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HARMONIC MAPS HAVING TANGENT BUNDLES WITH g-NATURAL METRICS AS SOURCE OR TARGET

Abstract. We produce new examples of harmonic maps, having as either source or target manifold the tangent bundle TM on a Riemannian manifold (M,g), equipped with a Riemannian g-natural metric G. In particular, we study the harmonicity of the canonical projection $\pi : (TM,G) \to (M,g)$, and of the identity map $(TM,G) \to (TM,g^S)$ and conversely, g^S being the Sasaki metric on TM. A corresponding study is made for the unit tangent sphere bundle T_1M, equipped with a Riemannian g-natural metric $\tilde{G}$.

1. Introduction

Let (M,g), (M',g') be Riemannian manifolds, with M compact, and consider a smooth map $f : (M,g) \to (M',g')$. The *energy* of f is defined as the integral

$$\mathscr{E}(f) := \int_M e(f)\, dv_g,$$

where $e(f) = \frac{1}{2}\|f_*\|^2 = \frac{1}{2}\mathrm{tr}_g f^*g'$ is the so-called *energy density* of f. With respect to a local orthonormal basis of vector fields $\{e_1,\dots,e_n\}$ on M, one has

$$e(f) = \frac{1}{2}\sum_{i=1}^{n} g'(f_*e_i, f_*e_i).$$

Critical points of the energy functional $\mathscr{E}$ on $C^\infty(M,M')$ are known as *harmonic maps*. They have been characterized in [10] as maps for which the *tension field* $\tau(f) = \mathrm{tr}\nabla df$ vanishes. When M is not compact, a map $f : (M,g) \to (M',g')$ is said to be harmonic if $\tau(f) = 0$. We refer the reader to [9, 18] for further details and results about the energy functional.

It is particularly interesting to investigate the harmonicity of maps between Riemannian manifolds that are naturally constructed from one another. A classical example is the tangent bundle TM on a Riemannian manifold (M,g), equipped with the *Sasaki metric* g^S. Nouhaud [15] proved that the only vector fields V defining harmonic maps from a compact Riemannian manifold (M,g) to (TM,g^S) are the parallel vector fields. The same result was obtained independently by Ishihara [12], who also gave an explicit expression of the tension field associated to a vector field. It is well known that the canonical projection $\pi : (TM,g^S) \to (M,g)$ is harmonic (see for example [16]). Oniciuc [16] proved the same result when TM is equipped with the *Cheeger–Gromoll metric* g^{CG}, and also proved the harmonicity of the canonical projection $\pi_1 : (T_1M,\widetilde{g^S}) \to (M,g)$, where $\widetilde{g^S}$ denotes the Sasaki metric on the unit tangent

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sphere bundle T_1M. Han and Yim [11] characterized unit vector fields which define harmonic maps from (M,g) to $(T_1M,\widetilde{g^S})$, by determining the associated tension field.

The Sasaki metric g^S (as well as the Cheeger–Gromoll metric) is only one possible choice inside a very large family of Riemannian metrics on TM, known as *Riemannian g-natural metrics*. As their name suggests, those metrics are constructed in a very "natural" way from a Riemannian metric g over M. The introduction of g-natural metrics converts the classification of second order natural transformations of Riemannian metrics on manifolds to that of metrics on tangent bundles, by work of O. Kowalski and M. Sekizawa [14]. Other presentations of the basic result from [14] and more details about the concept of naturality can be found in [13]. The set of g-natural metrics, which depend on six smooth functions from $\mathbb{R}^+$ to $\mathbb{R}$, has been completely described in [7].

In [2], the present authors and D. Perrone studied when a vector field V on a Riemannian manifold (M,g) defines a harmonic map $V:(M,g)\to(TM,G)$, where G is an arbitrary Riemannian g-natural metric. Equipping the unit tangent sphere bundle T_1M with an arbitrary induced Riemannian g-natural metric $\tilde{G}$, the harmonicity of $V:(M,g)\to(T_1M,\tilde{G})$ was discussed in [3], while [4] studied the harmonicity of the *geodesic flow* $\tilde{\xi}:(T_1M,\tilde{G})\to(T_\rho T_1M,\tilde{\tilde{G}})$.

In this paper, we study the harmonicity of the canonical projection $\pi:(TM,G)\to(M,g)$, where G is an arbitrary Riemannian g-natural metric. We also determine necessary and sufficient conditions for the harmonicity of G with respect to g^S, that is, of the identity map from (TM,G) into (TM,g^S), and vice versa. Finally, a corresponding study is made for the canonical projection $\pi_1:(T_1M,\tilde{G})\to(M,g)$, and for the identity map from $(T_1M,\tilde{G})$ to $(T_1M,\widetilde{g^S})$ and vice versa. In this way, we establish large classes of examples of harmonic maps, defined either from or to tangent bundles equipped with g-natural Riemannian metrics and thus possessing a highly nontrivial geometry.

The paper is organized in the following way. The basic information about Riemannian g-natural metrics on TM and T_1M is given in Section 2. In Section 3 we discuss the harmonicity of $\pi:(TM,G)\to(M,g)$, while in Section 4 we investigate when the identity map $(TM,G)\to(TM,g^S)$ and $(TM,g^S)\to(TM,G)$ is harmonic. The corresponding studies for the unit tangent sphere bundle T_1M, equipped with a Riemannian g-natural metric $\tilde{G}$, are given in Sections 5 and 6 respectively.

2. Preliminaries on Riemannian g-natural metrics

Let (M,g) be an n-dimensional Riemannian manifold and ∇ its Levi-Civita connection. At any point (x,u) of its *tangent bundle* TM, the tangent space of TM splits into the horizontal and vertical subspaces with respect to ∇:

$$(TM)_{(x,u)} = H_{(x,u)} \oplus V_{(x,u)}.$$

For any vector $X\in M_x$, there exists a unique vector $X^h\in H_{(x,u)}$ (the *horizontal lift* of X to $(x,u)\in TM$), such that $\pi_*X^h = X$, where $\pi:TM\to M$ is the natural projection. The *vertical lift* of a vector $X \in M_x$ to $(x,u) \in TM$ is a vector $X^v \in V_{(x,u)}$ such that $X^v(df) = Xf$, for all functions f on M. Here we consider 1-forms df on M as functions on TM (i.e., $(df)(x,u) = uf$). The map $X \mapsto X^h$ is an isomorphism between the vector spaces M_x and $H_{(x,u)}$. Similarly, the map $X \mapsto X^v$ is an isomorphism between M_x and $V_{(x,u)}$. Each tangent vector $\tilde{Z} \in (TM)_{(x,u)}$ can be written in the form $\tilde{Z} = X^h + Y^v$, where $X, Y \in M_x$ are uniquely determined vectors.

Horizontal and vertical lifts of vector fields on M can be defined in an obvious way and are vector fields uniquely defined on TM.

We can refer to [7] for the description of the class of g-natural metrics on the tangent bundle of a Riemannian manifold (M,g). All g-natural metrics are characterized as follows.

PROPOSITION 1 ([7]). *Let (M,g) be a Riemannian manifold of dimension n and G be a g-natural metric on TM. Then there are six smooth functions α_i, $\beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1,2,3$, such that for every u, X, Y $\in M_x$, we have*

$$(1) \quad \begin{cases} G_{(x,u)}(X^h,Y^h) = (\alpha_1+\alpha_3)(r^2)g_x(X,Y) + (\beta_1+\beta_3)(r^2)g_x(X,u)g_x(Y,u), \\ G_{(x,u)}(X^h,Y^v) = G_{(x,u)}(X^v,Y^h) = \alpha_2(r^2)g_x(X,Y) + \beta_2(r^2)g_x(X,u)g_x(Y,u), \\ G_{(x,u)}(X^v,Y^v) = \alpha_1(r^2)g_x(X,Y) + \beta_1(r^2)g_x(X,u)g_x(Y,u), \end{cases}$$

where $r^2 = g_x(u,u)$. For $n = 1$, the same holds with $\beta_i = 0$, $i = 1,2,3$.

Notation. In the sequel, we shall use the following notation. For all $t \in \mathbb{R}^+$,

- $\phi_i(t) = \alpha_i(t) + t\,\beta_i(t)$,
- $\alpha(t) = \alpha_1(t)(\alpha_1+\alpha_3)(t) - \alpha_2^2(\mathrm{t})$,
- $\phi(t) = \phi_1(t)(\phi_1+\phi_3)(t) - \phi_2^2(t)$.

PROPOSITION 2 ([7]). *A g-natural metric G on TM is Riemannian if and only if its defining functions α_i, β_i satisfy the inequalities*

$$(2) \qquad \alpha_1(t) > 0, \quad \phi_1(t) > 0, \quad \alpha(t) > 0, \quad \phi(t) > 0,$$

for all $t \in \mathbb{R}^+$. For $n = 1$, (2) *reduces to $\alpha_1(t) > 0$ and $\alpha(t) > 0$, for all $t \in \mathbb{R}^+$.*

CONVENTION 1. a) Throughout the paper, when we consider an arbitrary Riemannian g-natural metric G on TM, we implicitly suppose that it is defined by the functions α_i, $\beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1,2,3$, given in Proposition 1 and satisfying (2).

b) Unless otherwise stated, all real functions α_i, β_i, ϕ_i, α and ϕ and their derivatives are evaluated at $r^2 := g_x(u,u)$.

c) We consider the Riemannian curvature R of g with the sign convention $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

Remark. In literature, there are some well known Riemannian metrics on the tangent bundle, which turn out to be special cases of Riemannian g-natural metrics. In particular:

- the *Sasaki metric* g^S is obtained for $\alpha_1(t) = 1$ and $\alpha_2(t) = \alpha_3(t) = \beta_1(t) = \beta_2(t) = \beta_3(t) = 0$.
- the *Cheeger–Gromoll metric* g^{CG} (see [8]) is obtained when $\alpha_2(t) = \beta_2(t) = 0$, $\alpha_1(t) = \beta_1(t) = -\beta_3(t) = 1/(1+t)$ and $\alpha_3(t) = t/(1+t)$.

Since $\alpha_2 = \beta_2 = 0$, by (1) it follows that g^S and g^{CG} are examples of Riemannian g-natural metrics on TM for which horizontal and vertical distributions are mutually orthogonal.

The Levi-Civita connection $\bar{\nabla}$ of an arbitrary Riemannian g-natural metric G on TM, can be described as follows:

PROPOSITION 3 ([6]). *Let (M,g) be a Riemannian manifold of dimension n, ∇ its Levi-Civita connection and R its curvature tensor. Let G be a Riemannian g-natural metric on TM. Then the Levi-Civita connection $\bar{\nabla}$ of (TM,G) is characterized by*

$$\begin{array}{llcl}
(i) & (\bar{\nabla}_{X^h}Y^h)_{(x,u)} & = & (\nabla_X Y)^h_{(x,u)} + h\{A(u;X_x,Y_x)\} + v\{B(u;X_x,Y_x)\}, \\
(ii) & (\bar{\nabla}_{X^h}Y^v)_{(x,u)} & = & (\nabla_X Y)^v_{(x,u)} + h\{C(u;X_x,Y_x)\} + v\{D(u;X_x,Y_x)\}, \\
(iii) & (\bar{\nabla}_{X^v}Y^h)_{(x,u)} & = & h\{C(u;Y_x,X_x)\} + v\{D(u;Y_x,X_x)\}, \\
(iv) & (\bar{\nabla}_{X^v}Y^v)_{(x,u)} & = & h\{E(u;X_x,Y_x)\} + v\{F(u;X_x,Y_x)\},
\end{array}$$

for all vector fields X, Y on M and $(x,u) \in TM$. Here A,B,C,D,E,F are defined, for all u, X, $Y \in M_x$, $x \in M$, by:

$$\begin{aligned}
A(u;X,Y) = {} & A_1[R(X,u)Y + R(Y,u)X] + A_2[g_x(Y,u)X + g_x(X,u)Y] \\
& + A_3\, g_x(R(X,u)Y,u)u + A_4\, g_x(X,Y)u + A_5\, g_x(X,u)g_x(Y,u)u,
\end{aligned}$$

$$A_1 = -\frac{\alpha_1\alpha_2}{2\alpha}, \qquad A_2 = \frac{\alpha_2(\beta_1+\beta_3)}{2\alpha}, \qquad A_3 = \frac{\alpha_2\{\alpha_1[\phi_1(\beta_1+\beta_3)-\phi_2\beta_2]+\alpha_2(\beta_1\alpha_2-\beta_2\alpha_1)\}}{\alpha\phi},$$

$$A_4 = \frac{\phi_2(\alpha_1+\alpha_3)'}{\phi}, \; A_5 = \frac{\alpha\phi_2(\beta_1+\beta_3)'+(\beta_1+\beta_3)\{\alpha_2[\phi_2\beta_2-\phi_1(\beta_1+\beta_3)]+(\alpha_1+\alpha_3)(\alpha_1\beta_2-\alpha_2\beta_1)\}}{\alpha\phi};$$

next:

$$\begin{aligned}
B(u;X,Y) = {} & B_1 R(X,u)Y + B_2 R(X,Y)u + B_3[g_x(Y,u)X + g_x(X,u)Y] \\
& + B_4 g_x(R(X,u)Y,u)u + B_5 g_x(X,Y)u + B_6 g_x(X,u)g_x(Y,u)u,
\end{aligned}$$

$$B_1 = \frac{\alpha_2^2}{\alpha}, \qquad B_2 = -\frac{\alpha_1(\alpha_1+\alpha_3)}{2\alpha}, \qquad B_3 = -\frac{(\alpha_1+\alpha_3)(\beta_1+\beta_3)}{2\alpha},$$

$$B_4 = \frac{\alpha_2\{\alpha_2[\phi_2\beta_2-\phi_1(\beta_1+\beta_3)]+(\alpha_1+\alpha_3)(\beta_2\alpha_1-\beta_1\alpha_2)\}}{\alpha\phi}, \qquad B_5 = -\frac{(\phi_1+\phi_3)(\alpha_1+\alpha_3)'}{\phi},$$

$$B_6 = \frac{-\alpha(\phi_1+\phi_3)(\beta_1+\beta_3)'+(\beta_1+\beta_3)\big\{(\alpha_1+\alpha_3)[(\phi_1+\phi_3)\beta_1-\phi_2\beta_2]+\alpha_2[\alpha_2(\beta_1+\beta_3)-(\alpha_1+\alpha_3)\beta_2]\big\}}{\alpha\phi};$$

next:

$$C(u;X,Y) = C_1 R(Y,u)X + C_2 g_x(X,u)Y + C_3 g_x(Y,u)X + C_4 g_x(R(X,u)Y,u)u + C_5 g_x(X,Y)u + C_6 g_x(X,u)g_x(Y,u)u,$$

$$C_1 = -\frac{\alpha_1^2}{2\alpha}, \qquad C_2 = \frac{\alpha_1(\beta_1+\beta_3)}{2\alpha}, \qquad C_3 = \frac{\alpha_1(\alpha_1+\alpha_3)' - \alpha_2(\alpha_2' - \frac{\beta_2}{2})}{\alpha},$$

$$C_4 = \frac{\alpha_1\{\alpha_2(\alpha_2\beta_1 - \alpha_1\beta_2) + \alpha_1[\phi_1(\beta_1+\beta_3) - \phi_2\beta_2]\}}{2\alpha\phi}, \qquad C_5 = \frac{\phi_1(\beta_1+\beta_3) + \phi_2(2\alpha_2' - \beta_2)}{2\phi},$$

$$C_6 = \frac{\alpha\phi_1(\beta_1+\beta_3)' + \{\alpha_2(\alpha_1\beta_2 - \alpha_2\beta_1) + \alpha_1[\phi_2\beta_2 - (\beta_1+\beta_3)\phi_1]\}[(\alpha_1+\alpha_3)' + \frac{\beta_1+\beta_3}{2}]}{\alpha\phi} + \frac{\left\{\alpha_2[\beta_1(\phi_1+\phi_3) - \beta_2\phi_2] - \alpha_1[\beta_2(\alpha_1+\alpha_3) - \alpha_2(\beta_1+\beta_3)]\right\}\left(\alpha_2' - \frac{\beta_2}{2}\right)}{\alpha\phi};$$

next:

$$D(u;X,Y) = D_1 R(Y,u)X + D_2 g_x(X,u)Y + D_3 g_x(Y,u)X + D_4 g_x(R(X,u)Y,u)u + D_5 g_x(X,Y)u + D_6 g_x(X,u)g_x(Y,u)u,$$

$$D_1 = \frac{\alpha_1\alpha_2}{2\alpha}, \qquad D_2 = -\frac{\alpha_2(\beta_1+\beta_3)}{2\alpha}, \qquad D_3 = \frac{-\alpha_2(\alpha_1+\alpha_3)' + (\alpha_1+\alpha_3)(\alpha_2' - \frac{\beta_2}{2})}{\alpha},$$

$$D_4 = \frac{\alpha_1\{(\alpha_1+\alpha_3)(\alpha_1\beta_2 - \alpha_2\beta_1) + \alpha_2[\phi_2\beta_2 - \phi_1(\beta_1+\beta_3)]\}}{2\alpha\phi}, \quad D_5 = -\frac{\phi_2(\beta_1+\beta_3) + (\phi_1+\phi_3)(2\alpha_2' - \beta_2)}{2\alpha\phi},$$

$$D_6 = \frac{-\alpha\phi_2(\beta_1+\beta_3)' + \left\{(\alpha_1+\alpha_3)(\alpha_2\beta_1 - \alpha_1\beta_2) + \alpha_2[\phi_1(\beta_1+\beta_3) - \phi_2\beta_2]\right\}\left[(\alpha_1+\alpha_3)' + \frac{\beta_1+\beta_3}{2}\right]}{\alpha\, phi} + \frac{\left\{(\alpha_1+\alpha_3)[\beta_2\phi_2 - \beta_1(\phi_1+\phi_3)] + \alpha_2[\beta_2(\alpha_1+\alpha_3) - \alpha_2(\beta_1+\beta_3)]\right\}\left(\alpha_2' - \frac{\beta_2}{2}\right)}{\alpha\phi};$$

next:

$$E(u;X,Y) = E_1[g_x(Y,u)X + g_x(X,u)Y] + E_2 g_x(X,Y)u + E_3 g_x(X,u)g_x(Y,u)u,$$

$$E_1 = \frac{\alpha_1(\alpha_2' + \frac{\beta_2}{2}) - \alpha_2\alpha_1'}{\alpha}, \qquad E_2 = \frac{\phi_1\beta_2 - \phi_2(\beta_1 - \alpha_1')}{\phi},$$

$$E_3 = \frac{\alpha(2\phi_1\beta_2' - \phi_2\beta_1') + 2\alpha_1'\left\{\alpha_1[\alpha_2(\beta_1+\beta_3) - \beta_2(\alpha_1+\alpha_3)] + \alpha_2[\beta_1(\phi_1+\phi_3) - \beta_2\phi_2]\right\}}{\alpha\phi} + \frac{(2\alpha_2' + \beta_2)\left\{\alpha_1[\phi_2\beta_2 - \phi_1(\beta_1+\beta_3)] + \alpha_2(\alpha_1\beta_2 - \alpha_2\beta_1)\right\}}{\alpha\phi};$$

and finally:

$$F(u;X,Y) = F_1[g_x(Y,u)X + g_x(X,u)Y] + F_2 g_x(X,Y)u + F_3 g_x(X,u)g_x(Y,u)u,$$

$$F_1 = \frac{-\alpha_2(\alpha_2' + \frac{\beta_2}{2}) + (\alpha_1+\alpha_3)\alpha_1'}{\alpha}, \qquad F_2 = \frac{(\phi_1+\phi_3)(\beta_1 - \alpha_1') - \phi_2\beta_2}{\phi}$$

$$F_3 = \frac{\alpha[(\phi_1+\phi_3)\beta_1' - 2\phi_2\beta_2'] + 2\alpha_1'\{\alpha_2[\beta_2(\alpha_1+\alpha_3) - \alpha_2(\beta_1+\beta_3)] + (\alpha_1+\alpha_3)[\beta_2\phi_2 - \beta_1(\phi_1+\phi_3)]\}}{\alpha\phi} + \frac{(2\alpha_2' + \beta_2)\{\alpha_2[\phi_1(\beta_1+\beta_3) - \phi_2\beta_2] + (\alpha_1+\alpha_3)(\alpha_2\beta_1 - \alpha_1\beta_2)\}}{\alpha\phi}.$$

For $n = 1$, the same holds with $\beta_i = 0$, $i = 1,2,3$.

Next, we recall that the *tangent sphere bundle of radius* $r>0$ over a Riemannian manifold (M,g) is the hypersurface $T_rM=\{(x,u)\in TM \mid g_x(u,u)=r^2\}$. The tangent space at a point $(x,u)\in T_rM$ is given by

$$(T_rM)_{(x,u)}=\left\{X^h+Y^v \;\middle|\; X\in M_x,\ Y\in\{u\}^{\perp}\subset M_x\right\}.$$

When $r=1$, T_1M is called *the unit tangent (sphere) bundle*.

For the restrictions to T_rM of Riemannian g-natural metrics, we have

PROPOSITION 4 ([5]). *Let $r>0$ and (M,g) be a Riemannian manifold. For every Riemannian metric $\tilde{G}$ on T_rM induced from a Riemannian g-natural metric G on TM, there exist four constants a, b, c and d, with $a>0$, $a(a+c)-b^2>0$ and $a(a+c+dr^2)-b^2>0$, such that*

$$\begin{cases}\tilde{G}_{(x,u)}(X_1^h,X_2^h)=(a+c)g_x(X_1,X_2)+dg_x(X_1,u)g_x(X_2,u),\\ \tilde{G}_{(x,u)}(X_1^h,Y_1^v)=\tilde{G}_{(x,u)}(Y_1^v,X_1^h)=bg_x(X_1,Y_1),\\ \tilde{G}_{(x,u)}(Y_1^v,Y_2^v)=ag_x(Y_1,Y_2),\end{cases}$$

for all $(x,u)\in T_1M$ and X_i, $Y_i\in M_x$, $i=1,2$, with Y_i orthogonal to u.

We shall call such a metric an *induced Riemannian g-natural metric on T_1M*.

Using the Schmidt's orthonormalization process, a simple calculation shows that the vector field on TM defined by

$$N^G_{(x,u)}=\frac{1}{\sqrt{(a+c+d)\phi}}\left[-b\,u^h+(a+c+d)\,u^v\right],$$

for all $(x,u)\in TM$, is normal to T_1M and unitary at any point of T_1M.

We now define the "tangential lift" X^{t_G} (with respect to G) of a vector $X\in M_x$ to $(x,u)\in T_1M$ as the tangential projection of the vertical lift of X to (x,u) (with respect to N^G), that is,

$$X^{t_G}=X^v-G_{(x,u)}(X^v,N^G_{(x,u)})\,N^G_{(x,u)}=X^v-\sqrt{\frac{\phi}{a+c+d}}g_x(X,u)\,N^G_{(x,u)}.$$

If $X\in M_x$ is orthogonal to u, then $X^{t_G}=X^v$. Note that if $b=0$, then X^{t_G} coincides with the classical tangential lift X^t defined for the case of the Sasaki metric. In the general case,

$$X^{t_G}=X^t+\frac{b}{a+c+d}g(X,u)u^h. \tag{3}$$

The tangent space $(T_1M)_{(x,u)}$ of T_1M at (x,u) is spanned by vectors of the form X^h and Y^{t_G}, where X, $Y\in M_x$. Hence, $\tilde{G}$ on T_1M is completely determined by

$$\begin{cases}\tilde{G}_{(x,u)}(X^h,Y^h)=(a+c)g_x(X,Y)+dg_x(X,u)g_x(Y,u),\\ \tilde{G}_{(x,u)}(X^h,Y^{t_G})=\tilde{G}_{(x,u)}(X^{t_G},Y^h)=bg_x(X,Y),\\ \tilde{G}_{(x,u)}(X^{t_G},Y^{t_G})=ag_x(X,Y)-\frac{\phi}{a+c+d}g_x(X,u)g_x(Y,u),\end{cases} \tag{4}$$

for all $(x,u) \in T_1M$ and $X, Y \in M_x$. We now have

PROPOSITION 5 ([1]). *The Levi-Civita connection $\tilde{\nabla}$ of $\tilde{G}$ is given at a point $(x,u) \in T_1M$ by*

$$
\begin{aligned}
(\tilde{\nabla}_{X^h} Y^h)_{(x,u)} = &\Big\{ (\nabla_X Y)_x - \frac{ab}{2\alpha}\big[R(X_x,u)Y_x + R(Y_x,u)X_x\big] + \frac{bd}{2\alpha}\big[g(X_x,u)Y_x + g(Y_x,u)X_x\big] \\
&+ \frac{b}{(a+c+d)\alpha}\Big[(ad+b^2)\,g(R(X_x,u)Y_x,u) - d(a+c+d)\,g(X_x,u)g(Y_x,u)\Big]u \Big\}^h \\
&+ \Big\{ \frac{b^2}{\alpha} R(X_x,u)Y_x - \frac{a(a+c)}{2\alpha} R(X_x,Y_x)u - \frac{(a+c)d}{2\alpha}\big[g(Y_x,u)X_x + g(X_x,u)Y_x\big] \\
&+ \frac{1}{\alpha}\Big[-b^2 g(R(X_x,u)Y_x,u) + d(a+c)\,g(Y_x,u)g(X_x,u)\Big]u \Big\}^{t_G},
\end{aligned}
$$

$$
\begin{aligned}
(\tilde{\nabla}_{X^h} Y^{t_G})_{(x,u)} = &\Big\{ -\frac{a^2}{2\alpha} R(Y_x,u)X_x - \frac{ab^2}{2(a+c+d)\alpha} g(Y_x,u)R(X_x,u)u + \frac{ad}{2\alpha} g(X_x,u)Y_x \\
&+ \frac{db^2}{2(a+c+d)\alpha} g(Y_x,u)X_x + \frac{1}{2(a+c+d)\alpha}\Big[a(ad+b^2)\,g(R(X_x,u)Y_x,u) \\
&+ d\alpha\, g(X_x,Y_x) - ad(2(a+c)+d)\,g(X_x,u)g(Y_x,u)\Big]u \Big\}^h \\
&+ \Big\{ (\nabla_X Y)_x + \frac{ab}{2\alpha} R(Y_x,u)X_x - \frac{b(\alpha-b^2)}{2(a+c+d)\alpha} g(Y_x,u)R(X_x,u)u - \frac{bd}{2\alpha} g(X_x,u)Y_x \\
&- \frac{(a+c)bd}{2(a+c+d)\alpha} g(Y_x,u)X_x + \frac{b}{2(a+c+d)\alpha}\Big[-a(a+c+d)\,g(R(X_x,u)Y_x,u) \\
&+ d(2(a+c)+d)\,g(X_x,u)g(Y_x,u)\Big]u \Big\}^{t_G},
\end{aligned}
$$

$$
\begin{aligned}
(\tilde{\nabla}_{X^{t_G}} Y^h)_{(x,u)} = &\Big\{ -\frac{a^2}{2\alpha} R(X_x,u)Y_x + \frac{b}{a+c+d} g(X_x,u)\,\nabla_u Y + \frac{ad}{2\alpha} g(Y_x,u)X_x \\
&- \frac{ab^2}{2(a+c+d)\alpha} g(X_x,u)R(Y_x,u)u + \frac{db^2}{2(a+c+d)\alpha} g(X_x,u)Y_x \\
&+ \frac{1}{2(a+c+d)\alpha}\Big[a(ad+b^2)\,g(R(X_x,u)Y_x,u) \\
&+ d\alpha\, g(X_x,Y_x) - ad(2(a+c)+d)\,g(X_x,u)g(Y_x,u)\Big]u \Big\}^h \\
&+ \Big\{ \frac{ab}{2\alpha} R(X_x,u)Y_x + \frac{a(a+c)b}{2(a+c+d)\alpha} g(X_x,u)R(Y_x,u)u - \frac{bd}{2\alpha} g(Y_x,u)X_x
\end{aligned}
$$

$$-\frac{(a+c)bd}{2(a+c+d)\alpha}g(X_x,u)Y_x+\frac{b}{2(a+c+d)\alpha}[-a(a+c+d)g(R(X_x,u)Y_x,u)$$
$$\left.+d(2(a+c)+d)g(X_x,u)g(Y_x,u)]u\right\}^{t_G},$$

$$(\tilde{\nabla}_{X^{t_G}}Y^{t_G})_{(x,u)} = \frac{b}{2(a+c+d)\alpha}\Big\{-a^2[g(X_x,u)R(Y_x,u)u+g(Y_x,u)R(X_x,u)u]$$
$$\left.+ad\,g(X_x,u)Y_x+(2\alpha+ad)g(Y_x,u)X_x-\frac{2d\phi}{a+c+d}g(X_x,u)g(Y_x,u)u\right\}^h$$
$$+\frac{1}{2(a+c+d)\alpha}\Big\{2b\alpha g(X_x,u)\nabla_u Y+ab^2[g(X_x,u)R(Y_x,u)u$$
$$\left.+g(Y_x,u)R(X_x,u)u]-b^2d\,g(X_x,u)Y_x-(2(a+c+d)\alpha-b^2d)g(Y_x,u)X_x\right\}^{t_G},$$

for all $(x,u)\in T_1M$ *and* X, Y *vector fields on* M.

CONVENTION 2. The tangential lift to $(x,u)\in T_1M$ of the vector u is given by $u^{t_G}=\frac{b}{a+c+d}u^h$ and so, it is a horizontal vector. Hence, the tangent space $(T_1M)_{(x,u)}$ coincides with

$$\{X^h+Y^{t_G}\mid X\in M_x,\ Y\in\{u\}^{\perp}\subset M_x\}.$$

For this reason, the operation of tangential lift from M_x to a point $(x,u)\in T_1M$ will be always applied only to vectors of M_x which are orthogonal to u.

3. Harmonicity of the canonical projection $\pi:(TM,G)\to(M,g)$

Let (M,g) be a Riemannian manifold of dimension n and (TM,G) its tangent bundle, equipped with an arbitrary Riemannian g-natural metric G. We shall calculate the tension field of the map $\pi:(TM,G)\to(M,g)$, in order to decide when π is harmonic.

If $(E_I;\ I=1,\dots,2n)$ is a local frame (not necessarily orthonormal) on an open subset $W\subset TM$, then the tension field on W is defined by

$$\tau(\pi)\restriction_W=\sum_{I,J=1}^{2n}G^{IJ}\nabla d\pi(E_I,E_J), \tag{5}$$

where (G^{IJ}) is the inverse matrix of the matrix $(G(E_I,E_J))$. In order to calculate $\nabla d\pi(E_I,E_J)$, it is convenient to choose the vector fields E_I on W which are π-related with some local vector fields on M, since in this case

$$\nabla d\pi(E_I,E_J)=\nabla_{\pi_*E_I}(\pi_*E_J)-\pi_*(\bar{\nabla}_{E_I}E_J).$$

Fix $(x,u) \in TM$ and consider an orthonormal moving frame $(e_i;\ i=1,\dots,n)$ on an open subset $U \subset M$, such that $r(e_n)_x = u$, where $r = \|u\|$. On $\pi^{-1}(U)$, we put

$$E_i = e_i^h, \quad E_{n+i} = e_i^v, \qquad i=1,\dots,n. \tag{6}$$

It is easy to see that $(E_I;\ I=1,\dots,2n)$ is a local frame on $\pi^{-1}(U)$, such that E_i is π-related with e_i and E_{n+i} is π-related with the zero section on U, for all $i=1,\dots,n$. Thus,

$$\begin{aligned}
&\nabla d\pi(E_i,E_j) = \nabla_{e_i} e_j - \pi_*(\bar{\nabla}_{e_i^h} e_j^h), \qquad &&\nabla d\pi(E_i,E_{n+j}) = -\pi_*(\bar{\nabla}_{e_i^h} e_j^v),\\
&\nabla d\pi(E_{n+i},E_j) = -\pi_*(\bar{\nabla}_{e_i^v} e_j^h), &&\nabla d\pi(E_{n+i},E_{n+j}) = -\pi_*(\bar{\nabla}_{e_i^v} e_j^v).
\end{aligned} \tag{7}$$

Using Proposition 3, formulas (7) calculated at (x,u) become

(8)

$$\nabla d\pi_{(x,u)}(E_i,E_j) = -A(u;(e_i)_x,(e_i)_x), \qquad \nabla d\pi_{(x,u)}(E_i,E_{n+j}) = -C(u;(e_i)_x,(e_i)_x),$$

$$\nabla d\pi_{(x,u)}(E_{n+i},E_j) = -C(u;(e_i)_x,(e_i)_x), \quad \nabla d\pi_{(x,u)}(E_{n+i},E_{n+j}) = -E(u;(e_i)_x,(e_i)_x).$$

We now consider the inverse matrix $(G^{IJ}_{(x,u)})$, in order to calculate the tension field $\tau(\pi)_{(x,u)}$ of π. Since

$$(G_{(x,u)}(E_I,E_J)) = \begin{pmatrix}
(\alpha_1+\alpha_3)I_{n-1} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \alpha_2 I_{n-1} & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\\
0\cdots 0 & \phi_1+\phi_3 & 0\cdots 0 & \phi_2\\
\alpha_2 I_{n-1} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \alpha_1 I_{n-1} & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\\
0\cdots 0 & \phi_2 & 0\cdots 0 & \phi_1
\end{pmatrix},$$

it is easily seen that

$$(G^{IJ}_{(x,u)}) = \begin{pmatrix}
\frac{\alpha_1}{\alpha} I_{n-1} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & -\frac{\alpha_2}{\alpha} I_{n-1} & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\\
0\cdots 0 & \frac{\phi_1}{\phi} & 0\cdots 0 & -\frac{\phi_2}{\phi}\\
-\frac{\alpha_2}{\alpha} I_{n-1} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \frac{\alpha_1+\alpha_3}{\alpha} I_{n-1} & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\\
0\cdots 0 & -\frac{\phi_2}{\phi} & 0\cdots 0 & \frac{\phi_1+\phi_3}{\phi}
\end{pmatrix}. \tag{9}$$

Substituting from (8) and (9) into (5), we obtain

$$\tau(\pi)_{(x,u)} = \frac{1}{\alpha}\sum_{i=1}^{n}[-\alpha_1 A(u;(e_i)_x,(e_i)_x)) + 2\alpha_2 C(u;(e_i)_x,(e_i)_x)) \\ -(\alpha_1+\alpha_3)E(u;(e_i)_x,(e_i)_x))] \\ +\Big(\frac{\alpha_1}{\alpha}-\frac{\phi_1}{\phi}\Big)A(u;(e_n)_x,(e_n)_x)) - 2\Big(\frac{\alpha_2}{\alpha}-\frac{\phi_2}{\phi}\Big)C(u;(e_n)_x,(e_n)_x)) \\ +\Big(\frac{\alpha_1+\alpha_3}{\alpha}-\frac{\phi_1+\phi_3}{\phi}\Big)E(u;(e_n)_x,(e_n)_x)). \tag{10}$$

We then use the expressions of A, C and E from Proposition 3 and the fact that $r\,(e_n)_x = u$ and (10) becomes

$$\tau(\pi)_{(x,u)} = \tfrac{2}{\alpha}[\alpha_1 A_1 - \alpha_2 C_1]\,Qu + \Big\{\tfrac{1}{\alpha}[\alpha_1 A_3 - 2\alpha_2 C_4]g(Qu,u) \\ +\tfrac{1}{\phi}[-\phi_1(2A_2+A_4+r^2A_5)+2\phi_2(C_2+C_3+C_5+r^2C_6) \\ -(\phi_1+\phi_3)(2E_1+E_2+r^2E_3)] + \tfrac{n-1}{\alpha}[-\alpha_1 A_4 + 2\alpha_2 C_5 - (\phi_1+\phi_3)E_2]\Big\},$$

where Q is the Ricci operator associated to g and A_i,B_i,E_i are evaluated at r^2. Next, a long but routine calculation shows that

$$\alpha_1 A_1 - \alpha_2 C_1 = 0, \quad \alpha_1 A_3 - 2\alpha_2 C_4 = 0, \quad 2A_2 + A_4 + r^2 A_5 = \tfrac{\phi_2}{\phi}(\phi_1+\phi_3)', \\ C_2 + C_3 + C_5 + r^2 C_6 = \tfrac{\phi_1}{\phi}(\phi_1+\phi_3)', \quad 2E_1 + E_2 + r^2 E_3 = \tfrac{1}{\phi}(2\phi_1\phi_2' - \phi_2\phi_1'), \\ -\alpha_1 A_4 + 2\alpha_2 C_5 - (\phi_1+\phi_3)E_2 = \tfrac{1}{\phi}\Big\{-\phi_2\alpha' + \phi_1[\alpha_2(\beta_1+\beta_3) - (\alpha_1+\alpha_3)\beta_2] \\ \phi_2\big[(\alpha_1+\alpha_3)\beta_1 - \alpha_2\beta_2\big]\Big\}.$$

Substituting these equations into the previous expression for $\tau(\pi)_{(x,u)}$, we obtain

$$\tau(\pi)_{(x,u)} = \Big\{\frac{1}{\phi^2}\big[\phi_2\phi' - 2\phi\phi_2'\big] + \frac{n-1}{\alpha\phi}\{-\phi_2\alpha' + \phi_1\big[\alpha_2(\beta_1+\beta_3) - (\alpha_1+\alpha_3)\beta_2\big] \\ +\phi_2[(\alpha_1+\alpha_3)\beta_1 - \alpha_2\beta_2]\}\Big\}\,u.$$

In this way, we proved the following

THEOREM 1. *Let (M,g) be a Riemannian manifold of dimension n and (TM,G) its tangent bundle, equipped with an arbitrary Riemannian g-natural metric G. The canonical projection $\pi : (TM,G) \to (M,g)$ is harmonic if and only if the functions defining the metric G satisfy*

$$(n-1)\phi\{-\phi_2\alpha' + \phi_1[\alpha_2(\beta_1+\beta_3) - (\alpha_1+\alpha_3)\beta_2] \\ +\phi_2[(\alpha_1+\alpha_3)\beta_1 - \alpha_2\beta_2]\} + \alpha[\phi_2\phi' - 2\phi\phi_2'] = 0. \tag{11}$$

We explicitly note that for any arbitrary choice of (for example) $\alpha_1, \alpha_2, \alpha_3, \beta_1$, β_2 defining G, (11) gives a first order linear differential equation which fixes the remaining function β_3. The standard existence theorem ensures that (11) admits solutions, depending on an arbitrary real parameter. Therefore, Theorem 1 yields

COROLLARY 1. *Riemannian g-natural metrics G on the tangent bundle TM, for which* $\pi : (TM,G) \to (M,g)$ *is harmonic, form a class depending on five arbitrary smooth functions (satisfying* (2)*) and a real parameter.*

In the special case when $\alpha_2 = \beta_2 = 0$, equation (11) is trivially satisfied. So, we have the following

COROLLARY 2. *Let* (M,g) *be a Riemannian manifold of dimension n, whose tangent bundle TM is equipped with an arbitrary Riemannian g-natural metric G, with respect to which horizontal and vertical distributions are orthogonal. Then* $\pi : (TM,G) \to (M,g)$ *is harmonic.*

Corollary 2 extends to a large family of Riemannian g-natural metrics, depending on four functions of one variable, the classical result about the harmonicity of $\pi : (TM, g^S) \to (M,g)$, and the result proved in [16] concerning the harmonicity of $\pi : (TM, g^{CG}) \to (M,g)$.

Another interesting class of Riemannian metrics on TM to which Corollary 2 applies, is the non-classical family of metrics studied by Oproiu in [17], and described as follows: for any of such a metric, there exist two smooth functions $v, w : \mathbb{R}^+ \to \mathbb{R}$, such that (see also [7])

$$\begin{cases} \alpha_1(t) = \frac{1}{v(t/2)}, & \alpha_2(t) = 0, \quad (\alpha_1 + \alpha_3)(t) = v(t/2), \\ \beta_1(t) = -\frac{w(t/2)}{v(t/2)[v(t/2)+tw(t/2)]}, & \beta_2(t) = 0, \quad (\beta_1 + \beta_3)(t) = w(t/2). \end{cases}$$

As regards explicit examples of Riemannian g-natural metrics on TM, whose horizontal and vertical distributions are not orthogonal but have a harmonic canonical projection, one can easily deduce the following from Theorem 1.

THEOREM 2. *Let* (M,g) *be a Riemannian manifold of dimension n, TM its tangent bundle and G be a Riemannian g-natural metric whose defining functions satisfy* $\beta_1 = \beta_2 = \beta_3 = 0$*.*

a) If $n = 2$*, then the canonical projection* $\pi : (TM,G) \to (M,g)$ *is harmonic if and only if* α_2 *is constant;*

b) If $n > 2$*, then the canonical projection* $\pi : (TM,G) \to (M,g)$ *is harmonic if and only if*

- *either* $\alpha_2 = 0$ *identically, or*
- $\alpha_2(t) \neq 0$ *for all t and there exists a real constant* $K > 0$*, such that* $\alpha = K|\alpha_2|^{\frac{-2}{n-2}}$*.*

In particular, Theorem 2 implies that if $n > 2$ and G is a Riemannian g-natural metric satisfying $\beta_1 = \beta_2 = \beta_3 = 0$ and

(i) either $\alpha_1 = a$, $\alpha_2 = b$, $\alpha_3 = c$, for three real constants $a > 0$, $b \neq 0$ and c (satisfying $a(a+c) - b^2 > 0$), or

(ii) $\alpha_1(t) = \sqrt{1 + Ke^{-2t}}\,e^{\frac{n-2}{n-1}t}$, $\alpha_2(t) = e^{\frac{n-2}{n-1}t}$, $\alpha_3 = 0$, for a real constant $K > 0$,

then $\pi : (TM, G) \to (M, g)$ is harmonic.

4. Harmonicity of G with respect to g^{S} and conversely

In this section, we shall study when an arbitrary Riemannian g-natural metric G on TM is harmonic with respect to the Sasaki metric g^{S} and conversely, that is, the harmonicity of the identity maps $\mathrm{id}_{g^{\mathrm{S}}G} : (TM, g^{\mathrm{S}}) \to (TM, G)$ and $\mathrm{id}_{Gg^{\mathrm{S}}} : (TM, G) \to (TM, g^{\mathrm{S}})$, calculating their tension fields.

We fix $(x,u) \in TM$ and consider an orthonormal moving frame $(e_i;\ i = 1, \ldots, n)$ on an open subset $U \subset M$, such that $r\,(e_n)_x = u$, where $r = \|u\|$. On $\pi^{-1}(U)$, we consider the local moving frame $(E_I;\ I = 1, \ldots, 2n)$ given by (6). The tension field of $\mathrm{id}_{g^{\mathrm{S}}G}$ at (x,u) is given by

$$\tau_{(x,u)}(\mathrm{id}_{g^{\mathrm{S}}G}) = \sum_{i=1}^{2n} (\overline{\nabla}_{E_I} E_I - \nabla^{s}_{E_I} E_I)_{(x,u)}, \tag{12}$$

where ∇^s is the Levi-Civita connection of the Sasaki metric. Using Proposition 3, equation (12) becomes

$$\begin{aligned} \tau_{(x,u)}(\mathrm{id}_{g^{\mathrm{S}}G}) \ &= \ h\Big\{ \sum_{i=1}^{n} [A(u;e_i,e_i) - A^0(u;e_i,e_i) + E(u;e_i,e_i) - E^0(u;e_i,e_i)] \Big\} \\ &\quad + v\Big\{ \sum_{i=1}^{n} [B(u;e_i,e_i) - B^0(u;e_i,e_i) + F(u;e_i,e_i) - F^0(u;e_i,e_i)] \Big\}, \end{aligned}$$

where A^0, B^0, E^0, F^0 are the F-tensor fields associated to the Sasaki metric. Hence,

$$\begin{aligned} &\tau_{(x,u)}(\mathrm{id}_{g^{\mathrm{S}}G}) \\ &\quad = h\Big\{ -2A_1Qu + [-A_3 g(Qu,u) + 2A_2 + r^2A_5 + nA_4]u + [2E_1 + r_3^E + nE_2]u \Big\} \\ &\qquad + v\Big\{ B_1Qu + [-B_4 g(Qu,u) + 2B_3 + r^2B_6 + nB_5]u + [2F_1 + r_3^F + nF_2]u \Big\} \end{aligned}$$

and so, $\tau_{(x,u)}(\mathrm{id}_{g^{\mathrm{S}}G}) = 0$ if and only if

$$\begin{cases} 2A_1Qu = [-A_3 g(Qu,u) + 2A_2 + r^2A_5 + nA_4 + 2E_1 + r_3^E + nE_2]u, \\ B_1Qu = -[-B_4 g(Qu,u) + 2B_3 + r^2B_6 + nB_5 + 2F_1 + r_3^F + nF_2]u. \end{cases} \tag{13}$$

We take the scalar product of both equations in (13) by u. Calculating $g(Qu,u)$ from both of them, a routine calculation gives

$$\begin{aligned}&\alpha_2\phi_1 g(Qu,u)\\ &\quad = r^2\{\phi_2\phi_3' + 2\phi_1\phi_2' + (n-1)[\phi_1\beta_2 - \phi_2(\beta_1 - \alpha_1' - (\alpha_1+\alpha_3)')]\}\\ &\quad = r^2\{(\phi_1+\phi_3)\phi_3' + 2\phi_2\phi_2' + (n-1)[\phi_2\beta_2 - (\phi_1+\phi_3)(\beta_1 - \alpha_1' - (\alpha_1+\alpha_3)')]\},\end{aligned}$$

and so

$$\phi_3' = (n-1)[\beta_1 - \alpha_1' - (\alpha_1+\alpha_3)'] \tag{14}$$

and

$$-\alpha_2 g(Qu,u) = r^2\{2\phi_2' + (n-1)\beta_2\},$$

which, used into (13), leads to conclude that the tension field of the map $\mathrm{id}_{g^S G}$ vanishes identically if and only if (14) holds and

$$-\alpha_2 Qu = [2\phi_2' + (n-1)\beta_2]u. \tag{15}$$

If $\alpha_2 = 0$, then (15) gives $2\phi_2' + (n-1)\beta_2 = 0$, that is,

$$(n+1)\beta_2(t) + t\beta_2'(t) = 0, \quad \text{for all} \quad t \in \mathbb{R}^+. \tag{16}$$

We now prove that β_2 vanishes identically on $\mathbb{R}^+$. In fact, if $\beta_2 \neq 0$, we can consider the open set $I = \{t \in \mathbb{R}^+,\ \beta_2(t) \neq 0\} \neq \emptyset$. A connected component J of I is an open interval of I. If we put $t_0 = \inf J$, then $\beta_2(t_0) = 0$. Integrating (16), we get $\beta_2 = Kt^{-(n+1)}$ on J, for some $K \in \mathbb{R}$. The continuity of β_2 at t_0 then implies $K = 0$, which contradicts $\beta_2 \neq 0$ on J. Therefore, $\beta_2 = 0$.

Next, if $\alpha_2(t_1) \neq 0$ for some $t_1 \in \mathbb{R}^+$, we consider the open subset $I' = \{t \in \mathbb{R}^+,\ \alpha_2(t) \neq 0\} \neq \emptyset$ and a connected component $J' \subset I'$. By virtue of (15), we have

$$Qu = \frac{2\phi_2' + (n-1)\beta_2}{\alpha_2}u, \quad \text{for all} \quad u \quad \text{such that} \quad \|u\|^2 \in J'.$$

The linearity of Q then implies that the function $\frac{2\phi_2' + (n-1)\beta_2}{\alpha_2} = K$ is constant on J', and $Qu = Ku$, for all u such that $\|u\|^2 \in J'$. Again the linearity of Q then yields $Qu = Ku$ for all $u \in TM$. In this way, we proved the following

THEOREM 3. *Let (M,g) be a Riemannian manifold of dimension n and G an arbitrary Riemannian g-natural metric G on TM. The identity map $\mathrm{id}_{g^S G} : (TM, g^S) \to (TM, G)$ is harmonic if and only if* (14) *holds and*

- *either horizontal and vertical distributions are orthogonal with respect to G, or*
- *(M,g) is an Einstein manifold, with $Qu = Ku$ for all u, and*

$$2\phi_2' + (n-1)\beta_2 = K\alpha_2. \tag{17}$$

Note that (14) fixes β_3 in function of some of the remaining defining functions of G, as a solution of a first order linear differential equation. By Theorem 3, if (M,g) is a Riemannian manifold which is not Einstein, then $\mathrm{id}_{g^S G}$ is harmonic if and only if $\alpha_2 = \beta_2 = 0$ and (14) holds. Hence, in this case, Riemannian g-natural metrics G for which $\mathrm{id}_{g^S G}$ is harmonic, form a class depending on three smooth functions and a real parameter. When (M,g) is Einstein, g-natural metrics G for which $\mathrm{id}_{g^S G}$ is harmonic, depend on four smooth functions and a real parameter, since in this case (14) and (17) must hold. Some explicit examples are given in the following

COROLLARY 3. *Let (M,g) be an Einstein manifold, with $Qu = 2\lambda u$ for all u, $\lambda > 0$, and G a Riemannian g-natural metric on TM, whose defining functions α_i, β_i are given by*

$$\begin{array}{lll} \alpha_1(t) = Ke^{\lambda t}, & \alpha_2(t) = K'e^{\lambda t}, & \alpha_3 = 0, \\ \beta_1(t) = 2\lambda K e^{\lambda t}, & \beta_2 = 0, & \beta_3 = 0, \end{array}$$

for some real numbers K and K', satisfying $K > |K'|$. Then, the identity map $\mathrm{id}_{g^S G}$: $(TM, g^S) \to (TM, G)$ is harmonic.

Note that the inequalities $\lambda \geq 0$ and $K > |K'|$ are the necessary and sufficient conditions for G to be Riemannian.

As a special subclass of Riemannian g-natural metrics on TM of Corollary 1, we can quote the linear combination, with constant factors, of the classical lifts g^S and g^h of g, that is, $G = a g^S + b h$, where $a > |b|$.

The study of the harmonicity of $\mathrm{id}_{G g^S} : (TM, G) \to (TM, g^S)$, for an arbitrary Riemannian g-natural metric G on TM, is significantly more difficult. As before, to calculate the tension field of $\mathrm{id}_{G g^S}$, we fix $(x,u) \in TM$ and consider an orthonormal moving frame $(e_i;\ i = 1,\dots,n)$ on an open subset $U \subset M$, such that $r(e_n)_x = u$, where $r = \|u\|$. On $\pi^{-1}(U)$, we consider the local moving frame $(E_I;\ I = 1,\dots,2n)$ given by (6). Using (9), the tension field of $\mathrm{id}_{TM} : (TM, G) \to (TM, g^S)$ at the point (x,u) is given by

$$\begin{aligned} \tau_{(x,u)}(\mathrm{id}_{G g^S}) := {} & \sum_{i=1}^{n-1} \frac{\alpha_1}{\alpha} (\nabla^s_{e_i^h} e_i^h - \bar{\nabla}_{e_i^h} e_i^h)_{(x,u)} + \frac{\phi_1}{\phi} (\nabla^s_{e_n^h} e_n^h - \bar{\nabla}_{e_n^h} e_n^h)_{(x,u)} \\ & - \sum_{i=1}^{n-1} \frac{\alpha_2}{\alpha} [(\nabla^s_{e_i^h} e_i^v - \bar{\nabla}_{e_i^h} e_i^v) + (\nabla^s_{e_i^v} e_i^h - \bar{\nabla}_{e_i^v} e_i^h)]_{(x,u)} \\ & - \frac{\phi_2}{\phi} [(\nabla^s_{e_n^h} e_n^v - \bar{\nabla}_{e_n^h} e_n^v) + (\nabla^s_{e_n^v} e_n^h - \bar{\nabla}_{e_n^v} e_n^h)]_{(x,u)} \\ & + \sum_{i=1}^{n-1} \frac{\alpha_1 + \alpha_3}{\alpha} (\nabla^s_{e_i^v} e_i^v - \bar{\nabla}_{e_i^v} e_i^v)_{(x,u)} + \frac{\phi_1 + \phi_3}{\phi} (\nabla^s_{e_n^v} e_n^v - \bar{\nabla}_{e_n^v} e_n^v)_{(x,u)}. \end{aligned}$$

Using Proposition 3, long calculations then give

$$\tau_{(x,u)}(\mathrm{id}_{Gg^S}) = -\Big\{\frac{1}{\phi^2}\left[\phi_2\phi_1(\phi_1+\phi_3)'+\phi_2(\phi+\phi_3)\phi_1'-2\phi_1(\phi_1+\phi_3)\phi_2'\right]$$
$$+\frac{n-1}{\alpha\phi}\Big[\ \phi_2\alpha'-\phi_1[\alpha_2(\beta_1+\beta_3)-(\alpha_1+_3)\beta_2]-\phi_2[(\alpha_1+\alpha_3)\beta_1-\alpha_2\beta_2]\Big]\Big\}u^h$$
$$-\Big\{\frac{1}{\phi^2}\Big[(\phi_2^2-\phi)(\phi_1+\phi_3)'+(\phi+\phi_3)^2\phi_1'-2\phi_2(\phi_1+\phi_3)\phi_2'\Big]+\frac{n-1}{\alpha\phi}\Big[-(\phi_1+\phi_3)\alpha'$$
$$+\phi_2[\alpha_2(\beta_1+\beta_3)-(\alpha_1+_3)\beta_2]+(\phi_1+\phi_3)[(\alpha_1+\alpha_3)\beta_1-\alpha_2\beta_2]\Big]\Big\}u^v.$$

Hence, we have the following

THEOREM 4. *Let (M,g) be a Riemannian manifold of dimension n and G an arbitrary Riemannian g-natural metric on TM. The identity map* $\mathrm{id}_{Gg^S} : (TM,G) \to (TM,g^S)$ *is harmonic if and only if the following conditions are satisfied:*

$$0 = \frac{1}{\phi}\left[\phi_2\phi_1(\phi_1+\phi_3)'+\phi_2(\phi+\phi_3)\phi_1'-2\phi_1(\phi_1+\phi_3)\phi_2'\right]$$
$$+\frac{n-1}{\alpha}\Big[\phi_2\alpha'-\phi_1[\alpha_2(\beta_1+\beta_3)-(\alpha_1+_3)\beta_2]$$
$$-\phi_2[(\alpha_1+\alpha_3)\beta_1-\alpha_2\beta_2]\Big], \tag{18}$$

$$0 = \frac{1}{\phi}\Big[(\phi_2^2-\phi)(\phi_1+\phi_3)'+(\phi_1+\phi_3)^2\phi_1'-2\phi_2(\phi_1+\phi_3)\phi_2'\Big]$$
$$+\frac{n-1}{\alpha}\Big[-(\phi_1+\phi_3)\alpha'+\phi_2[\alpha_2(\beta_1+\beta_3)-(\alpha_1+\alpha_3)\beta_2]$$
$$+(\phi_1+\phi_3)[(\alpha_1+\alpha_3)\beta_1-\alpha_2\beta_2]\Big]. \tag{19}$$

The class of Riemannian g-natural metrics G satisfying (18)–(19) is very large, since these equations can be used to determine two of the defining functions of G by means of the remaining four defining functions. However, the complexity of conditions (18) and (19) does not permit to give them an easy geometrical interpretation.

In order to find examples, we shall restrict ourselves to some special cases. The first we investigate is the one where horizontal and vertical distributions are orthogonal with respect to G. Then, (18) is automatically satisfied, while (19) reduces to

$$(\phi_1+\phi_3)'-\frac{\phi_1+\phi_3}{\phi_1}\phi_1' = \frac{n-1}{\alpha}\Big[-(\phi_1+\phi_3)\alpha'+\ (\phi_1+\phi_3)(\alpha_1+\alpha_3)\beta_1\Big].$$

Assuming also that this equation does not depend on the dimension n of M, it gives

$$\begin{cases} \phi_1(\phi_1+\phi_3)' = (\phi_1+\phi_3)\phi_1', \\ -(\phi_1+\phi_3)\alpha'+(\phi_1+\phi_3)(\alpha_1+\alpha_3)\beta_1 = 0. \end{cases} \tag{20}$$

Since $\alpha_2 = \beta_2 = 0$, by (2) we necessarily have $\phi_1+\phi_3 > 0$ and $\alpha_1+\alpha_3 > 0$. Thus, (20) reduces to

$$\phi_1(\phi_1+\phi_3)' = (\phi_1+\phi_3)\phi_1', \qquad \beta_1 = \frac{\alpha'}{\alpha_1+\alpha_3}. \tag{21}$$

Integrating the first equation in (21), we get $\phi_1 = K_1(\phi_1+\phi_3)$ for a real constant $K_1 > 0$. Solving $\phi_1 = K_1(\phi_1+\phi_3)$ with respect to β_3, we then have

$$\beta_3 = \frac{1-K_1}{K_1 t}\phi_1 - \frac{1}{t}\alpha_3, \qquad \beta_1 = \frac{(\alpha(\alpha_1+\alpha_3))'}{\alpha_1+\alpha_3}. \tag{22}$$

Hence, we can state the following

PROPOSITION 6. *For any Riemannian g-natural metric G such that horizontal and vertical distributions are orthogonal and* (22) *hold, the identity map* $\mathrm{id}_{TM} : (TM,G) \to (TM,g^S)$ *is harmonic.*

Next, we want to find further examples of Riemannian g-natural metrics to which Theorem 4 applies, but whose horizontal and vertical distributions are not orthogonal. A reasonable assumption is that conditions (18)–(19) do not depend on the dimension n of the base manifold M. In this case, from (18)–(19) we get

$$\begin{cases} \phi_2\phi_1(\phi_1+\phi_3)' + \phi_2(\phi+\phi_3)\phi_1' - 2\phi_1(\phi_1+\phi_3)\phi_2' = 0, \\ \phi_2\alpha' - \phi_1[\alpha_2(\beta_1+\beta_3) - (\alpha_1+_3)\beta_2] - \phi_2[(\alpha_1+\alpha_3)\beta_1 - \alpha_2\beta_2] = 0, \\ (\phi_2^2-\phi)(\phi_1+\phi_3)' + (\phi+\phi_3)^2\phi_1' - 2\phi_2(\phi_1+\phi_3)\phi_2' = 0, \\ -(\phi_1+\phi_3)\alpha' + \phi_2[\alpha_2(\beta_1+\beta_3) - (\alpha_1+\alpha_3)\beta_2] + (\phi_1+\phi_3)[(\alpha_1+\alpha_3)\beta_1 - \alpha_2\beta_2] = 0. \end{cases}$$

There are plenty of examples of Riemannian g-natural metrics on TM satisfying this set of equations. In particular, it is easy to check that $\mathrm{id}_{TM} : (TM,G) \to (TM,g^S)$ is harmonic in the following situations:

- G is a linear combination, with constant factors, of the classical lifts g^S, g^h and g^v of g; that is, there exist three constants a,b,c, satisfying $a(a+c)-b^2 > 0$, such that $\alpha_1 = a$, $\alpha_2 = b$, $\alpha_3 = c$, while $\beta_i = 0$ for all $i = 1,2,3$.
- $G = e^{\frac{t}{2}} G^0_{\lambda,\mu}$, where λ,μ are real constants satisfying $1+\mu > \lambda^2$, and $G^0_{\lambda,\mu}$ is the Riemannian g-natural metric defined by

$$\begin{cases} G^0_{\lambda,\mu}(X^h,Y^h) &=& g(X,Y) + g(X,u)g(Y,u), \\ G^0_{\lambda,\mu}(X^h,Y^v) &=& \mu(g(X,Y) + g(X,u)g(Y,u)), \\ G^0_{\lambda,\mu}(X^v,Y^v) &=& \lambda(g(X,Y) + g(X,u)g(Y,u)). \end{cases}$$

5. Harmonicity of the canonical projection $\pi_1 : (T_1M,\tilde{G}) \to (M,g)$

In this section, we study the harmonicity of the canonical projection $\pi_1 : (T_1M,\tilde{G}) \to (M,g)$, where $\tilde{G}$ is an arbitrary induced Riemannian g-natural $\tilde{G}$ on T_1M. By Proposition 4, there exist four constants a, b, c and d, with $a > 0$, $a(a+c)-b^2 > 0$ and $a(a+c+d)-b^2 > 0$, such that (4) is satisfied.

We fix $(x,u) \in T_1M$ and consider an orthonormal moving frame $(e_i;\ i=1,\dots,n)$ on an open subset $U \subset M$, such that $(e_n)_x = u$. Notice that we shall use the classical tangential lift e_i^t of the local vector fields e_i instead of tangential lifts $e_i^{t_G}$, since if $b \neq 0$, then $e_i^{t_G}$ does not project onto a vector field. If we consider on $\pi_1^{-1}(U)$ the vector fields

$$\tilde{E}_i = e_i^h, \quad \tilde{E}_{n+j} = e_j^t; \qquad i=1,\dots,n, \quad j=1,\dots,n-1,$$

then $(\tilde{E}_I;\ I=1,\dots,2n-1)$ is not a local frame on the whole $\pi^{-1}(U)$. In fact, $(e_1)_x \in \pi_1^{-1}(U)$ and $(\tilde{E}_{n+1})_{(e_1)_x} = 0$. However, $((\tilde{E}_I)_{(x,u)};\ I=1,\dots,2n-1)$ is a basis of $(T_1M)_{(x,u)}$. So, there exists an open set $W \subset \pi_1^{-1}(U)$, such that $(\tilde{E}_I;\ I=1,\dots,2n-1)$ is a local frame on W. Hence, since π_1 is an open mapping, then $U' := \pi_1(W)$ is an open set of U.

Now, it is easy to see that $\tilde{E}_i \restriction_W$ is π_1-related with $e_i \restriction_{U'}$ and $\tilde{E}_{n+j} \restriction_W$ is π_1-related with the zero section on U', for all $i=1,\dots,n$ and $j=1,\dots,n-1$. We deduce that, for all $i,j=1,\dots,n$ and $k,l=1,\dots,n-1$,

$$\begin{aligned}
&\nabla d\pi_1(\tilde{E}_i,\tilde{E}_j) = \nabla_{e_i}e_j - \pi_*(\widetilde{\nabla}_{e_i^h}e_j^h), \qquad && \nabla d\pi(\tilde{E}_i,\tilde{E}_{n+k}) = -\pi_*(\widetilde{\nabla}_{e_i^h}e_k^t),\\
&\nabla d\pi(\tilde{E}_{n+k},\tilde{E}_i) = -\pi_*(\widetilde{\nabla}_{e_k^t}e_i^h), && \nabla d\pi(\tilde{E}_{n+k},\tilde{E}_{n+l}) = -\pi_*(\widetilde{\nabla}_{e_k^t}e_l^t).
\end{aligned} \tag{23}$$

Using Proposition 5, the identity (3) and the facts that $g((e_i)_x,u) = 0$, $i=1,\dots,n-1$, and $(e_n)_x = u$, the expression of (23) at (x,u) becomes

(24)

$$\begin{aligned}
\nabla d\pi_{(x,u)}(\tilde{E}_i,\tilde{E}_j) &= \frac{ab}{2\alpha}[R((e_i)_x,u)(e_j)_x + R((e_j)_x,u)(e_i)_x]\\
&\quad - \frac{bd}{2\alpha}[g((e_i)_x,u)(e_j)_x + g((e_j)_x,u)(e_i)_x]\\
&\quad - \frac{b}{(a+c+d)\alpha}[(ad+b^2)\,g(R((e_i)_x,u)(e_j)_x,u)\\
&\qquad\qquad - d(a+c+d)\,g((e_i)_x,u)g((e_j)_x,u)]u,\\
\nabla d\pi_{(x,u)}(\tilde{E}_i,\tilde{E}_{n+k}) &= \frac{a^2}{2\alpha}R((e_k)_x,u)(e_i)_x - \frac{ad}{2\alpha}g((e_i)_x,u)\,(e_k)_x\\
&\quad - \frac{1}{2(a+c+d)\alpha}\,[a(ad+b^2)\,g(R((e_i)_x,u)(e_k)_x,u) + d\alpha\delta_{ik}]u,\\
\nabla d\pi_{(x,u)}(\tilde{E}_{n+k},\tilde{E}_i) &= \frac{a^2}{2\alpha}R((e_k)_x,u)(e_i)_x - \frac{ad}{2\alpha}g((e_i)_x,u)\,(e_k)_x\\
&\quad - \frac{1}{2(a+c+d)\alpha}\,[a(ad+b^2)\,g(R((e_i)_x,u)(e_k)_x,u) + d\alpha\delta_{ik}]u,\\
\nabla d\pi_{(x,u)}(\tilde{E}_{n+k},\tilde{E}_{n+l}) &= \frac{b}{a+c+d}\delta_{kl}\,u.
\end{aligned}$$

The tension field of π_1 on W is defined by

$$(25) \qquad \tau(\pi_1)\restriction_W = \sum_{I,J=1}^{2n-1} \tilde{G}^{IJ} \nabla d\pi_1(\tilde{E}_I, \tilde{E}_J);$$

where $(\tilde{G}^{IJ})$ is the inverse matrix of the matrix $(\tilde{G}(\tilde{E}_I, \tilde{E}_J))$. We shall calculate the tension field of π_1 at (x,u). For this, note that the matrix $(\tilde{G}_{(x,u)}(\tilde{E}_I, \tilde{E}_J))$ is expressed as

$$(\tilde{G}_{(x,u)}(\tilde{E}_I, \tilde{E}_J)) = \begin{pmatrix} (a+c)I_{n-1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & bI_{n-1} \\ 0 \cdots 0 & a+c+d & 0 \cdots 0 \\ bI_{n-1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & aI_{n-1} \end{pmatrix}$$

and its inverse is given by

$$(\tilde{G}^{IJ}) = \begin{pmatrix} \frac{a}{\alpha}I_{n-1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & -\frac{b}{\alpha}I_{n-1} \\ 0 \cdots 0 & \frac{1}{a+c+d} & 0 \cdots 0 \\ -\frac{b}{\alpha}I_{n-1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \frac{a+c}{\alpha}I_{n-1} \end{pmatrix}.$$

Substituting from the last matrix expression and the identities (24) into (25), we obtain

$$\tau(\pi_1)_{(x,u)} = (n-1)\frac{b}{\alpha}\, u,$$

which implies at once the following

THEOREM 5. *Let (M,g) be a Riemannian manifold of dimension $n > 1$ and $(T_1M, \tilde{G})$ its unit tangent bundle, equipped with an arbitrary induced Riemannian g-natural metric $\tilde{G}$. The canonical projection $\pi_1 : (T_1M, \tilde{G}) \to (M,g)$ is harmonic if and only if the horizontal and tangential distributions of T_1M are orthogonal with respect to $\tilde{G}$.*

Theorem 5 includes as a very special case the result by Oniciuc [16] concerning the harmonicity of $\pi_1 : (T_1M, \widetilde{g^S}) \to (M,g)$, and gives yet another interesting geometrical meaning to the orthogonality of the horizontal and tangential distributions of $(T_1M, \tilde{G})$.

6. Harmonicity of $\tilde{G}$ with respect to $\widetilde{g^S}$ and conversely

In this section, we make use of techniques very similar to the ones used in Section 4. For this reason, we shall omit the details about how the formulas are deduced. The tension fields of $\mathrm{id}_{\widetilde{g^S}\tilde{G}} : (T_1M, \widetilde{g^S}) \to (T_1M, \tilde{G})$ and $\mathrm{id}_{\tilde{G}\tilde{g}_s} : (T_1M, \tilde{G}) \to (T_1M, \widetilde{g^S})$ turn out to be given, respectively, by

$$\begin{aligned}
\tau_{(x,u)}(\mathrm{id}_{\widetilde{g^S}\tilde{G}}) &= \frac{b}{\alpha} h\Big\{aQu - \frac{ad+b^2}{\alpha} g(Qu,u)u\} - \frac{b^2}{\alpha} v\{Qu - g(Qu,u)u\Big\}, \\
\tau_{(x,u)}(\mathrm{id}_{\tilde{G}\tilde{g}_s}) &= -\frac{b}{\alpha} h\{Qu\}.
\end{aligned}$$

Therefore, we easily deduce the following results.

THEOREM 6. *Let (M,g) be a Riemannian manifold and $\tilde{G}$ an arbitrary induced Riemannian g-natural metric on T_1M. The identity map $\mathrm{id}_{\widetilde{g^S}\tilde{G}} : (TM, \widetilde{g^S}) \to (T_1M, \tilde{G})$ is harmonic if and only if the horizontal and vertical distributions of T_1M are orthogonal with respect to $\tilde{G}$.*

THEOREM 7. *Let (M,g) be a Riemannian manifold and $\tilde{G}$ an arbitrary induced Riemannian g-natural metric on T_1M.*

i) If (M,g) is Ricci-flat, then the identity map $\mathrm{id}_{\tilde{G}\widetilde{g^S}} : (TM, \tilde{G}) \to (T_1M, \widetilde{g^S})$ is always harmonic.

ii) If (M,g) is not Ricci-flat, then $\mathrm{id}_{\tilde{G}\widetilde{g^S}} : (TM, \tilde{G}) \to (T_1M, \widetilde{g^S})$ is harmonic if and only if horizontal and tangential distributions of T_1M are orthogonal with respect to $\tilde{G}$.

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