \omega^j_i = 0), \tag{89}$$

$$d\omega^i_j = -\omega^i_k \wedge \omega^k_j + \tfrac{1}{2} R^M_{ijhk} \theta^h \theta^k. \tag{90}$$

Given an orthonormal frame $u = (x, u_1, \ldots, u_m)$ of $O(M)$, we have

$$R^M_{ijhk}(u) = R^M(u_i, u_j, u_h, u_k) = ((\nabla^M_{[u_i, u_j]} - \nabla^M_{u_i} \nabla^M_{u_j} + \nabla^M_{u_j} \nabla^M_{u_i}) u_h, u_k). \tag{91}$$

Similarly,

$$d\tilde{\theta}^A = -\tilde{\omega}^A_B \wedge \tilde{\theta}^B, \qquad (\tilde{\omega}^A_B + \tilde{\omega}^B_A = 0), \tag{92}$$

$$d\tilde{\omega}^A_B = -\tilde{\omega}^A_C \wedge \tilde{\omega}^B_C + \tfrac{1}{2} R^N_{ABCD} \tilde{\theta}^C \wedge \tilde{\theta}^D. \tag{93}$$

Identifying M with its image $f(M)$ in N, the bundle of Darboux frames $O(N, M)$ along f is the bundle on M defined as follows. An element

$$u = (x, u_1, \ldots, u_m, u_{m+1}, \ldots, u_n), \quad x = \delta(u),$$

of $O(N,M)$ (where δ is the canonical projection of $O(N,M)$ on M) is such that

$$u' = (x, u_1, \dots u_m), \qquad u'' = (x, u_{m+1}, \dots, u_n)$$

are orthonormal frames of respectively T_xM and $T_xM^{\perp}$ (the subspace of T_xN orthogonal to T_xM with respect to the metric of T_xN).

The structure group $O(m)\times O(n-m)$ of $O(N,M)$ is naturally immersed in $O(n)$ as follows:

$$(a', a'') \in O(m)\times O(n-m) \mapsto \begin{pmatrix} a' & 0 \\ 0 & a'' \end{pmatrix} \in O(n).$$

Let $s : O(N,M) \to O(M)$ and $\kappa : O(N,M) \to O(N)$ denote the submersion and the natural immersion defined by the diagram

$$\begin{array}{ccccc} O(M) & \xleftarrow{s} & O(N,M) & \xrightarrow{\kappa} & N \\ \Big\downarrow \pi & & \Big\downarrow \delta & & \Big\downarrow \tilde{\pi} \\ M & \longleftarrow & M = f(M) & \longrightarrow & N \end{array}$$

Observe that (we refer the reader to [13, vol. II, p. 3–4]):

$$\kappa^*\tilde{\theta}^i = s^*\theta^i = \overline{\theta}^i, \qquad i = 1, \dots, m, \tag{94}$$

$$\kappa^*\tilde{\theta}^\alpha = 0, \qquad \alpha = m+1, \dots, n. \tag{95}$$

If we set

$$\kappa^*(\tilde{\omega}^A_B) = \overline{\omega}^A_B,$$

then differentiating (94), we obtain

$$\overline{\omega}^i_j = \kappa^*\tilde{\omega}^i_j = s^*\omega^i_j.$$

The forms $(\omega', \omega'') = (\overline{\omega}^i_j, \overline{\omega}^\alpha_\beta)$, representing the $\mathfrak{o}(m)$ and the $\mathfrak{o}(n-m)$ components of $\kappa^*\tilde{\omega}$, define a connection on $O(N,M)$. Given a local section χ of

$$O(N,M) \xrightarrow{\delta} M,$$

i.e., a local field of frames

$$\chi(x) = (x, e_1, \dots, e_m, e_{m+1}, \dots, e_n) \tag{96}$$

adapted along the immersion f, it obviously follows that

$$(\chi^*\overline{\omega}^i_j)(X) = (\nabla^N_X e_j, e_i) = (\nabla^M_X e_j, e_i)$$

$$(\chi^*\overline{\omega}^\alpha_\beta)(X) = (\nabla^N_X e_\beta, e_\alpha) = (\nabla^\perp_X e_\beta, e_\alpha),$$

where $\nabla^\perp$ is the connection on $T^\perp M$ singled out by $\omega'' = (\overline{\omega}^\alpha_\beta)$.

Differentiating equation (95) we have

$$0 = \kappa^*(d\tilde{\theta}^\alpha) = -\kappa^*\tilde{\omega}_i^\alpha \wedge \kappa^*\tilde{\theta}^i = \overline{\omega}_i^\alpha \wedge \overline{\theta}^i,$$

from which *Cartan's Lemma* implies that

$$\overline{\omega}_i^\alpha = h_{ij}^\alpha \overline{\theta}^j, \qquad h_{ij}^\alpha = h_{ji}^\alpha,$$

where h_{ij}^α are the components of the second fundamental form h of the immersion f of M in N, i.e.,

$$h_{ij}^\alpha(u) = (\nabla_{u_i}^N u_j, u_\alpha).$$

Differentiating the equations

$$\kappa^*\tilde{\omega}_j^i = \overline{\omega}_j^i \tag{97}$$

$$\kappa^*\tilde{\omega}_\beta^\alpha = \overline{\omega}_\beta^\alpha \tag{98}$$

$$\kappa^*\tilde{\omega}_i^\alpha = h_{ij}^\alpha \overline{\theta}^j, \tag{99}$$

we obtain the *equations of Gauss, Ricci, Codazzi*. Indeed

$$\begin{aligned} \kappa^* d\tilde{\omega}_j^i &= \kappa^*(-\tilde{\omega}_k^i \wedge \tilde{\omega}_j^k - \tilde{\omega}_\alpha^i \wedge \tilde{\omega}_j^\alpha + \tfrac{1}{2}R_{ijAB}^N \tilde{\theta}^A \wedge \tilde{\theta}^B) \\ &= -\overline{\omega}_k^i \wedge \overline{\omega}_j^k + h_{ih}^\alpha h_{jk}^\alpha \overline{\theta}^h \wedge \overline{\theta}^k + \tfrac{1}{2}R_{ijhk}^N \overline{\theta}^h \wedge \overline{\theta}^k. \end{aligned}$$

From another point of view,

$$\begin{aligned} \kappa^* d\tilde{\omega}_j^i &= s^* d\overline{\omega}_j^i = s^*(-\omega_k^i \wedge \omega_j^k + \tfrac{1}{2}R_{ijhk}^N \theta^h \wedge \theta^k) \\ &= -\overline{\omega}_k^i \wedge \overline{\omega}_j^k + \tfrac{1}{2}R_{ijhk}^N \overline{\theta}^h \wedge \overline{\theta}^k. \end{aligned}$$

Comparing the last expressions we obtain the following (Gauss equations):

$$R_{ijhk}^M = R_{ijhk}^N + h_{ih}^\alpha h_{jk}^\alpha - h_{ik}^\alpha h_{jh}^\alpha. \tag{100}$$

Similarly, from (98), we have

$$\begin{aligned} \kappa^* d\tilde{\omega}_\beta^\alpha &= \kappa^*(-\tilde{\omega}_i^\alpha \wedge \tilde{\omega}_\beta^i - \tilde{\omega}_\gamma^\alpha \wedge \tilde{\omega}_\beta^\gamma + \tfrac{1}{2}R_{\alpha\beta AB}^N \tilde{\theta}^A \wedge \tilde{\theta}^B) \\ &= h_{ih}^\alpha h_{jk}^\beta \overline{\theta}^h \wedge \overline{\theta}^k - \overline{\omega}_\gamma^\alpha \wedge \overline{\omega}_\beta^\gamma + \tfrac{1}{2}R_{\alpha\beta hk}^N \overline{\theta}^h \wedge \overline{\theta}^k. \end{aligned}$$

Then setting

$$d\overline{\omega}_\beta^\alpha = -\overline{\omega}_\gamma^\alpha \wedge \overline{\omega}_\beta^\gamma + \frac{1}{2}R_{\alpha\beta hk}^\perp \overline{\theta}^h \wedge \overline{\theta}^k,$$

where $R^\perp$ is the curvature tensor of the connection $\nabla^\perp$ on $T^\perp M \to M$, we find the Ricci equations

$$R_{\alpha\beta hk}^\perp = R_{\alpha\beta hk}^N + h_{ih}^\alpha h_{ik}^\beta - h_{ik}^\alpha h_{ih}^\beta. \tag{101}$$

Finally, differentiating equation (99), we have

$$\begin{aligned}\kappa^* d\tilde{\omega}_i^\alpha &= \kappa^*(-\tilde{\omega}_h^\alpha \wedge \tilde{\omega}_i^h - \tilde{\omega}_\beta^\alpha \wedge \tilde{\omega}_i^\beta + \tfrac{1}{2} R^N_{\alpha iAB} \tilde{\theta}^A \wedge \tilde{\theta}^B) \\ &= -h^\alpha_{hk} \overline{\theta}^k \wedge \overline{\omega}_i^h - h^\beta_{ik} \overline{\omega}_\beta^\alpha \wedge \overline{\theta}^k + \tfrac{1}{2} R^N_{\alpha ijk} \overline{\theta}^h \wedge \overline{\theta}^k.\end{aligned}$$

From another point of view, we have

$$d(h^\alpha_{ij}\overline{\theta}^j) = dh^\alpha_{ij} \wedge \overline{\theta}^j - h^\alpha_{ih} \overline{\omega}_j^h \wedge \overline{\theta}^j.$$

Comparing the last equations we obtain:

$$(dh^\alpha_{ik} - h^\alpha_{ik}\overline{\omega}_k^h - h^\alpha_{hk}\overline{\omega}_i^h + h^\beta_{ik}\overline{\omega}_\beta^\alpha - \frac{1}{2} R^N_{\alpha ihk}\overline{\theta}^h) \wedge \overline{\theta}^k = 0.$$

Cartan's Lemma implies

$$dh^\alpha_{ik} - h^\alpha_{ik}\overline{\omega}_k^h - h^\alpha_{hk}\overline{\omega}_i^h + h^\beta_{ik}\overline{\omega}_\beta^\alpha - \frac{1}{2} R^N_{\alpha ihk}\overline{\theta}^h = A^\alpha_{ikj}\overline{\theta}^j,$$

(102) $$A^\alpha_{ikj} = A^\alpha_{ijk}.$$

If we set

(103) $$dh^\alpha_{ik} - h^\alpha_{ik}\overline{\omega}_k^h - h^\alpha_{hk}\overline{\omega}_i^h + h^\beta_{ik}\overline{\omega}_\beta^\alpha = h^\alpha_{ikj}\overline{\theta}^j,$$

with $h^\alpha_{ikj} = h^\alpha_{kij}$ then, because of the symmetry of the second fundamental form, we have

(104) $$h^\alpha_{ikj} = A^\alpha_{ikj} + \frac{1}{2} R^N_{\alpha ijk}.$$

The relations expressed by (102) and (104) lead to the *Codazzi equations*

(105) $$h^\alpha_{ikj} = h^\alpha_{ijk} + R^N_{\alpha ijk}.$$

In fact, the first term of (103) gives an expression of the covariant differential of the second fundamental form h, since (given a frame $u \in O(N,M)$), (103) implies that

(106) $$\begin{aligned}h^\alpha_{ikj}(u) &= (\nabla^\perp_{u_j}(h(u_i,u_k)) - h(\nabla^M_{u_j}u_i,u_k) - h(u_i,\nabla^M_{u_j}u_k),\ u_\alpha) \\ &= ((\overline{\nabla}_{u_j}h)(u_i,u_k),\ u_\alpha).\end{aligned}$$

The mean curvature vector H of the Riemannian immersion f is given by

(107) $$H = \frac{1}{m} h(u_i,u_i) = \frac{1}{m} h^\alpha_{ii} u_\alpha,$$

and together with equation (106), we obtain

$$h^\alpha_{iij} = m(\nabla^\perp_{u_j}H, u_\alpha) = m\nabla^\perp_{u_j}H^\alpha.$$

It follows that H is parallel (in the normal bundle) if and only if

$$h^{\alpha}_{iij} = 0. \tag{108}$$

Equation (105), upon setting $k = i$ and summing over i, implies that

$$h^{\alpha}_{iij} = h^{\alpha}_{iji} + R^{N}_{\alpha iji} = h^{\alpha}_{jii} + R^{N}_{\alpha iji}. \tag{109}$$

For this reason, *if N has constant sectional curvature*, the conditions

$$h^{\alpha}_{iij} = 0, \qquad h^{\alpha}_{jii} = 0, \qquad \nabla^{\perp} H = 0 \tag{110}$$

are equivalent.

Appendix B. The tension field of a map

Let (M,g) and (N,h) be Riemannian manifolds of respective dimensions m,n. Let $\{e_i\}$ and $\{e_A\}$ be local orthonormal local frames on M and N, and let $\{\theta^i\}$, $\{\theta^A\}$ be the corresponding dual coframes and $\{\omega^i_j\}$, $\{\omega^A_B\}$ the local forms reppresenting the Levi-Civita connections with respect to these local frames. We therefore have

$$\begin{aligned} d\theta^i &= -\omega^i_j \wedge \theta^j, & (\omega^i_j + \omega^j_i) &= 0, \\ \theta^i(Z) &= (X, e_i), & \omega^i_j(X) &= (\nabla^N_X e_i, e_j). \end{aligned} \tag{111}$$

Similarly,

$$\begin{aligned} d\theta^A &= -\omega^A_B \wedge \theta^B, & (\omega^A_B + \omega^B_A) &= 0, \\ \theta^A(Z) &= (Z, e_A), & \omega^A_B(Z) &= (\nabla^N_Z e_B, e_A). \end{aligned} \tag{112}$$

Consider a smooth map $f : M \to N$. Its differential df can be viewed either as a map from TM to TN determined by

$$df(x,X) = (f(x), df_x(X)), \qquad X \in T_x M,$$

or as a $f^{-1}(TN)$-valued 1-form on M.

Setting

$$df(e_i) = a^A_i e_A, \tag{113}$$

it turns out that

$$f^*\theta^A = a^A_i \theta^i. \tag{114}$$

Differentiating equations (114) we have

$$f^*(-\omega^A_B \wedge \theta^B) = da^A_j \wedge \theta^j - a^A_j \omega^i_j \wedge \theta^j.$$

Equivalently,

$$(da_j^A - a_i^A\omega_j^i + a_j^B f^*\omega_B^A)\wedge\theta^j = 0,$$

from which by *Cartan's Lemma* we obtain

$$Da_j^A \equiv da_j^A - a_i^A\omega_j^i + a_j^B f^*\omega_B^A = a_{jk}^A\theta^k, \tag{115}$$

with

$$a_{jk}^A = a_{kj}^A.$$

The functions a_{jk}^A are the components of the covariant differential Ddf, also called the second fundamental quadratic form of the map f; the differential Ddf may also be introduced as the $f^{-1}(TN)$-valued symmetric 2-form defined by

$$Ddf(X,Y) = (D_Y df)X = D_Y(dfX) - df\nabla_Y^M X, \tag{116}$$

where D is a metric connection on $f^{-1}(TN)\to M$ induced by the Levi-Civita connection on N.

It is easy to see that equations (115) and (116) are actually equivalent; indeed (116) implies

$$\begin{aligned} Ddf(e_j,e_k) &= D_{e_k}(a_j^A e_A) - df(\omega_j^i(e_k)e_i) \\ &= e_k(a_j^A)e_A + a_j^B\omega_B^A(dfe_k)e_A - \omega_j^i(e_k)a_i^A e_A \\ &= (da_j^A + a_j^B f^*\omega_B^A - a_i^A\omega_j^i)(e_k)e_A = a_{jk}^A e_A, \end{aligned}$$

with the same coefficients a_{jk}^A as in equation (115). It is obvious that the computation of the covariant differential Ddf according to (115) (put in evidence by Chern–Goldberg [5]), is particulary useful when orthonormal coframes on the manifolds are chosen.

The map $f: M\to N$ is called *totally geodesic* if $Ddf = 0$, equivalently if $a_{jk}^A = 0$.

The *tension field* τf is the trace of Ddf, i.e., the $f^{-1}(TN)$-valued field on M defined by

$$\tau(f) = Ddf(e_j,e_j) = a_{jj}^A e_A. \tag{117}$$

Furthermore the map f is called *harmonic* if $\tau f = 0$, i.e., $a_{jj}^A = 0$.

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