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A FIELD OF GENERALISED PUISEUX SERIES FOR TROPICAL GEOMETRY

Abstract. In this paper we define a field \mathbb{K} of characteristic zero with valuation whose value group is $(\mathbb{R}, +)$, and we show that this field of *generalised Puiseux series* is algebraically closed and complete with respect to the norm induced by its valuation. We consider this field to be a good candidate for the base field for tropical geometry.

1. Introduction

In order to study the geometric properties of a variety, say $V = V(I) \subseteq (C^*)^n$ with $I \triangleleft C[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, it is common to study deformations of the variety, or rather of its defining equations, i.e. we replace the ideal I by an ideal $I_t \triangleleft C[[t]][x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ such that $I = \{f|_{t=0} \mid f \in I_t\}$. The generic fibre of this family is then defined by the ideal which I_t generates over the quotient field $\text{Quot}(C[[t]])$ of the power series ring $C[[t]]$. Unfortunately, this field is not algebraically closed. If we are interested in the geometric properties of the general fibre it thus is natural to pass to the algebraic closure of this field, which is the field

$$C\{\{t\}\} = \bigcup_{N=1}^{\infty} C\left[\left[t^{\frac{1}{N}}\right]\right] = \left\{ \sum_{k=m}^{\infty} a_k \cdot t^{\frac{k}{N}} \mid m \in \mathbb{Z}, N \in \mathbb{N}, a_k \in C \right\}$$

of Puiseux series over C . This field comes with a valuation

$$\text{val} : C\{\{t\}\}^* \longrightarrow \mathbb{Q} : \sum_{k=m}^{\infty} a_k \cdot t^{\frac{k}{N}} \mapsto \min \left\{ \frac{k}{N} \mid a_k \neq 0 \right\}$$

sending a Puiseux series to its order. Given an ideal $J \triangleleft C\{\{t\}\}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and its variety $V(J) \subseteq (C\{\{t\}\}^*)^n$ the idea of tropical geometry is to try to understand $V(J)$ better by just looking at its image under the n -fold Cartesian product of the valuation map

$$\text{val} : (C\{\{t\}\}^*)^n \longrightarrow \mathbb{Q}^n : (p_1, \dots, p_n) \mapsto (\text{val}(p_1), \dots, \text{val}(p_n)),$$

or rather its closure, say $\text{Trop}(V(J))$, in \mathbb{R}^n under the Euclidean topology. (Depending on whether they prefer max over min people sometimes use the negative of this function val for the process of tropicalisation.) Due to the Theorem of Bieri-Groves (see [2, Thm. A], [18, Thm. 2.4], [6, Thm. 2.2.5]) and the Lifting Lemma (see [6, Thm. 2.13], [18, Thm. 2.1], [5, Thm. 4.2], [9, Thm. 2.13]) this object turns out to be piecewise linear and its points, say ω , can be characterised by the fact that the t -initial ideal of

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J with respect to ω is monomial free (see e.g. [9]). Taking into account how crude the valuation map is, that is, how much information it ignores (e.g. [13, Thm. 4.2] shows that each fibre of the restriction of val to $V(J)$ is dense in $V(J)$ as soon as it is non-empty), it is surprising how much valuable information is preserved (see e.g. [6, 17, 19, 5, 7, 12, 16, 10, 3]).

Forgetting motivation as to why the field $C\{\{t\}\}$ should be an interesting field to start with, one can replace $C\{\{t\}\}$ by any field K with a valuation whose value group is dense in \mathbb{R} with respect to the Euclidean topology, and study the tropicalisation of varieties $(K^*)^n$ via the n -fold Cartesian product of the valuation map. The Lifting Lemma holds in any case (see [18, Thm. 2.1], [5, Thm. 4.2]), and it seems somehow more natural to choose a field where the valuation map is surjective onto \mathbb{R} , so that $\text{Trop}(V(J))$ coincides with $\text{val}(V(J))$ and no topological closure is necessary, which also leads to a larger class of tropical varieties, e.g. points with non-rational coordinates. To this extend other authors (see e.g. [13], [3, Chap. 4.2]) use the following field of a generalised Laurent series,

$$K = \left\{ \sum_{\alpha \in A} a_{\alpha} \cdot t^{\alpha} \mid A \subset \mathbb{R} \text{ well-ordered, } a_{\alpha} \in C \right\},$$

with the obvious addition and multiplication, and where the valuation of generalised Laurent series is again given by its order. This field is indeed algebraically closed and complete (see [14, Thm. 2]), and its value group is \mathbb{R} . However, it seems a rather big step from the field $C\{\{t\}\}$ to this field K by passing to exponent sets A which are arbitrary well-ordered sets. In this paper we want to introduce an alternative field \mathbb{K} which contains $C\{\{t\}\}$ and is contained in K , which has a valuation with value group \mathbb{R} and which is also algebraically closed and complete. In comparison with $C\{\{t\}\}$ it thus has the advantage of completeness and that no topological closure is necessary when tropicalising, and in comparison with K it has the advantage that the exponents of the generalised Laurent series considered are simply sequences of real numbers diverging to infinity.

In Section 2 we define \mathbb{K} and formulate the main result, namely that \mathbb{K} is algebraically closed. Section 3 is then devoted to the proof of this result. Finally in Section 4 we show that \mathbb{K} is complete.

2. \mathbb{K} is algebraically closed

We are now defining this new field \mathbb{K} and we state its main properties.

DEFINITION 1. 1. We use the symbol

$$\alpha_n \nearrow \infty$$

to denote a sequence $(\alpha_n)_{n \in \mathbb{N}}$ of real numbers which is strictly monotonously increasing and unbounded, and we call the sequence *smiub*. Note that such a sequence is determined uniquely by the set $\{\alpha_n \mid n \in \mathbb{N}\}$.

2. We define the set \mathbb{M} to be

$$\mathbb{M} = \{ \{ \alpha_n \mid n \in \mathbb{N} \} \mid \alpha_n \nearrow \infty \} \cup \{ A \mid A \subset \mathbb{R}, \#A < \infty \},$$

which is basically the union of all smiub-sequences and of all finite sequences.

3. Given a set $A \in \mathbb{M}$ and $a_\alpha \in C^*$ for $\alpha \in A$ we use the short hand notation

$$(1) \quad \sum_{\alpha \in A} a_\alpha \cdot t^\alpha$$

in order to denote the function

$$f : \mathbb{R} \longrightarrow C : \alpha \mapsto \begin{cases} a_\alpha, & \text{if } \alpha \in A, \\ 0, & \text{else,} \end{cases}$$

and we call A the support of f . The set of all function $f : \mathbb{R} \rightarrow C$ of this type is denoted by \mathbb{K} , i.e.

$$\mathbb{K} = \left\{ \sum_{\alpha \in A} a_\alpha \cdot t^\alpha \mid A \in \mathbb{M} \right\}.$$

Note that we allow the set A to be empty, so that the constant zero function is contained in \mathbb{K} . We call the elements of \mathbb{K} generalised Puiseux series.

4. If $A, B \subset \mathbb{R}$ we set $A * B = \{ \alpha + \beta \mid \alpha \in A, \beta \in B \}$.

REMARK 1. 1. Note that the representation (1) of a function $f \in \mathbb{K}$ is unique, and it is either a generalised Laurent polynomial

$$f = \sum_{n=0}^k a_{\alpha_n} \cdot t^{\alpha_n},$$

or it is a generalised Laurent series

$$f = \sum_{n=0}^{\infty} a_{\alpha_n} \cdot t^{\alpha_n},$$

where the exponents are real numbers forming a smiub-sequence $\alpha_n \nearrow \infty$. In particular with the notation from above we obviously have

$$C\{\{t\}\} \subset \mathbb{K} \subset K.$$

We will, however, not bother too much about the *uniqueness* of the representation and spoil it by allowing the coefficients to be zero in order to make the notation simpler.

2. For $A, B \in \mathbb{M}$ one easily sees that $A \cup B \in \mathbb{M}$ and $A * B \in \mathbb{M}$. Moreover, for any fixed element $\gamma \in A * B$ there is only a finite number of pairs $(\alpha, \beta) \in A \times B$ such that $\alpha + \beta = \gamma$.

3. Sometimes we will have to access the value of a function $f \in \mathbb{K}$ for $\alpha = 0$ where f is given as an algebraic expression involving several elements of \mathbb{K} . We then will use the short hand notation

$$f|_{t=0} = f(0).$$

DEFINITION 2. For $f = \sum_{\alpha \in A} a_\alpha \cdot t^\alpha$, $g = \sum_{\beta \in B} b_\beta \cdot t^\beta \in \mathbb{K}$ we define the functions

$$f + g : \mathbb{R} \longrightarrow \mathbb{C} : \alpha \mapsto f(\alpha) + g(\alpha)$$

and

$$f \cdot g = \sum_{\gamma \in A * B} \left(\sum_{\alpha \in A, \beta \in B : \alpha + \beta = \gamma} a_\alpha \cdot b_\beta \right) \cdot t^\gamma.$$

REMARK 2. With the notation of Definition 2 we obviously have that

$$f + g = \sum_{\gamma \in A \cup B} (a_\gamma + b_\gamma) \cdot t^\gamma,$$

if we use the convention that $a_\gamma = 0$ if $\gamma \notin A$ and $b_\gamma = 0$ whenever $\gamma \notin B$, and both $f + g$ and $f \cdot g$ are elements of \mathbb{K} . In particular, $(\mathbb{K}, +, \cdot)$ is a subfield of K . Moreover, the valuation on K induces the valuation

$$\text{val} : (\mathbb{K}^*, \cdot) \longrightarrow (\mathbb{R}, +) : f \mapsto \min\{\alpha \in \mathbb{R} \mid f(\alpha) \neq 0\}$$

on \mathbb{K} , i.e. val is a group homomorphism such that

$$\text{val}(f + g) \geq \min\{\text{val}(f), \text{val}(g)\}$$

for $f, g \in \mathbb{K}^*$. We call $\text{lc}(f) = f(\text{val}(f))$ the leading coefficient of f , and as usual we extend the valuation to the whole of \mathbb{K} by $\text{val}(0) = \infty$.

REMARK 3. If $f_0, \dots, f_n \in \mathbb{K}$ with $\text{val}(f_i) > 0$ for all $i = 0, \dots, n$ and $G \in \mathbb{C}[[z_0, \dots, z_n]]$ is a formal power series, then we may actually substitute z_i by f_i in order to receive an element $G(f_0, \dots, f_n)$ in \mathbb{K} .

The aim of this paper is to prove the following theorem.

THEOREM 1. *The field \mathbb{K} is algebraically closed.*

We do neither claim any originality for the definition of the field, nor for the fact that it is algebraically closed. In fact, the field can be viewed as a special case of a much wider classes of fields studied in [14] respectively [15], and they also show that the fields in question are algebraically closed. [14, Thm. 2] and [15, (5.2)] both reduce this fact to general results in the ramification theory of non-archimedean valued fields. We want to present a different proof. The basic idea is as follows: Given a non-constant polynomial over \mathbb{K} we have to find a root. Using the Weierstraß' Preparation Theorem

(see e.g. [8, Kap. I,§4]) we reduce to the situation where the t -Newton polygon (see Notation 1) has only a single lower face connecting the two coordinate axes, and to this polynomial we apply an adaptation of the classical Newton–Puiseux algorithm (see e.g. [4, Thm. 5.1.14]). The idea for the reduction step is due to Marina Viazovska.

REMARK 4. 1. If we replace the base field C in the definition of \mathbb{K} by any algebraically closed field of characteristic zero, then Theorem 1 holds with the same proof.

2. If we replace the base field C by a field of positive characteristic p Theorem 1 holds no longer. The Artin-Schreyer polynomial

$$F = y^p - y - \frac{1}{t}$$

has the roots

$$\bar{y} = -k + \sum_{i=1}^{\infty} t^{-\frac{1}{p^i}} \in K \setminus \mathbb{K},$$

for $k = 0, \dots, p - 1$ (see [1]). The algebraic closure of the quotient field of the formal power series ring, i.e. the analogue of $C\{\{t\}\}$ in this situation is studied in [11]. Since already the square of any of the roots of the above polynomial F has a support which can no longer be written as a single ascending sequence, there is no nice substitution of the analogue of K for tropical geometry in positive characteristic.

3. The valuation ring

$$R_{\text{val}} = \{f \in \mathbb{K} \mid \text{val}(f) \geq 0\}$$

of \mathbb{K} is a non-noetherian local ring of dimension one with maximal ideal $\mathfrak{m} = \langle t^\alpha \mid \alpha \in \mathbb{R}_{>0} \rangle$.

4. The field extension $C \subset \mathbb{K}$ has infinite transcendence degree, since whenever $\alpha_1, \dots, \alpha_n$ are algebraically independent over \mathbb{Q} then $t^{\alpha_1}, \dots, t^{\alpha_n}$ are algebraically independent over C .

3. The proof of the main result Theorem 1

Let us fix some notation before we start with the actual proof.

NOTATION 1. Let $F = \sum_{i=0}^n f_i \cdot y^i \in \mathbb{K}[y]$ be a polynomial of degree n . We define the t -support of F as the set

$$t\text{-supp}(F) = \{(i, \text{val}(f_i)) \mid i = 0, \dots, n, f_i \neq 0\} \subset \mathbb{R}^2,$$

and we call the convex hull, say $N(F)$, of $t\text{-supp}(F)$ the t -Newton polygon of F .

Assume now that $\text{val}(f_i) \geq 0$ for all i and fix a real number ω . We then call

$$\text{ord}_\omega(F) = \min\{\text{val}(f_i) + \omega \cdot i \mid f_i \neq 0\}$$

the ω -order of F , and we define the t -initial form of F with respect to ω as

$$\mathbf{t-in}_\omega(F) = \sum_{i: \text{val}(f_i) + \omega \cdot i = \text{ord}_\omega(F)} \text{lc}(f_i) \cdot y^i \in \mathbb{C}[y].$$

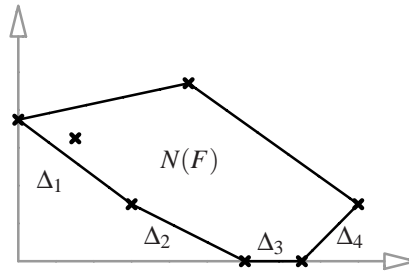
EXAMPLE 1. Consider the polynomial

$$F = (2t + t^{\frac{4}{3}}) \cdot y^6 + y^5 + \frac{1}{1 - t^{\frac{1}{2}}} \cdot y^4 - t^\pi \cdot y^3 + t \cdot y^2 + (t^{\frac{4e}{5}} - t^4) \cdot y + 3t^{\frac{5}{2}} \in \mathbb{K}[y].$$

The t -support of F is

$$\mathbf{t-supp}(F) = \{(0, 2.5), (1, 0.8 \cdot e), (2, 1), (3, \pi), (4, 0), (5, 0), (6, 1)\},$$

and thus the t -Newton polygon $N(F)$ of F looks as follows:



It has four lower faces Δ_1 , Δ_2 , Δ_3 , and Δ_4 , and the slope of Δ_1 is $-\frac{3}{4}$. If we choose $\omega = \frac{3}{4}$ then $\text{ord}_\omega(F) = \frac{5}{2}$ and $\mathbf{t-in}_\omega(F) = y^2 + 3$.

Proof of Theorem 1. Consider a non-constant polynomial

$$F_0 = \sum_{i=0}^n f_{i,0} \cdot y^i \in \mathbb{K}[y]$$

with coefficients $f_{i,0} \in \mathbb{K}$. We have to show that there is a $\bar{y} \in \mathbb{K}$ such that $F_0(\bar{y}) = 0$. For this we first want to show that we may assume that the coefficients of F_0 satisfy certain assumptions.

If $f_{0,0} = 0$ then $\bar{y} = 0$ will do, so that we may assume

$$(2) \quad f_{0,0} \neq 0.$$

Multiplying F_0 by $t^{-\min\{\text{val}(f_{i,0}) \mid i=0, \dots, n\}}$ does not change the set of roots of F_0 but it allows us to assume that

$$(3) \quad \text{val}(f_{i,0}) \geq 0 \quad \text{for all } i = 1, \dots, n$$

and that the minimum

$$r = \min\{i \mid \text{val}(f_{i,0}) = 0\}$$

exists.

We claim that we may actually assume

$$(4) \quad r \geq 1.$$

Suppose the contrary, i.e. $r = 0$. If $\text{val}(f_{n,0}) > 0$ then we can replace F_0 by $G = y^n \cdot F_0\left(\frac{1}{y}\right) = \sum_{i=0}^n f_{n-i,0} \cdot y^i \in \mathbb{K}[y]$ which is a polynomial whose constant coefficient has positive valuation, and if we find a root y' of G , necessarily non-zero, then $\bar{y} = \frac{1}{y'}$ is a root of F_0 . If instead also $\text{val}(f_{n,0}) = 0$ this replacement would not help. However, in this situation

$$h = F_0|_{t=0} = \sum_{i=0}^n f_{i,0}(0) \cdot y^i \in \mathbb{C}[y]$$

is a polynomial of degree n with non-zero constant term. Since \mathbb{C} is algebraically closed there is a $0 \neq c \in \mathbb{C}$ such that $h(c) = 0$. If we then set

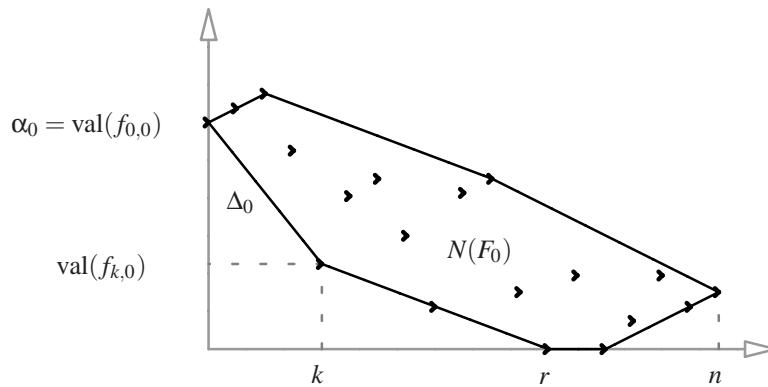
$$G = F_0(y+c) = \sum_{i=0}^n \left(\sum_{j=i}^n f_{j,0} \cdot \binom{j}{i} \cdot c^{j-i} \right) \cdot y^i,$$

the constant coefficient, say $g_0 \in \mathbb{K}$, of this polynomial satisfies $g_0(0) = h(c) = 0$ and has thus positive valuation. We may again replace F_0 by G , and if y' is a root of G then $\bar{y} = y' + c$ is a root of F_0 . This shows the claim.

We are now ready to show by induction on $n+r$ that a polynomial F_0 satisfying the conditions (2), (3) and (4) has a root $\bar{y} \in \mathbb{K}$ such that $\text{val}(\bar{y}) > 0$.

If $n = r = 1$ there is nothing to show, and we may assume that $n > 1$.

Due to the above assumptions the t -Newton polygon of F_0 looks basically as follows:



Here we simply set $\alpha_0 = \text{val}(f_{0,0})$ and choose k such that the point $(k, \text{val}(f_{k,0}))$ is the second end point of the lower face, say Δ_0 , of the t -Newton polygon emanating from the vertex $(0, \alpha_0)$. By our assumptions we have necessarily

$$(5) \quad k \leq r.$$

If we now set

$$\omega_0 = \frac{\alpha_0 - \text{val}(f_{k,0})}{k} > 0,$$

then $-\omega_0$ is the slope of the above mentioned Δ_0 . With this notation we can write the t -initial form F_0 with respect to ω_0 as follows:

$$\text{t-in}_{\omega_0}(F_0) = \sum_{i=0}^n (t^{i\omega_0 - \alpha_0} \cdot f_{i,0})|_{t=0} \cdot y^i \in C[y].$$

In particular, the degree of the t -initial form with respect to ω_0 is

$$(6) \quad \deg(\text{t-in}_{\omega_0}(F_0)) = k,$$

and the constant coefficient is $\text{lc}(f_{0,0}) \neq 0$. Since C is algebraically closed we can choose a non-zero root of $\text{t-in}_{\omega_0}(F_0)$, or more precisely

$$\exists 0 \neq c_0 \in C \quad \text{and} \quad 0 < r' \leq r \quad \text{s.t.} \quad \text{t-in}_{\omega_0}(F_0) = (y - c_0)^{r'} \cdot g, \quad g(c_0) \neq 0,$$

i.e. c_0 is a root of multiplicity r' of $\text{t-in}_{\omega_0}(F_0)$ and $r' \leq r$ follows from (5) and (6).

Having found this root c_0 we transform F_0 into a new polynomial

$$F_1 = t^{-\alpha_0} \cdot F_0(t^{\omega_0} \cdot (y + c_0)) = \sum_{i=0}^n f_{i,1} \cdot y^i \in \mathbb{K}[y].$$

The coefficients $f_{i,1}$ of F_1 are just

$$(7) \quad f_{i,1} = \sum_{j=i}^n f_{j,0} \cdot t^{j\omega_0 - \alpha_0} \cdot \binom{j}{i} \cdot c_0^{j-i}$$

for $i = 0, \dots, n$. In particular they have all non-negative valuation. But for the first r' coefficients we know more, namely

$$f_{i,1}(0) = \frac{1}{i!} \cdot \frac{\partial^i \text{t-in}_{\omega_0}(F_0)}{\partial y^i}(c_0) \begin{cases} = 0, & \text{if } 0 \leq i \leq r' - 1, \\ \neq 0, & \text{if } i = r'. \end{cases}$$

Note that here we use that the characteristic of the ground field is zero! It follows that the number r' defined above plays the same role for F_1 as r does for F_0 , i.e.

$$r' = \min\{i \mid \text{val}(f_{i,1}) = 0\},$$

and as we have seen before r' satisfies the inequalities

$$1 \leq r' \leq k \leq r.$$

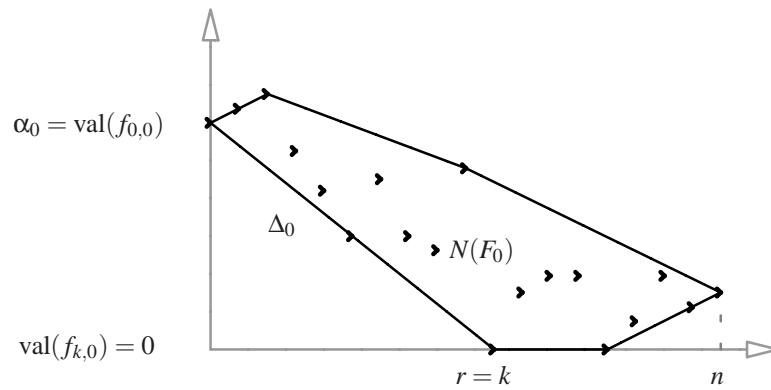
If we find a root $y' \in \mathbb{K}$ of F_1 then $\bar{y} = t^{\omega_0} \cdot (y' + c_0) \in \mathbb{K}$ will be a root of F_0 with $\text{val}(\bar{y}) = \omega_0 + \min\{0, \text{val}(y')\}$.

In particular, if $f_{0,1} = 0$ then $y' = 0$ will do and we are done since then $\text{val}(\bar{y}) \geq \omega_0 > 0$. We may therefore assume that $f_{0,1} \neq 0$, so that F_1 satisfies the assumption (2),

(3) and (4). Thus, if $r' < r$ we are done by induction since $\deg(F_1) = n$, and we may assume therefore that

$$(8) \quad r' = r.$$

Note that this forces $k = r$, i.e. the t -Newton polygon of F_0 actually looks as follows,



and the lower face Δ_0 of the t -Newton polygon of F_0 emanating from $(0, \alpha_0)$ connects the two coordinate axes.

We now claim that in this situation we may indeed assume that

$$(9) \quad r = n.$$

For this define the polynomial

$$F' = \frac{F_1}{f_{r,0}(0)} = \frac{t^{-\alpha_0}}{f_{r,0}(0)} \cdot F_0(t^{\omega_0} \cdot (y + c_0)) = y^r + \sum_{i=0}^n f'_i \cdot y^i \in \mathbb{K}[y],$$

and note that $f'_i \in \mathbb{K}$ with $\text{val}(f'_i) > 0$ for all $i = 0, \dots, n$ since c_0 is a root of $t\text{-in}_{\omega_0}(F_0)$ of order r . Moreover, we consider the polynomial

$$F'' = y^r + \sum_{i=0}^n z_i \cdot y^i \in \mathcal{C}[[z_0, \dots, z_n, y]]$$

as a formal power series over \mathcal{C} , which then is *regular of order r in y* in the sense of the Weierstraß' Preparation Theorem (see e.g. [8, Kap. I, §4] or [4, Thm. 3.2.4]). The latter theorem thus implies that there exists a unit $U \in \mathcal{C}[[z_0, \dots, z_n, y]]^*$ and a Weierstraß polynomial $P = y^r + \sum_{i=0}^{r-1} p_i \cdot y^i \in \mathcal{C}[[z_0, \dots, z_n]][y]$ with $p_i(0) = 0$ for all $i = 0, \dots, r-1$ such that

$$F'' = U \cdot P.$$

Since the f'_i have strictly positive valuation we can substitute the z_i by f'_i in U and P to get an invertible power series

$$U' = U(f'_0, \dots, f'_n, y) \in \mathbb{K}[[y]]^*$$

and a polynomial

$$P' = P(f'_0, \dots, f'_n, y) = y^r + \sum_{i=0}^{r-1} p_i(f'_0, \dots, f'_n) \cdot y^i \in \mathbb{K}[y]$$

with

$$(10) \quad \text{val}(p_i(f'_0, \dots, f'_n)) > 0 \quad \text{for all } i = 0, \dots, r-1.$$

If $p_0(f'_0, \dots, f'_n) = 0$ then $F'(0) = U'(0) \cdot P'(0) = 0$ and thus $\bar{y} = c_0 \cdot t^{\omega_0}$ is a root of F_0 with $\text{val}(\bar{y}) = \omega_0 > 0$, so that we are done. Otherwise P' satisfies the conditions (2), (3) and (4). Thus, if $r < n$ then $r+r < r+n$ and by induction there exists a $y' \in \mathbb{K}$ such that $P'(y') = 0$ with $\text{val}(y') > 0$. Since its valuation is positive we can substitute y' into U' and get an element $U'(y') \in \mathbb{K}$. But then $F'(y') = U'(y') \cdot P'(y') = 0$, and hence

$$\bar{y} = t^{\omega_0} \cdot (y' + c_0) \in \mathbb{K}$$

is a root of F_0 with $\text{val}(\bar{y}) = \omega_0 > 0$. This proves the claim.

We finally claim that under the assumption (9) we can also assume

$$(11) \quad f_{n,0} = 1.$$

For this note that $r = n$ implies that $\text{val}(f_{n,0}) = 0$, i.e. $f_{n,0}$ is a unit in the valuation ring of \mathbb{K} and $\frac{1}{f_{n,0}}$ has valuation zero as well. Thus, replacing F_0 by $\frac{F_0}{f_{n,0}}$ does not effect the conditions (2), (3), (4), or (9). This shows the claim.

Note that if F_0 satisfies (9) and (11) then by (7) and (8) F_1 satisfies the corresponding conditions as well. Thus, applying the same procedure to F_1 and going on by recursion we may assume that we produce for each $v \in \mathbb{N}$ a polynomial

$$F_v = \sum_{i=0}^n f_{i,v} \cdot y^i \in \mathbb{K}[y]$$

satisfying the corresponding versions of (2), (3), (4), (9), and (11), and we produce a root $0 \neq c_v \in C$ of $t\text{-in}_{\omega_v}(F_v)$ of order

$$n = \min\{i \mid \text{val}(f_{i,v}) = 0\}$$

such that for $i = 0, \dots, n$

$$(12) \quad f_{i,v} = \sum_{j=i}^n f_{j,v-1} \cdot t^{j \cdot \omega_{n-1} - \alpha_{v-1}} \cdot \binom{j}{i} \cdot c_{v-1}^{j-i},$$

where for each $v \in \mathbb{N}$

$$\alpha_v = \text{val}(f_{0,v}) > 0$$

and

$$(13) \quad \omega_v = \frac{\alpha_v}{n} > 0$$

is the negative of the slope of the lower face, say Δ_v , of the t -Newton polygon of F_v connecting the two coordinate axes by joining the points $(0, \alpha_v)$ and $(n, 0)$. Note that for this we use the fact that if at some point $F_v(0) = 0$ then

$$\bar{y} = \sum_{i=0}^{v-1} c_i \cdot t^{\omega_0 + \dots + \omega_i} = t^{\omega_0} \cdot \left(c_0 + t^{\omega_1} \cdot (c_1 + \dots + t^{\omega_{v-2}} \cdot (c_{v-2} + t^{\omega_{v-1}} \cdot c_{v-1}) \dots) \right)$$

is a root of F_0 of valuation $\omega_0 > 0$.

That way we obviously construct a generalised Laurent series

$$\bar{y} = \sum_{i=0}^{\infty} c_i \cdot t^{\omega_0 + \dots + \omega_i}$$

in the field K , and it remains to show that indeed $\bar{y} \in \mathbb{K}$ and $F_0(\bar{y}) = 0$.

Let us first address the issue that $\bar{y} \in \mathbb{K}$. By (13) we know that $n \cdot \omega_v - \alpha_v = 0$ and $(n-1) \cdot \omega_v - \alpha_v = -\omega_v$, so that (12) and the fact that $f_{n,v} = 1$ imply that

$$f_{n-1,v+1} = f_{n-1,v} \cdot t^{-\omega_v} + n \cdot c_v,$$

or equivalently

$$f_{n-1,v} = -n \cdot c_v \cdot t^{\omega_v} + t^{\omega_v} \cdot f_{n-1,v+1}.$$

Doing a descending induction on v we deduce

$$f_{n-1,0} = -n \cdot \sum_{i=0}^v c_i \cdot t^{\omega_0 + \dots + \omega_i} + t^{\omega_0 + \dots + \omega_v} \cdot f_{n-1,v+1}.$$

Since the valuation of $f_{n-1,v+1}$ is strictly positive it follows that the first $v+1$ summands of $f_{n-1,0}$ coincide with $-n \cdot \sum_{i=0}^v c_i \cdot t^{\omega_0 + \dots + \omega_i}$, and since this holds for each $v \in \mathbb{N}$ we necessarily have

$$\bar{y} = -\frac{f_{n-1,0}}{n} \in \mathbb{K}.$$

Note that we here again use that the characteristic of C is zero.

In order to show that $F_0(\bar{y}) = 0$ we set

$$\bar{y}_v = \sum_{i=v}^{\infty} c_i \cdot t^{\omega_v + \dots + \omega_i},$$

so that

$$F_v(\bar{y}_v) = t^{\alpha_v} \cdot F_{v+1}(\bar{y}_{v+1}).$$

But this equation together with a simple induction shows that

$$\text{val}(F_0(\bar{y})) = \sum_{i=0}^v \alpha_i + \text{val}(F_{v+1}(\bar{y}_{v+1}))$$

for each $v \in \mathbb{N}$. Since the coefficients of F_{v+1} all have non-negative valuation and since $\text{val}(\bar{y}_{v+1}) > 0$ it follows that the last summand is non-negative, and therefore

$$\text{val}(F_0(\bar{y})) \geq n \cdot \sum_{i=0}^v \omega_i \xrightarrow{v \rightarrow \infty} \infty.$$

This implies that $F_0(\bar{y}) = 0$ and finishes the proof. \square

4. The field \mathbb{K} is complete

DEFINITION 3. *The valuation on \mathbb{K} induces via the exponential map the norm*

$$|\cdot| : \mathbb{K} \longrightarrow \mathbb{R} : f \mapsto \exp(-\text{val}(f))$$

on \mathbb{K} , where we use the convention that $\exp(-\infty) = 0$. It satisfies the strong triangle inequality

$$|f + g| \leq \max\{|f|, |g|\}.$$

As usual we call a sequence $(f_n)_{n \in \mathbb{N}}$ in \mathbb{K} a Cauchy sequence with respect to $|\cdot|$ if for all $\varepsilon > 0$ there exists an $N(\varepsilon) \in \mathbb{N}$ such that $|f_n - f_m| < \varepsilon$ for all $n, m \geq N(\varepsilon)$. And we call a sequence $(f_n)_{n \in \mathbb{N}}$ in \mathbb{K} convergent with respect to $|\cdot|$ if there is an $f \in \mathbb{K}$ such that for all $\varepsilon > 0$ there exists an $N(\varepsilon) \in \mathbb{N}$ such that $|f_n - f| < \varepsilon$ for all $n \geq N(\varepsilon)$.

Compared with the field $C\{\{t\}\}$ of Puiseux series the field \mathbb{K} has the advantage that it is complete with respect to the norm induced by the valuation.

PROPOSITION 1. *$(\mathbb{K}, |\cdot|)$ is complete, i.e. every Cauchy sequence is convergent.*

Proof. Let $(f_n)_{n \in \mathbb{N}}$ be a Cauchy sequence. Given any positive integer M we set $\varepsilon_M = \exp(-M) > 0$. Thus there is an $N(\varepsilon_M) \in \mathbb{N}$ such that

$$\exp(-M) = \varepsilon_M > |f_n - f_m| = \exp(-\text{val}(f_n - f_m))$$

for all $n, m \geq N(\varepsilon_M)$, or equivalently

$$\text{val}(f_n - f_m) > M.$$

This implies that $f_n(\alpha) = f_m(\alpha)$ for all $\alpha \leq M$ and for all $n, m \geq N(\varepsilon_M)$. Without loss of generality we may assume that $N(\varepsilon_M) \geq N(\varepsilon_{M'})$ whenever $M \geq M'$. We may therefore define a function $f : \mathbb{R} \longrightarrow C$ by

$$f(\alpha) = f_{N(\varepsilon_M)}(\alpha)$$

if $\alpha \geq M$, and obviously $(f_n)_{n \in \mathbb{N}}$ converges to this function f . □

REMARK 5. 1. The statement of Proposition 1 remains true if we replace in the definition of \mathbb{K} the field C by any other field. It is independent of the characteristic.

2. If we replace in the definition of \mathbb{K} the domain \mathbb{R} of the elements in \mathbb{K} by \mathbb{Q} we get the completion of $C\{\{t\}\}$ with respect to the norm induced by the valuation. With the same proof as in Theorem 1 this field is algebraically closed. But note that the value group is still only \mathbb{Q} .

References

- [1] ABHYANKAR S. Two notes on formal power series. *Proc. Amer. Math. Soc.* 7 (1956), 903–905.
- [2] BIERI R. AND GROVES J. R. J. The geometry of the set of characters induced by valuations. *J. Reine Angew. Math.* 347 (1984), 168–195.
- [3] BÖHM J. *Mirror symmetry and tropical geometry*. PhD thesis, Universität des Saarlandes, 2007.
- [4] DE JONG T. AND PFISTER G. *Local analytic geometry*. Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Braunschweig, 2000.
- [5] DRAISMA J. A tropical approach to secant dimensions. *J. Pure Appl. Algebra* 212, 2 (2008), 349–363.
- [6] EINSIEDLER M., KAPRANOV M. AND LIND D. Non-Archimedean amoebas and tropical varieties. *J. Reine Angew. Math.* 601 (2006), 139–157.
- [7] GATHMANN A. Tropical algebraic geometry. *Jahresber. Deutsch. Math.-Verein.* 108, 1 (2006), 3–32.
- [8] GRAUERT H. AND REMMERT R. *Analytische Stellenalgebren, Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, vol. 176*. Springer-Verlag, Berlin, 1971.
- [9] JENSEN A. N., MARKWIG H. AND MARKWIG T. An algorithm for lifting points in a tropical variety. *Collect. Math.* 59, 2 (2008), 129–165.
- [10] KATZ E. M. H. AND T. M. jinvariant.lib, a SINGULAR 3.0 library for computations with hbinvariants in tropical geometry, 2007. <http://www.singular.uni-kl.de/~keilen/de/jinvariant.html>.
- [11] KEDLAYA K. S. The algebraic closure of the power series field in positive characteristic. *Proc. Amer. Math. Soc.* 129, 12 (2001), 3461–3470 (electronic).
- [12] MIKHALKIN G. Tropical geometry and its applications. In *International Congress of Mathematicians. Vol. II*. Eur. Math. Soc., Zürich, 2006, pp. 827–852.
- [13] PAYNE S. Fibers of tropicalization. *Math. Z.* 262, 2 (2009), 301–311.
- [14] RAYNER F. J. Algebraically closed fields analogous to fields of Puiseux series. *J. London Math. Soc. (2)* 8, 2 (1974), 504–506.
- [15] RIBENBOIM P. Fields: algebraically closed and others. *Manuscripta Math.* 75, 2 (1992), 115–150.
- [16] SHUSTIN E. A tropical calculation of the Welschinger invariants of real toric del Pezzo surfaces. *J. Algebraic Geom.* 15, 2 (2006), 285–322.
- [17] SPEYER D. *Tropical geometry*. PhD thesis, University of California, Berkeley, 2005.
- [18] SPEYER D. AND STURMFELS B. The tropical Grassmannian. *Adv. Geom.* 4, 3 (2004), 389–411.
- [19] TABERA L. F. Tropical constructive Pappus’ theorem. *Int. Math. Res. Not.*, 39 (2005), 2373–2389.

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