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THE ITALIAN CONTRIBUTION TO THE FOUNDATION AND DEVELOPMENT OF CONTINUED FRACTIONS*

Abstract. The role of Italian mathematicians in the foundation and the development of the theory and applications of continued fractions is emphasized.

1. Introduction

A continued fraction is an expression of the form

$$b_0+\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cfrac{a_3}{b_3+\ddots}}}$$

where $b_0, b_1, b_2, \ldots$ and $a_1, a_2, a_3, \ldots$ are numbers (arithmetical continued fraction) or functions of a complex variable (analytic continued fraction). An abbreviated notation is

$$b_0+\frac{a_1}{\left|\, b_1\right.}+\frac{a_2}{\left|\, b_2\right.}+\frac{a_3}{\left|\, b_3\right.}+\cdots$$

The history of continued fractions is as long as the history of numbers themselves since, in fact, Euclid's algorithm for computing the greatest common divisor of two integers leads to a (terminating, and not infinite) continued fraction. They also implicitly appear in the approximation of π given by Archimedes, in the solution of Diophantine and Pell's equations, in various approximations of the square root of a number, and in the famous quadrature of the circle, problems which all go back to antiquity.

When truncating such a continued fraction after the term with index n, and after reduction to the same denominator, one gets an ordinary fraction $C_n = A_n/B_n$. C_n is called a *convergent* of the continued fraction even if the sequence (C_n) does not converge. The partial numerators A_n and the partial denominators B_n can be computed by the following recurrence relations

$$\begin{aligned} A_n &= b_nA_{n-1}+a_nA_{n-2}, \qquad A_1=1, \quad A_0=b_0,\\ B_n &= b_nB_{n-1}+a_nB_{n-2}, \qquad B_1=0, \quad B_0=1, \end{aligned}$$

which were first given by Bhascara II, an Indian mathematician who was born in Vijayapura (in the present state of Mysore) in 1115 and became the head of the observatory of Ujjain where he died in 1185. These relations can be found in his book *Lilavati*,

*A Lezione Lagrangiana given in Turin on 11 June 2009

which means "beautiful" or "charming" and was the name of one of his daughters, written around 1150. In Europe, they will be rediscovered only 500 years later (in 1655) by the English mathematician John Wallis (Ashford, 23 November 1616 – Oxford, 28 October 1703), who gave them his *Arithmetica Infinitorum*.

On continued fractions, see [31].

2. About rabbits, and other things

We consider a couple of newly born rabbits, a male and a female. Rabbits are able to reproduce at the age of one month, and gestation lasts also one month. We assume that, each time, the female gave birth to one male and one female. At the end of the first month, we always have only one pair of rabbits. At the end of the second month, the female gave birth to one male and one female, and so we now have two couples. At the end of the third month, the first female gave birth to a new couple, but the second pair has no offspring. Thus, in total, we have three couples. At the end of the forth month, the first and the second female gave birth to a couple, and we now have five couples. And so on...

The question is to find how many couples we have after n months. Its answer is given by the sequence

$$1,1,2,3,5,8,13,21,34,55,89,144,233,377,610,987,\ldots$$

This problem was settled and solved by an Italian mathematician from Pisa who was living in the twelfth century : Leonardo Fibonacci (Pisa, c. 1170 – Pisa, c. 1250). His name is a contraction of "filius Bonacci", but he was also called Leonardo Bigollus. In Tuscan dialect "Bigollo" is difficult to translate, but one interpretation is "absent-minded". Leonardo was a merchant who travelled quite widely in the East, visiting Egypt, Syria, Greece and Sicily. He was in contact with the Arabic mathematical writings. In 1202 he wrote a book entitled *Liber abaci*, revised in 1228, but only published in 1857.

How might we build this sequence of numbers, called a *Fibonacci sequence*? Let u_n be the number of couples of rabbits at month n. At the beginning of the story, we have no rabbit at all, and we set $u_0 = 0$. The first month, we start with one couple. Thus, at month 1, we have $u_1 = 1$. Since rabbits are becoming adult at the age of one month, we have no additional rabbits at month 2, and thus $u_2 = 1$. At the end of month 3, the couple of rabbits gives birth to a new couple, and so $u_3 = 2$. And so on...

The general argument is as follows. The number u_{n+1} of couples at month $n+1$ is equal to the number of couples at month n plus the number of couples born during month $n+1$. It is also the number of couples at month n plus the number of adult couples at month n, or, in other words the number of couples at month n plus the number of couples born at month $n-1$. That is, finally,

$$\begin{aligned}
u_0 &= 0,\\
u_1 &= 1,\\
u_2 &= u_1 + u_0 = 1 + 0 = 1,\\
u_3 &= u_2 + u_1 = 1 + 1 = 2,\\
u_4 &= u_3 + u_2 = 2 + 1 = 3,\\
u_5 &= u_4 + u_3 = 3 + 2 = 5,
\end{aligned}$$

and, more generally,

$$u_{n+1} = u_n + u_{n-1}, \quad \text{for} \quad n = 1, 2, \ldots$$

This recurrence relation appears in many other natural phenomena such as the genealogy of drones, the spirals of seashells, pinecones, sunflowers, the optimal arrangement of pistils, phyllotaxy which is the study of the repartition of leaves on the stem of a plant, multiple reflections, the hydrogen atom, etc.

Let us construct a Fibonacci sequence, starting from two arbitrary numbers, for example, $u_0 = 1$ and $u_1 = 3$. With the preceding recurrence relation, we obtain $4, 7, 11, 18, 29, 47, 76, 123, 199, \ldots$ Computing the ratio of each number to the preceding one, we get

n	u_{n+1}/u_n		value
0	$3/1$	$=$	3
1	$4/3$	$=$	$1.333\ldots$
2	$7/4$	$=$	$1.750\ldots$
3	$11/7$	$=$	$1.571\ldots$
4	$18/11$	$=$	$1.6363\ldots$
7	$76/47$	$=$	$1.6170\ldots$
10	$322/199$	$=$	$1.61809\ldots$
14	$2207/1364$	$=$	$1.618035\ldots$

We see that these ratios converge to the famous *golden section* (sectio aurea) $(1+\sqrt{5})/2 = 1.618033988\ldots$ This golden section is also called the *Divine Proportione*, and is the title of the famous book by Luca Pacioli (Borgo San Sepolcro, Umbria, 1445 – Roma, 1517) published in 1509. He wrote (Chap. VII):

> *When a segment is divided according to the proportion with a mean point and two extremal ones, if we add to its longest part (a) the half of the total length $(a+b)$, the square of their sum will always be 5 times the square of the half of this total length.*

Translating this sentence into mathematical formulae gives

$$\left(a + \frac{a+b}{2}\right)^2 = 5\left(\frac{a+b}{2}\right)^2,$$

that is

$$a^2 - ab - b^2 = 0.$$

Setting

$$p = \frac{a}{b},$$

this relation becomes

$$p^2 - p - 1 = 0,$$

an equation whose positive zero is

$$\frac{1+\sqrt{5}}{2}.$$

A sequence of ratios with the same limit would have been obtained starting from other values of u_0 and u_1.

This number was the subject of an enormous literature. Aesthetical qualities in art (architecture, painting, music) were attributed to it. Some mystical interpretations were also given, and it is found in many mathematical problems as well. It fascinated, and is still fascinating.

Fibonacci numbers satisfy $u_2 = u_1 + u_0$. Dividing both sides by u_1, we get

$$\frac{u_2}{u_1} = 1 + \frac{u_0}{u_1} = 1 + \frac{1}{u_1/u_0}.$$

Similarly, we have $u_3 = u_2 + u_1$ and, dividing by u_2, we obtain

$$\frac{u_3}{u_2} = 1 + \frac{u_1}{u_2} = 1 + \frac{1}{u_2/u_1} = 1 + \frac{1}{1 + \dfrac{1}{u_1/u_0}}$$

after replacing u_2/u_1 by the expression above. If we go on, we get

$$\frac{u_4}{u_3} = 1 + \frac{1}{u_3/u_2} = 1 + \frac{1}{1 + \dfrac{1}{1 + \dfrac{1}{u_1/u_0}}}$$

and then

$$\frac{u_5}{u_4} = 1 + \frac{1}{1 + \dfrac{1}{1 + \dfrac{1}{1 + \dfrac{1}{u_1/u_0}}}}$$

and so on. Continuing indefinitely from $u_0 = u_1 = 1$, we obtain the continued fraction

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}}}}$$

which converges to $(1+\sqrt{5})/2$.

It was Robert Simson (West Kilbrich, Scotland, 14 September 1687 – Glasgow, 1st October 1768) who proved, in 1753, that the sequence $1,2,3,5,8,13,\ldots$ studied by Fibonacci consists in the successive convergents of this continued fraction [41].

Fibonacci's numbers and related questions form a research topic which is still extremely active, and a journal is entirely devoted to it, the *Fibonacci Quaterly* published by *The Fibonacci Association*.

3. Ascending continued fractions

Ascending continued fractions came to Europe during the Middle Ages. Leonardo Fibonacci introduced this kind of fractions in *Pars sexta, septimi capituli: De disgregatione partium in singulis partibus* of his book *Liber abaci*, with the notation

$$\begin{aligned}\frac{e}{f}\,\frac{c}{d}\,\frac{a}{b} &= \cfrac{c+\cfrac{e}{f}}{a+\cfrac{d}{b}}\\ &= \frac{afd+cf+e}{bdf} = \frac{a}{b}+\frac{c}{d}\frac{1}{b}+\frac{e}{f}\frac{1}{b}\frac{1}{d}.\end{aligned}$$

He called them *fractiones in gradibus*. The process of writing from the right to the left is probably due to the Arabic influence. He also exhibited $\frac{1}{5}\frac{0}{12}\frac{0}{20}$ as equal to $\frac{1}{2}\frac{0}{6}\frac{0}{10}\frac{0}{10}$, these two ascending continued fractions representing $1/1200$. Fibonacci also gave two other notations, one of them he described as follows:

> *And if on the line there should be many fractions and the line itself terminated in a circle, then its fractions would denote other than what has been stated, as in this* $\frac{2}{3}\frac{4}{5}\frac{6}{7}\frac{8}{9}O$ *the line of which denotes the fractions eight-ninths of a unit, and six-sevenths of eight-ninths, and four-fifths of six-sevenths of eight-ninths, and two-thirds of four-fifths of six-sevenths of eight-ninths of an integral unit.*

This is

$$\cfrac{192+\cfrac{384}{3}}{8+\cfrac{48+\cfrac{5}{7}}{9}}.$$

In fact, Fibonacci did not need such a tedious representation, but it was the tradition to expose the subject in mathematical treatises. He was the first to give an abstract theory of ascending continued fractions that was not related to any system of fractional units, and to replace the Egyptian conversion table by a rigorous method. His method, known outside Italy as the *Practica Italiana*, was presented in almost all the famous arithmetical treatises of the 16th century.

This type of ascending continued fraction is related to measurement problems, and it goes back to the Egyptians and the Rhind papyrus, which is at the British Museum in London. This papyrus is not a mathematical book in the modern sense, containing rules for solving various problems, but it does consist of numerical examples and a table for reducing fractions whose numerator is two into a sum of fractions with numerators equal to one (unitary fractions). The reason why the Egyptians only used unitary fractions (with the exception of 2/3) is because of their notation. Such fractions were represented by drawing a sort of horizontal bar above the hieroglyphic signs for representing integers. This method of representing fractions as a sum of unitary fractions was connected with the units of measurement used in the everyday life. For example, the unit of capacity was called the *hekat* (approximatively 292.24 cubic inches). It was divided into $1/2, 1/4, 1/8, 1/16, 1/32$ and $1/64$, each of which was represented by a different hieroglyphic sign. These signs were usually arranged in the celebrated Horus eye, as it is called.

Now if we consider the problem of dividing 5 hekats (volume units) of wheat between 12 persons, we begin by dividing each hekat into 3 parts and giving one of them to each person. Three parts remain. Each of these is divided into 4 parts one of which is given to each person, that is 1/4 of 1/3 of a hekat. Thus, each person has received

$$\frac{5}{12}=\frac{1}{3}+\frac{1}{12}$$

of the total amount of wheat. But

$$\frac{5}{12}=\frac{1}{3}+\frac{1}{4}\frac{1}{3}=\frac{1+\dfrac{1}{4}}{3},$$

which is an ascending continued fraction.

This method of counting goes back to the early ages of humanity, and it is related to the fractional units of measure which were widely used in various countries and for various purposes. For example the ancient Romans used the *as*, also called the *libra* (= 12 unciæ), the *uncia* (= 8 dramma), the *dramma* or *drachma* (= 3 scrupuli),

and the *scrupulum* as their weights. For example let us convert 6 unciæ, 5 dramma, 1 scrupulum into libræ. We have

$$\begin{aligned} 6U+5D+1S &= 6U+5D+\frac{1}{3}D \\ &= 6U+\left(5+\frac{1}{3}\right)D \\ &= 6U+\frac{5+\frac{1}{3}}{8}U = \left(6+\frac{5+\frac{1}{3}}{8}\right)U \\ &= \frac{6+\frac{5+\frac{1}{3}}{8}}{12}L = \left(\frac{6}{12}+\frac{5}{96}+\frac{1}{288}\right)L, \end{aligned}$$

which gives $160/288 = 5/9$ L. A scrupulum was around 1.137 g and, thus, a dramma was 3.411 g, the uncia 27.288 g, and the libra 327.456 g.

The universal medieval money system consisted of *libra* (= 20 solidi), the *solidus* (= 12 denarii) and the *denarius*. This system was preserved in the modern British pounds, shillings and pence. We can also cite the system formed by miles, furlongs, chains, yards, feet and inches.

4. The square root

A large part of mathematics was developed for the purpose of solving geometrical problems. It is the case for the computation of the square root of a number. Following the theorem of Pythagoras (6th century BC), the square of the hypothenuse of a rectangle triangle is equal to the sum of the squares of the two other sides. In order to obtain the length of this hypothenuse, it is necessary to extract the square root of a number.

Denote this number by A. Let us see how this computation leads to continued fractions (without respecting the historical chronology). Let a be the largest integer whose square a^2 is smaller than A. We subtract it from A. There is a remainder $r = A - a^2$. But we have

$$r = A - a^2 = (\sqrt{A}-a)(\sqrt{A}+a).$$

Thus, dividing both sides by $\sqrt{A}+a$, we get

$$\begin{aligned} \sqrt{A}-a &= \frac{r}{\sqrt{A}+a} \\ \sqrt{A} &= a+\frac{r}{\sqrt{A}+a}. \end{aligned}$$

In the first denominator, replace $\sqrt{A}$ by $a + r/(\sqrt{A}+a)$. This gives

$$\sqrt{A} = a+\frac{r}{2a+\frac{r}{a+\sqrt{A}}}.$$

The same replacement process of $\sqrt{A}$ by $a+r/(\sqrt{A}+a)$ can be used, again and again, in each new denominator. Thus, we obtain the continued fraction

$$\sqrt{A} = a + \frac{r}{\left\lfloor 2a \right\rfloor} + \frac{r}{\left\lfloor 2a \right\rfloor} + \frac{r}{\left\lfloor 2a \right\rfloor} + \cdots$$

The approximation $\sqrt{A} \simeq a + \dfrac{r}{2a}$ was universally known and used during the Antiquity and the Middle Ages. For example, it can also be found in the work of Paolo Dagomari (Prato, c. 1281 – between 1365 and 1372), who was an Italian mathematician, astronomer, astrologer and poet. His most important work is the *Trattato d'abbaco, d'astronomie e di segreti naturali e medicinali* written in 1339, where continued fractions, called "rotti infilzati" are mentioned as well as the preceding rule for finding the square root. Of course, the ancients did not proceed in that way. I only wanted to show how square roots are related to continued fractions.

The discovery of continued fractions ran over a period of 25 years, and it is due to two scholars from Bologna, the oldest University in the world.

Rafael Bombelli (Bologna, January 1526 – Roma ?, 1572), was an engineer and architect who studied with Pier Francesco Clementi da Corinaldo, who drained swamps and was later employed by Pope Paul III in the reclamation of the marshes of Foligno. Bombelli is the founder of complex numbers. In his book *L'algebra parte maggiore dell'arimetica divisa in tre libri* published by Giovanni Rossi in Bologna in 1572, but probably written with the help of his brother Ercole between 1560 and 1567, and also in its second edition published in 1579 under the title *L'algebra opera*, he gave an algorithm for extracting the square root of 13 which is completely equivalent to the continued fraction

$$\sqrt{13} = 3 + \frac{4}{\left\lfloor 6 \right\rfloor} + \frac{4}{\left\lfloor 6 \right\rfloor} + \cdots$$

Bombelli claimed that his work was based on the work of the great muslim mathematician Al-Khowarizmi (Khiva, c. 830 – ?), who lived in Bagdad and gave his name to the word "algorithm", and also on the work of Luca Pacioli, the author of the first mathematical encyclopedia of the Renaissance, and on the work of Fibonacci.

The real discoverer of continued fractions is Pietro Antonio Cataldi (Bologna, 15 April 1548 – Bologna, 11 February 1626), professor of mathematics and astronomy in Firenze, Perugia and Bologna. Cataldi follows the same method as Bombelli for computing the square root. He wrote a small booklet (140 pages) on this topics with the title *Trattato del modo brevissimo di trouare la radice quadra delli numeri et regole da approssimarsi di continuo al vero nelle radici de' numeri non quadrati, con le cause, & inuentioni loro, et anco il modo di pigliarne la radice cuba, applicando il tutto alle operationi militari & altre*. It was dedicated to Lodovico Mariscotti (with his portrait), and was published in Bologna in 1613 by B. Cochi. But, in fact, the book was finished in 1597 since, at the end of the book, it is written *Finij questa copia, data alla stampa, lunedi alli 11 agosto 1597 a' hore 21 1/2 nella stanza in s. Petronio vecchio, contrada di Bologna, essendodo fatta le prima bozza alcuni anni auanti*.

Cataldi applied the method to the computation of $\sqrt{18}$, and gave all the convergents of the continued fraction up to the 15th. He noticed that they are alternately larger and smaller than the exact value of the square root, and that they approximate it better and better. He also invented our modern notation for continued fractions. As noticed by Guglielmo Libri (Firenze, 2 January 1803 – Firenze, 28 Septembre 1869), Newton's method for solving the equation $x^2 - A = 0$ produces a subsequence of the convergents (the $(2^n - 1)$th, as was proved by A. Moret-Blanc in 1878) of this continued fraction. Around the same period, it was proved by Joseph Alfred Serret (Paris, 30 August 1819 – Versailles, 2 March 1885) that the convergents x_n of the continued fraction for $\sqrt{A}$ satisfy

$$x_{2n} = \frac{1}{2}\left(x_n + \frac{A}{x_n}\right),$$

which is exactly Heron's method for the square root. Related results by Padre Bellino Carrara S.J. (1889) must also be noted [7]. Let us also mention that, in 1606, Cataldi defined ascending continued fractions as *a quantity written or proposed in the form of a fraction of a fraction*.

For a more complete analysis of the works of Bombelli and Cataldi, we refer the reader to [5, pp. 61–70].

5. Lagrange

Joseph Louis (Giuseppe Ludovico) Lagrange (Torino, 25 January 1736 – Paris, 10 April 1813) made many contributions to continued fractions, and used them at many occasions. We will now describe the most important ones. On his life, consult [6]. His complete works have been published [27], and could also be found (and downloaded) from the numerical library *Gallica* of the *Bibliothèque Nationale de France* at `http://mathdoc.emath.fr/cgi-bin/oetoc?id=OE_LAGRANGE__1`

In 1769, Lagrange published a *Mémoire sur la résolution des équations numériques* where he gave a method for approximating a real zero of a polynomial by continued fractions [18]. Unlike the other methods, Lagrange's method cannot fail, and it rapidly became a classical one.

In February 1657, Pierre de Fermat (Beaumont-de-Lomagne, near Montauban, 17 August 1601 – Castres, 12 January 1665), claimed that Pell's equation $x^2 = Dy^2 + 1$ has infinitely many solutions if D is a positive nonsquare integer. But Fermat was known for not giving proofs. The first correct proof of the existence of an infinity of solutions, and their form as well, is due to Lagrange in 1769 [19]. The pair (x_1, y_1) solving Pell's equation and minimizing x is one of the convergents of the continued fraction for $\sqrt{D}$. It is called the *fundamental solution*, and it can be obtained by testing each successive convergent until a solution to Pell's equation is found. All the other solutions (x_i, y_i) can be computed from the fundamental one by the formula

$$x_i + y_i\sqrt{D} = (x_1 + y_1\sqrt{D})^i, \quad i = 2, 3, \ldots$$

Developing this formula, we get by induction

$$\begin{aligned}(x_1+y_1\sqrt{D})^{i+1} &= (x_1+y_1\sqrt{D})^i(x_1+y_1\sqrt{D})\\ &= x_1x_i+Dy_1y_i+\sqrt{D}(x_1y_i+y_1x_i),\end{aligned}$$

and, thus, the other solutions can also be obtained by the recurrence relations

$$\begin{aligned}x_{i+1} &= x_1x_i+Dy_1y_i,\\ y_{i+1} &= x_1y_i+y_1x_i.\end{aligned}$$

Any solution (x,y) approximates $\sqrt{D}$, and thus is a special case of a continued fraction approximation of a quadratic irrational. The relationship to continued fractions and their matrix interpretation implies that the solutions of Pell's equation form a semigroup which is a subset of the modular group. Therefore, Pell's equation is closely related to the theory of algebraic numbers of the form $\alpha = a+b\sqrt{D}$. Such an α is one of the solutions of the quadratic equation $t^2-2at+(a^2-Db^2)=0$. Any quadratic irrational number can be written in the form $(p+\sqrt{d})/q$, where p,d, and q are integers, $d>0$ is not a perfect square, and q divides p^2-d.

Major contributions to the theory of continued fractions are due to Leonhard Euler (Basel, 15 April 1707 – St. Petersburg, 18 September 1783). As early as 1731, he used them for Riccati's differential equation. His first arithmetical paper on the subject was entitled *De fractionihus continuis*, and it was published in 1737. Euler proved that every rational number can be developed into a finite continued fraction, that an irrational number gives rise to an infinite continued fraction, and that a periodic continued fraction is the zero of a quadratic equation. In 1770, Lagrange wrote a memoir where he proved the converse of this last result of Euler [20]. He wrote

> *Now I claim that the continued fraction which expresses the value of x* [the real positive irrational zero of a quadratic equation] *will always be necessarily periodic.*

In another paper dated 1770, Lagrange extended Huygens' and Saunderson's method to solve the Diophantine equation $py-qx=r$ [21].

An interesting problem treated by Lagrange in 1772, and again in 1775, is the solution of linear difference equations with constant coefficients [22, 23]. At this occasion, he made use of what is now known as the generating function of a sequence of numbers (c_n), namely

$$f(x) = c_0+c_1x+c_2x^2+\cdots$$

If f is a rational function with a numerator of degree $k-1$ and a denominator of degree k, then (c_n) satisfies a recurrence relation of order k. Lagrange was also interested in the inverse problem of searching for hidden periodicities in a sequence, and he proved that, if (c_n) is recurrent, its generating function is a rational function. His proof is based on continued fractions.

Lagrange also made use of continued fractions on several other occasions, and wrote about them in his *Leçons élémentaires sur les mathématiques données à l'École Normale en 1795* [26].

In his memoir *Essai d'analyse numérique sur la transformation des fractions*, published in 1798, he gave a method, very similar to that of Leonardo Fibonacci, for developing B/A into a sum of unitary fractions, thus leading to the simplest and most rapidly convergent series. He noticed that ascending and descending continued fractions proceed from the same idea [25]. Following Lagrange, these fractions were called Lambert's fractions by Giovanni Polvani (Spoleto, Umbria, 1892 – Milano, 11 August 1970), while Alfred Kunze used the name *Aufsteigende Kettenbrüche* which exactly corresponds to the name given by Fibonacci. Alfred Pringsheim (Ohlau, now Olawa, Lower Silesia, now Poland, 2 September 1850 – Zürich, 25 June 1941), the father-in-law of the writer Thomas Mann (Lübeck, 6 June 1875 – Zürich, 12 August 1955) who was awarded the Nobel Prize in literature in 1929, showed that the connection between ascending and descending continued fractions as treated by Lambert and Lagrange follows from a formula due to Euler [39].

Let us now discuss Padé approximation. We consider a formal power series

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots ,$$

and we are looking for a rational fraction with a numerator of degree p and a denominator of degree q such that the series expansion of this rational fraction coincides with f as far as possible, that is up to the term of degree $p+q$ inclusive. Such a rational fraction is called a *Padé approximant* of f. Padé approximants are related to continued fractions, Hankel determinants, orthogonal polynomials, and extrapolation methods as explained in [3]. On this connection, let us only mention that, in 1862, Nicola Trudi (Campobasso, 21 July 1811 – Napoli, 3 October 1884) studied the links between Hankel determinants and the determinantal expression for orthogonal polynomials (and, therefore, with Padé approximants).

Padé approximants were not invented by the French mathematician Henri Eugène Padé (Abbeville, 17 December 1863 - Aix-en-Provence, 9 July 1953). He merely studied them in detail in his Thesis, dated 1892, under the supervision of Charles Hermite (Dieuze, 24 December 1822 – Paris, 14 January 1901) [33]. Padé approximants are much older. They can be obtained by a division process of a power series expansion which is quite similar to Euclid's algorithm for the g.c.d., a procedure analyzed in [4]. In fact, many mathematicians were using Padé approximants without knowing their fundamental approximation-through-order property. Their real discovery is due to two prominent scientists : Johan Heinrich Lambert (Mulhouse, 1728 – Berlin, 1777), who derived them directly in 1758 by the definition given above [28], and Lagrange who obtained them by means of continued fractions (the denominators of successive Padé approximants satisfy recurrence relations similar to those for the convergents of a continued fraction, thus the connection between the two topics).

Indeed, in 1775, in a paper published in the *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Berlin*, Lagrange gave the solution of certain

differential equations in the form of continued fractions [24]. He wrote

> *Since the form of these continued fractions is not easy for algebraic manipulation, we shall reduce them to ordinary fractions...*

Then, he adds that these ordinary fractions *are exact up to the power of x that is the product of the two highest powers of x in the numerator and in the denominator*, which means up to the term $x^p x^q = x^{p+q}$ inclusive.

This is the birth certificate of Padé approximants.

As he did each year, Lagrange sent the volume containing his paper to Jean Le Rond d'Alembert (Paris, 16 November 1717 – Paris, 29 October 1783). In his accompanying letter, on 12 December 1778 (indeed, the publication was late!), he wrote

> *As usual, there is something from me, but nothing that merits your attention.*

This was not a prophetic view!

6. Miscellaneous contributions

Let us now briefly mention some contributions to continued fractions due to Italian mathematicians.

The first proof of the irrationality of $\sqrt{2}$, based on Proposition 2 of Euclid's tenth book, is due to Giovanni Alfonso Borelli (Castelnuova near Napoli, 28 January 1608 – Roma, 31 December 1679), who inserted it in his book *Euclid restitutus* published in 1658 in Pisa (second edition, Roma, 1695).

A method for computing zeros of polynomials was given by Paolo Ruffini (Valentano, 22 September 1765 – Modena, 10 May 1822) in his book *Teoria generale delle equazioni* published in 1799, whose Chapter 20 is entitled *Della soluzione per serie delle equazioni algebraiche col mezzo della frazioni continue, e riflessioni ulteriori intorno alle equazioni riducibili a grado inferiore*.

In 1840, Carlo D'Andrea (L'Aquila, 3 September 1802 – Napoli, 1885) proved the solvability of Pell's equation by means of continued fractions.

Salvatore Pincherle (Trieste, 11 March 1853 – Bologna, 11 July 1936) published numerous contributions on continued fractions, see [37]. His most important result concerns sufficient conditions for the convergence of continued fractions in 1889 [34]. One of his theorems was extended in 1890 by Dionisio Gambioli (Pergola, 11 September 1853 – Roma, 4 November 1941). Pincherle also studied in depth the minimal solution of three-term recurrence relations as those satisfied by continued fractions [36].

Using simultaneous approximations to several series using rational functions, Charles Hermite was able to prove, in 1873, that the number e is transcendental. In 1882, Ferdinand Lindemann (Hannover, 12 April 1852 – Munich, 6 March 1939) similarly proved the transcendence of π, thus ending in the negative a problem (the quadrature of the circle) that was open for more than 2000 years. Pincherle also made several

contributions to such approximations which are now called Padé–Hermite approximants, see [35].

In 1895, Emma Bortolotti (Bologna, 25 December 1867 – ?) proved that a necessary and sufficient condition that the zero of a quadratic equation with integer coefficients in the variable x can be developed into a periodic continued fraction is that the indeterminate equation of the second degree $Du^2 - v^2 = 1$ can be solved by integer polynomials, where D is the discriminant of the given equation. If the degree of D is odd, the second equation is obviously impossible to solve [2]. Emma was the sister of Ettore Bortolotti (Bologna, 6 March 1866 – Bologna, 17 February 1947), a student of Pincherle who made numerous contributions to continued fractions and approximations by rational functions, and who is also remarkable as an historian of mathematics; see, for example, [1].

Giovanni Frattini (Roma, 8 January 1852 – 21 July 1925) is well known for his contributions to group theory. The development of the square root into a continued fraction was the subject of several of his papers. We will mention only three of them. Let a be the largest integer smaller than $\sqrt{D}$, and set $(a+\sqrt{D})^n = P_n + Q_n\sqrt{D}$. The sequence (P_n/Q_n) is decreasing and it converges to $\sqrt{D}$. On the other side, consider the continued fraction expansion

$$\sqrt{D} = a_1 + \cfrac{1}{\left\lfloor a_2 \right\rfloor} + \cfrac{1}{\left\lfloor a_3 \right\rfloor} + \cdots$$

and let p_n/q_n be its convergents. In [12], Frattini proved that P_n/Q_n is closer to $\sqrt{D}$ than p_n/q_n. He obtained some of his results on group theory via the square root and continued fractions [13]. In 1904, he proved that if D is a positive integer or a polynomial with integer coefficients, then $x^2 = Dy^2 + 1$ is solvable if and only if $\sqrt{D} - a$ can be developed into a simple periodic continued fraction [14].

Continued fractions appear in two other papers by Frattini, one on chaos [15] and the other one on relativity [11].

7. Conclusion

The history of continued fractions is far from finished. Nowadays it remains quite an active field of research, with many applications in various branches. To name a few, let us mention: number theory where continued fractions play an important role in the transcendental characteristics of numbers; natural sciences, for the approximation of special functions used in particle physics, quantum chemistry, quantum physics, quantum electrodynamics, and more generally quantum mechanics; medical sciences were they have applications in diagnostics through magnetic resonance spectroscopy; economics for modelling time series data; complex analysis where they are used to derive new convergence and approximation results; numerical analysis, for example, in the treatment of the Gibbs phenomenon for Fourier and other orthogonal series.

Let us just mention a quite recent result. In his Thesis, defended in 2007, Jeroen Demeyer related the Chebyshev polynomials of the first kind T_k and those of the second

kind U_k to a form of Pell's equation, namely

$$T_k^2(x) - (x^2-1)U_{k-1}^2(x) = 1.$$

Thus, these polynomials can be generated by the standard technique for Pell's equation of taking powers of the fundamental solution

$$T_k(x) + U_{k-1}(x)\sqrt{x^2-1} = (x+\sqrt{x^2-1})^k.$$

It follows that

$$\begin{aligned} T_{k+1}(x) + U_k(x)\sqrt{x^2-1} &= (x+\sqrt{x^2-1})^k(x+\sqrt{x^2-1}) \\ &= (T_k(x) + U_{k-1}(x)\sqrt{x^2-1})(x+\sqrt{x^2-1}) \\ &= ((x^2-1)U_{k-1}(x) + xT_k(x)) + (T_k(x) + xU_{k-1}(x))\sqrt{x^2-1}, \end{aligned}$$

and finally we recover the well-known formulae

$$\begin{aligned} T_{k+1}(x) &= (x^2-1)U_{k-1}(x) + xT_k(x) \\ U_k(x) &= T_k(x) + xU_{k-1}(x), \end{aligned}$$

with $T_1(x) = x$ and $U_0(x) = 1$.

Continued fractions have been generalized in several ways, so as to define non-commutative continued fractions, vector continued fractions, matrix continued fractions, and simultaneous approximations.

Italian mathematicians have been prominent in the foundation and the early developments of continued fractions. Many important theoretical results are due to them, and they were also present at the first steps of Padé approximants. All these topics are of most relevance today.

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