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REPRESENTATION OF BILINEAR FORMS BY LINEAR OPERATORS IN NON-ARCHIMEDEAN HILBERT SPACE EQUIPPED WITH A KRULL VALUATION

Abstract. The paper considers representing bilinear forms by linear operators in the case of a Krull valuation. More precisely, after making some suitable assumptions, we prove that if φ is a non-degenerate bilinear form, then φ is representable by a unique linear operator A whose adjoint operator A^* exists.

1. Preliminaries

Let $\mathbb{N}_0 = \{1,2,3,....\}$ and let $\mathbb{N} = \{0,1,2,...\}$. Let $\mathbb{K}$ be a field and let Γ be a totally ordered Abelian additive group. The total ordering of Γ is denoted $\leq$ with the property that for $\gamma_1,\gamma_1,\gamma_3 \in \Gamma$, $\gamma_1 \leq \gamma_2$ if and only if $\gamma_1+\gamma_3 \leq \gamma_2+\gamma_3$. Additionally, we shall write $\gamma_1 < \gamma_2$ to mean that $\gamma_1 \leq \gamma_2$ but $\gamma_1 \neq \gamma_2$, and we write $\gamma_1 \geq \gamma_2$ and $\gamma_1 > \gamma_2$ to mean $\gamma_2 \leq \gamma_1$ and $\gamma_2 < \gamma_1$, respectively. Moreover, we write $\Gamma_\infty := \Gamma \cup \{\infty\}$ in which $\gamma + \{\infty\} = \{\infty\}$ for each $\gamma \in \Gamma_\infty$. Next we extend the total ordering to Γ_∞ by declaring that for all $\gamma \in \Gamma$, $\gamma < \{\infty\}$.

Typical examples of ordered Abelian groups include the additive group of integers $(\mathbb{Z},+,\leq)$, or the direct sum

$$\Gamma = \bigoplus_{j=1}^{\infty} \Gamma_j = \Gamma_1 \oplus \Gamma_2 \oplus ... \oplus \Gamma_n \oplus \Gamma_{n+1} \oplus ...,$$

where Γ_j is an isomorphic copy of the additive group $\mathbb{Z}$ of integers, for each $j \in \mathbb{N}_0$. Namely, Γ consists of all sequences $\gamma = (\gamma_j)_{j\in\mathbb{N}_0}$ with $\gamma_j \in \Gamma_j$ for which $\{j \in \mathbb{N}_0 : \gamma_j \neq 0\}$ is finite. The group Γ will be equipped with antilexicographic order, that is, if $0 \neq \gamma = (\gamma_j)_{j\in\mathbb{N}_0} \in \Gamma$ and $j_0 = \max\{j \in \mathbb{N}_0 : \gamma_j \neq 0\}$, then $\gamma > 0$ in Γ if and only if $\gamma_{j_0} > 0$ in Γ_{j_0}.

Let us mention that there exist totally ordered Abelian additive groups Γ, which are not subgroups of $\mathbb{R}$. In what follows, we give an example of such groups discussed in the remarkable book by Ribenboim [14]. Indeed, let G and H be nonzero subgroups of $\mathbb{R}$ and let $\Gamma = G \times H$ equipped with addition defined componentwise. Now equip Γ with the lexicographic order "$\leq$" as follows:

$$(f,g) < (h,l) \text{ if } (f < h \text{ or } f = h) \text{ and } g < l.$$

Clearly, it is not hard to see that the order "$\leq$" defined above is compatible with the operation of addition. Moreover, $\Gamma = G \times H$ defined above is not order-isomorphic to a subgroup of $\mathbb{R}$.

A map $v : \mathbb{K} \to \Gamma_\infty = \Gamma \cup \{\infty\}$ is said to be a Krull valuation provided that for all $\lambda, \mu \in \mathbb{K}$, the following conditions hold:

(P_1) $v(\lambda) = \{\infty\}$ if and only if $\lambda = 0$;

(P_2) $v(\lambda\mu) = v(\lambda) + v(\mu)$;

(P_3) $v(\lambda+\mu) \geq \min\Big\{v(\lambda), v(\mu)\Big\}$.

The group Γ is then called a value group for $\mathbb{K}$. The valuation v induces a topology on the field $\mathbb{K}$ by considering $\{O_\varepsilon : \varepsilon \in \Gamma\}$ as a neighborhood base of $0 \in \mathbb{K}$, where $O_\varepsilon = \{\lambda \in \mathbb{K} : v(\lambda) > \varepsilon\}$. Consequently, a sequence $(\lambda_j)_{j\in\mathbb{N}} \subset \mathbb{K}$ converges to 0 for this valuation topology if only if $v(\lambda_j) \to \infty$ as $j \to \infty$.

For the above-mentioned valuation, be have the following properties:

(P_4) $v(-\xi) = v(\xi)$ for each ξ;

(P_5) $v(\xi^{-1}) = -v(\xi)$ for each $\xi \in \mathbb{K} - \{0\}$;

(P_6) $v(\mu - \lambda) \geq \min\Big\{v(\mu), v(\lambda)\Big\}$;

(P_7) $v(\mu+\lambda) = \min\Big\{v(\mu), v(\lambda)\Big\}$ whenever $v(\mu) \neq v(\lambda)$.

Suppose that a valuation v and a value group Γ associated with $\mathbb{K}$ are given, as above. Fix once and for all a sequence $\omega = (\omega_j)_{j\in\mathbb{N}} \subset \mathbb{K}$ of nonzero terms. The space $c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma)$ is defined as the set of all $x = (x_j)_{j\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ such that $\omega_j x_j^2 \to 0$ as $j \to \infty$, that is,

$$\lim_{j\to\infty} \Big(v(\omega_j) + 2v(x_j)\Big) = \infty.$$

One defines an associated (non-Archimedean) norm $N : c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma) \to \Gamma_\infty$ as follows: for each $x = (x_j)_{j\in\mathbb{N}} \in c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma)$,

$$N(x) = \min_{j\in\mathbb{N}} \Big(2v(x_j) + v(\omega_j)\Big).$$

It is not hard to check that N satisfies the following properties:

(Q_1) $N(x) = \infty$ if and only if $x = 0$,

(Q_2) $N(\xi x) = 2v(\xi) + N(x)$;

(Q_3) $N(-x) = N(x)$;

(Q_4) $N(x+y) \geq \min\Big(N(x), N(y)\Big)$;

(Q_5) $N(x-y) \geq \min\Big(N(x), N(y)\Big)$;

valid for all $\xi \in \mathbb{K}$ and $x, y \in c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma)$. As an immediate consequence of (Q_4), we have the following: if $x, y \in c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma)$, then

(Q_6) $N(x+y) = \min\Big(N(x), N(y)\Big)$ whether $N(x) \neq N(y)$.

Indeed, suppose that $N(x) < N(y)$. (This by the way includes the case when $N(x) < \infty$ and $N(y) = \infty$.) It follows that

$$N(x+y) \geq \min\Big(N(x), N(y)\Big) = N(x).$$

Suppose $N(x+y) > N(x)$. Consequently,

$$N(x) = N(x+y-y) \geq \min\Big(N(x+y), N(y)\Big) > N(x),$$

which is impossible, and hence

$$N(x+y) = N(x) = \min\Big(N(x), N(y)\Big).$$

Another important consequence of (Q_3) is the following: if $x_j \in c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma)$ for all $j \in \mathbb{N}$, then

(Q_7) $N\Big(\sum_{j\in\mathbb{N}} x_j\Big) \geq \inf_{j\in\mathbb{N}} N(x_j)$ whenever the sum exists.

Similarly, from (P_3) we have the following: if $h_j \in \mathbb{K}$ for all $j \in \mathbb{N}$, then

(Q_8) $v\Big(\sum_{j\in\mathbb{N}} h_j\Big) \geq \inf_{j\in\mathbb{N}} v(x_j)$ whenever the sum exists.

For more on the Krull valuation and related issues, we refer the reader to the remarkable work of Keller and Ochsenius [10], Ochsenius [11] and Ochsenius and Schikhof [13].

One can check that the "normed" space $\Big(c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma), N\Big)$ is complete. Moreover, each $x = (x_j)_{j\in\mathbb{N}} \in c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma)$ can be written as

$$x = \sum_{i=0}^{\infty} x_i e_i \quad \text{with} \quad \lim_{i\to\infty} N(x_i e_i) = \infty,$$

where e_i is the sequence whose j-th term is 0 if $i \neq j$, and the i-th term is 1.

In particular, $N(e_j) = v(\omega_j)$ for each $j \in \mathbb{N}$. The system $(e_j)_{j\in\mathbb{N}}$ will be called an orthogonal basis for $c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma)$.

Similarly, an inner product (symmetric, non-degenerate, bilinear form) is also defined on $c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma)$ for all $x = (x_j)_{j\in\mathbb{N}}$, $y = (y_j)_{j\in\mathbb{N}} \in c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma)$ by

$$\langle x, y\rangle := \sum_{j=0}^{\infty} \omega_j x_j y_j,$$

with corresponding "Cauchy-Schwarz inequality" given by

$$2v(\langle x,y\rangle) \geq N(x)+N(y) \text{ for all } x,y \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma). \tag{1}$$

The space $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ equipped with the above-mentioned valuation N and inner product $\langle\cdot,\cdot\rangle$ is called a non-Archimedean Hilbert space.

2. Introduction

A bilinear form $\varphi : D(\varphi)\times D(\varphi) \to \mathbb{K}$ with domain $D(\varphi)$ is said to be *representable* (Definition 8) whether there exists a linear operator $A : D(A) \to c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ ($D(A)$ being the domain of A) such that

$$\varphi(x,y) = \langle Ax, y\rangle, \quad \forall x \in D(A),\ y \in D(\varphi).$$

An unbounded bilinear form $\varphi : D(\varphi)\times D(\varphi) \to \mathbb{K}$ whose domain $D(\varphi)$ contains all elements of the canonical basis $(e_i)_{i\in\mathbb{N}}$ will be called *stable*. The subclass of all these stable unbounded bilinear forms is denoted $\Sigma_S(c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)\times c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma))$. Similarly, the subclass of all bilinear forms whose domains do not contain the above-mentioned canonical basis will be called *unstable* and denoted $\Sigma_U(c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)\times c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma))$.

In a recent paper by Attimu and Diagana, that is, in [1], it was shown that if φ is a non-degenerate, symmetric bilinear form satisfying

$$\lim_{i\to\infty}\left(\frac{|\varphi(e_i,e_j)|}{\|e_i\|}\right) = \lim_{i\to\infty}\left(\frac{|\varphi(e_j,e_i)|}{\|e_i\|}\right) = 0, \quad \forall j \in \mathbb{N}, \tag{2}$$

then φ is uniquely representable. Moreover, if A denotes the (possibly unbounded) linear operator associated with φ, then its adjoint A^* does exist.

The main concern in this paper consists of studying representation theorems for (stable) bilinear forms in the case of a Krull valuation.

More precisely, it will be shown that a non-degenerate bounded bilinear form φ on $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)\times c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ is representable whenever

$$\lim_{i\to\infty}\Big[2v(\varphi(e_i,e_j)) - N(e_i)\Big] = \lim_{i\to\infty}\Big[2v(\varphi(e_j,e_i)) - N(e_i)\Big] = \infty \tag{3}$$

for all $j \in \mathbb{N}$.

Similarly, it will be shown that if in addition to (3),

$$\lim_{i\to\infty} N(e_i) = \gamma \in \Gamma, \tag{4}$$

then a possibly non-degenerate bilinear form φ on $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)\times c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$, not necessarily bounded, is representable. Moreover, if A denotes the linear operator on $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ associated with the form φ, then the adjoint A^* of A does exist.

In addition to the above-mentioned representation results for bilinear forms, we also establish a non-Archimedean version of the Riesz's representation theorem for a subclass of linear functionals on $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ in the case of a Krull valuation. Namely, it is shown if $F: c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma) \to \mathbb{K}$ is a linear functional such that

$$\lim_{i\to\infty}\Big(2v(F(e_i)) - N(e_i)\Big) = \infty,$$

then there exists a unique vector $x_0 \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ such that $F(x) = \langle x, x_0\rangle$ for each $x \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$.

To deal with the above-mentioned issues we introduce a new formalism of unbounded linear operators on the non-Archimedean Hilbert space $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ and that of (un)bounded bilinear forms on $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma) \times c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ in the case of a Krull valuation.

Representing (un)bounded bilinear forms by linear operators in the classical setting is a topic that arises in several fields such as quantum mechanics (through the study of form sums associated with the Hamiltonian), mathematical physics, symplectic geometry, and the study of weak solutions to some linear partial differential equations, see, e.g., [4, 8, 9]. In the non-Archimedean realm, one may expect some related applications in: (i) the study of weak solutions to some p-adic partial differential equations; and (ii) the study of a non-Archimedean version of the square root problem of Kato, which is of a great interest to the second author.

3. Linear operators on $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$

3.1. Bounded linear operators on $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$

In contrast with the classical definition of the boundedness of linear operators, we have:

DEFINITION 1. *One says that a linear operator $A: c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma) \to c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ is bounded if there exists $\gamma \in \Gamma$ such that*

$$N(Ax) \geq \gamma + N(x)$$

for each $0 \neq x \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$.

Equivalently, a linear operator A is bounded if and only if there exists $\gamma \in \Gamma$ such that $N(Ae_j) \geq \gamma + N(e_j)$ for each $j \in \mathbb{N}$.

The collection of bounded linear operators from $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ into $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ will be denoted $B(c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma))$. It can be easily checked that $B(c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma))$ is an algebra.

Note that a bounded linear operator A on $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ is continuous in the norm topology. However, continuous operators need not to be bounded. In contrast with the classical operator theory and except in some special cases, one cannot always assign a norm to bounded linear operators on $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$, as the norm takes its values in Γ, in which bounded sets may fail to have an infimum.

Let $A: c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma) \to c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ be a bounded linear operator. If

$$\sup\Big\{\gamma \in \Gamma : \;\; N(Ax) \geq \gamma + N(x) \;\text{ for each }\; 0 \neq x \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)\Big\}$$

exists, it is then called the norm of A and denoted $\|A\|$. Clearly, if $\|A\|$ exists, then

$$\|A\| = \inf_{x\neq 0}\Big(N(Ax) - N(x)\Big).$$

Given the orthogonal basis $(e_i)_{i\in\mathbb{N}}$ for $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$, define $e_i' \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)^*$ by

$$x = \sum_{i\in\mathbb{N}} x_i e_i, \quad e_i'(x) = x_i.$$

It turns out that $\|e_i'\| = -N(e_i) = -v(\omega_i)$. Moreover, every $x^* \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)^*$ can be expressed as $x^* = \sum_{i\in\mathbb{N}} \langle x^*, e_i\rangle e_i'$, and

$$\|x^*\| = \inf_{i\in\mathbb{N}}\Big[v(\langle x^*, e_i\rangle) - N(e_i)\Big] = \inf_{i\in\mathbb{N}}\Big[v(\langle x^*, e_i\rangle) - v(\omega_i)\Big].$$

Further, for $(u,v) \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma) \times c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)^*$, define the operator $\otimes$ as follows:

$$(v\otimes u)(x) = v(x)u, \text{ for all } x \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma).$$

PROPOSITION 1. *Let A be a bounded linear operator on $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$. There exists an infinite matrix $(a_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ with coefficients in $\mathbb{K}$, such that A can be written as a pointwise convergent sum, namely, $A = \sum_{i,j\in\mathbb{N}} a_{ij}\left(e_j'\otimes e_i\right)$ and for all $j\in\mathbb{N}$,*

$$\lim_{i\to\infty} N(a_{ij}e_i) = \infty.$$

Proof. Clearly for all j, $Ae_j = \sum_{i\in\mathbb{N}} a_{ij}e_i$ where $a_{ij}\in\mathbb{K}$, $\lim_{i\to\infty} N(a_{ij}e_i) = \infty$. Now for any $x = \sum_{j\in\mathbb{N}} x_j e_j \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$, we have

$$Ax = \sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}} a_{ij}x_j e_i = \sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}} a_{ij}\left(e_j'\otimes e_i\right)x.$$

□

PROPOSITION 2. *Let $A = \sum_{i,j\in\mathbb{N}} a_{ij}\left(e_j'\otimes e_i\right)$ be a bounded operator. If the norm $\|A\|$ exists, then*

$$\|A\| = \inf_{j\in\mathbb{N}}\Big(N(Ae_j) - N(e_j)\Big) = \inf_{i,j\in N}\Big[2v(a_{ij}) + N(e_i) - N(e_j)\Big].$$

Proof. Suppose $\|A\|$ exists. We first establish the first equality. Indeed, by definition,

$$\|A\| := \inf_{x\neq 0}\Big(N(Ax) - N(x)\Big) \le \inf_{j\in\mathbb{N}}\Big(N(Ae_j) - N(e_j)\Big).$$

Now

$$\begin{aligned}
N(Ax) &= N\left(\sum_{i\in\mathbb{N}} x_i A e_i\right) \\
&\geq \inf_{i\in\mathbb{N}} \Big(2v(x_i) + N(Ae_i)\Big) \\
&= \inf_{i\in\mathbb{N}} \Big[N(Ae_i) + 2v(x_i) - N(e_i) + N(e_i)\Big] \\
&\geq \inf_{i\in\mathbb{N}} \Big(N(Ae_i) - N(e_i)\Big) + \inf_{i\in\mathbb{N}} \Big(2v(x_i) + N(e_i)\Big) \\
&= \inf_{i\in\mathbb{N}} \Big(N(Ae_i) - N(e_i)\Big) + N(x),
\end{aligned}$$

from which we conclude that

$$\|A\| = \inf_{x\neq 0} \Big(N(Ax) - N(x)\Big) \geq \inf_{j\in\mathbb{N}} \Big(N(Ae_j) - N(e_j)\Big).$$

For the second equality, we have

$$\begin{aligned}
\inf_{j\in\mathbb{N}} \Big(N(Ae_j) - N(e_j)\Big) &= \inf_{j\in\mathbb{N}} \left(N\left(\sum_{i\in\mathbb{N}} a_{ij} e_i\right) - N(e_j)\right) \\
&= \inf_{i,j\in\mathbb{N}} \Big(2v(a_{ij}) + N(e_i) - N(e_j)\Big),
\end{aligned}$$

which completes the proof. □

As in the classical case, if $A \in B(c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma))$, an adjoint of A is an operator A^* satisfying $\langle Au, v\rangle = \langle u, A^* v\rangle$ for any u, v in $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$. If it exists, the adjoint A^* is also a bounded linear operator on $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$.

PROPOSITION 3. *Let* $A = \sum_{i,j\in\mathbb{N}} a_{ij}\,(e'_j \otimes e_i) \in B(c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma))$, *then the adjoint* A^* *exists if and only if for all* j, $\lim_{i\to\infty}\Big[2v(a_{ji}) - N(e_i)\Big] = \infty$. *In this situation,*

$$A^* = \sum_{i,j\in\mathbb{N}} \omega_i^{-1}\omega_j a_{ji}\,(e'_j \otimes e_i)\,.$$

Proof. Write $A^* = \sum_{i,j\in\mathbb{N}} b_{ij}\,(e'_j\otimes e_i)$, then A^* is the adjoint of A if and only if $\langle Ae_i, e_j\rangle = \langle e_i, A^* e_j\rangle$ for all $i, j \in \mathbb{N}$, that is,

$$\left\langle \sum_{l\in\mathbb{N}} a_{li} e_l, e_j \right\rangle = a_{ji}\omega_j = \left\langle e_i, \sum_{l\in\mathbb{N}} b_{lj} e_l \right\rangle = b_{ij}\omega_i, \qquad \forall i, j \in \mathbb{N}.$$

This is equivalent to $b_{ij} = \omega_i^{-1}\omega_j a_{ji}$ for all $i, j \in \mathbb{N}$. Moreover, for all j,

$$\lim_{i\to\infty} N(b_{ij} e_i) = \infty.$$

Now

$$\begin{aligned} N(b_{ij}e_i) &= 2v(b_{ij}) + N(e_i) \\ &= 2v(\omega_i^{-1}\omega_j a_{ji}) + N(e_i) \\ &= -2v(\omega_i) + 2v(\omega_j) + 2v(a_{ji}) + N(e_i) \\ &= 2v(\omega_j) + 2v(a_{ji}) - N(e_i). \end{aligned}$$

Therefore, from $v(\omega_j) = N(e_j) \neq \infty$, $\lim_{i\to\infty} N(b_{ij}e_i) = \infty$ if and only if, for all $j \in \mathbb{N}$,

$$\lim_{i\to\infty} \Big[2v(a_{ji}) - N(e_i)\Big] = \infty.$$

□

3.2. Unbounded linear operators on $c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma)$

DEFINITION 2. *A stable unbounded linear operator A from $c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma)$ into itself is a pair $(D(A), A)$ consisting of a subspace $D(A) \subset c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma)$ (called the domain of A) and a (possibly non continuous) linear transformation $A : c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma) \supset D(A) \to c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma)$. Namely, the domain $D(A)$ contains the basis $(e_i)_{i\in\mathbb{N}}$ and consists of all $x = (x_i)_{i\in\mathbb{N}} \in c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma)$ such $Ax = \sum_{i\in\mathbb{N}} x_i Ae_i$ converges in $c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma)$, that is,*

$$\begin{cases} D(A) := \left\{ x = (x_i)_{i\in\mathbb{N}} \in c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma) : \ \lim_{i\to\infty} N(u_i Ae_i) = \infty \right\}, \\ \qquad Ax = \sum_{i,j\in\mathbb{N}} a_{ij}\, e'_j(x)\, e_i \ \text{ for each } \ x \in D(A). \end{cases}$$

Using the proof of Proposition 3 one can easily see that the following holds.

PROPOSITION 4. *A stable unbounded linear operator*

$$\begin{cases} D(A) := \left\{ x = (x_i)_{i\in\mathbb{N}} \in c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma) : \ \lim_{i\to\infty} N(x_i Ae_i) = \infty \right\}, \\ \qquad Ax = \sum_{i,j\in\mathbb{N}} a_{ij}\, e'_j(x)\, e_i \ \text{ for each } \ x \in D(A), \end{cases}$$

has an adjoint A^* if and only if for all $j \in \mathbb{N}$,

$$\lim_{i\to\infty} \Big[2v(a_{ji}) - N(e_i)\Big] = \infty.$$

In this event the adjoint A^* of A is uniquely expressed by

$$\begin{cases} D(A^*) := \left\{ y = (y_i)_{i\in\mathbb{N}} \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma) : \ \lim_{i\to\infty} N(y_i A^* e_i) = \infty \right\}, \\ \\ A^* y = \sum_{i,j\in\mathbb{N}} a^*_{ij}\, e'_j(y) e_i \ \textit{ for each } \ y \in D(A^*), \end{cases}$$

where $a^*_{ij} = \dfrac{\omega_j a_{ji}}{\omega_i}$.

3.3. Bounded linear functionals on $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$

DEFINITION 3. *A linear functional $F : c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma) \to \mathbb{K}$ is said to be bounded if there exists $\gamma \in \Gamma$ such that*

$$v(F(x)) - N(x) \geq \gamma \ \textit{ for each } \ 0 \neq x \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma).$$

If $\sup \left\{ \gamma \in \Gamma : v\big(F(x)\big) - N(x) \geq \gamma \ \textit{ for each } \ 0 \neq x \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma) \right\}$ *exists, it is then called the norm of the continuous linear functional F and is defined by*

$$|||F||| = \inf_{x\in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma),\ x\neq 0} \Big[v(F(x)) - N(x) \Big].$$

Let us recall that the space of all continuous linear functionals on $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ is denoted $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)^*$ and called the (topological) dual of $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$. The space $(c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)^*, |||\cdot|||)$ is a Banach space over $\mathbb{K}$.

PROPOSITION 5. *Let $F \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)^*$. Then its norm $|||F|||$, if it exists, can be explicitly expressed as*

$$|||F||| = \inf_{i\in\mathbb{N}} \Big(v(F(e_i)) - N(e_i) \Big).$$

The next theorem constitutes a non-Archimedean version of the well-known Riesz representation theorem [9] in the case of a Krull valuation.

THEOREM 1. *Let $F : c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma) \to \mathbb{K}$ be a linear functional such that*

$$\lim_{i\to\infty} \Big(2v(F(e_i)) - N(e_i) \Big) = \infty. \tag{5}$$

Then there exists a unique $x_0 \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ such that

$$F(x) = \langle x, x_0 \rangle, \quad \textit{for all } x \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma).$$

Proof. If $x = \sum_{i\in\mathbb{N}} x_i e_i \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$, we then claim that $F(x) = \sum_{i\in\mathbb{N}} x_i F(e_i)$ is well-defined. Indeed, since $x \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$, then, $\lim_{i\to\infty} N(x_i e_i) = \infty$. Moreover, it is not hard to see that

$$2v(x_iF(e_i)) = 2v(x_i) + 2v(F(e_i)) = N(x_ie_i) + \Big(2v(F(e_i)) - N(e_i)\Big),$$

and hence

$$\lim_{i\to\infty} v(x_iF(e_i)) = \infty,$$

by using assumption (5).

Setting $x_0 = \sum_{i\in\mathbb{N}} \left(\frac{F(e_i)}{\omega_i}\right) e_i$ and using (5), one can see that

$$\lim_{i\to\infty} N\left[\left(\frac{F(e_i)}{\omega_i}\right)e_i\right] = \lim_{i\to\infty}\Big(2v(F(e_i)) - N(e_i)\Big) \to \infty,$$

and hence $x_0 \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$. Moreover, $F(x) = \langle x, x_0\rangle$ for each $x \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$.

Suppose that there exists another $y_0 \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ such that $F(x) = \langle x, y_0\rangle$ for each $x \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$. It follows that $\langle x_0 - y_0, u\rangle = 0$ for each $x \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$, that is, $x_0 - y_0 \perp c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$. In particular, $\langle x_0 - y_0, e_i\rangle = 0$ for each $i \in \mathbb{N}$, so all coordinates of $x_0 - y_0$ in the basis $(e_i)_{i\in\mathbb{N}}$ of $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ are zero, and hence $x_0 = y_0$. □

4. Bilinear forms on $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma) \times c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$

DEFINITION 4. *A mapping $\varphi : c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma) \times c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma) \to \mathbb{K}$ is said to be a bilinear form whenever $x \mapsto \varphi(x,y)$ is linear for each $y \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ and $y \mapsto \varphi(x,y)$ is linear for each $x \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$.*

Note that if $\varphi : c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma) \times c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma) \to \mathbb{K}$ is a bilinear form over the product $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma) \times c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$, then the sum

$$\varphi(x,y) = \sum_{i,j=0}^{\infty} \Omega_{ij}\, x_i y_j \tag{6}$$

where $\Omega_{ij} = \varphi(e_i, e_j)$ for all $i, j \in \mathbb{N}$, may or may not converge. However if both $x = (x_i)_{i\in\mathbb{N}}$ and $y = (y_i)_{i\in\mathbb{N}}$ are taken in $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ with

$$\lim_{i,j\to\infty}\Big(v(\Omega_{ij}) + 2v(x_i)\Big) = \infty \quad \text{and} \quad \lim_{i,j\to\infty}\Big(v(\Omega_{ji}) + 2v(y_i)\Big) = \infty, \tag{7}$$

then the sum in (6) converges.

4.1. Bounded bilinear forms

DEFINITION 5. *A non-Archimedean bilinear form* $\varphi : c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma) \times c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma) \to \mathbb{K}$ *is said to be bounded if there exists* $\gamma \in \Gamma$ *such that*

(8) $$2v(\varphi(x,y)) - N(x) - N(y) \geq \gamma \ \textit{ for all } \ x,y \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma).$$

If $\sup\{\gamma \in \Gamma : \ (8)$ *holds*$\}$ *exists, it is then called the norm of the bilinear form* φ *and is defined by*

$$\|\varphi\| = \inf_{x,y\neq 0}\Big(2v(\varphi(x,y)) - N(x) - N(y)\Big).$$

PROPOSITION 6. *Let* $\varphi : c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma) \times c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma) \to \mathbb{K}$ *be a bounded bilinear form. If* $\|\varphi\|$ *exists, it can then be explicitly expressed as*

$$\|\varphi\| = \inf_{i,j\in\mathbb{N}}\Big[2v(\varphi(e_i,e_j)) - N(e_i) - N(e_j)\Big].$$

Proof. Suppose $\|\varphi\|$ exists. The inequality

$$\|\varphi\| \leq \inf_{i,j\in\mathbb{N}}\Big[2v(\varphi(e_i,e_j)) - N(e_i) - N(e_j)\Big]$$

is a straightforward consequence of the definition of the norm $\|\varphi\|$ of φ.

Now suppose $x,y \neq 0$. In view of the above, one has

$$\begin{aligned} 2v(\varphi(x,y)) &= 2v\left(\sum_{i,j=0}^{\infty}\varphi(e_i,e_j)\, x_i y_j\right) \\ &\geq \inf_{i,j\in\mathbb{N}} 2v\Big(\varphi(e_i,e_j)x_i y_j\Big) \\ &= \inf_{i,j\in\mathbb{N}}\Big[\Big(2v(\varphi(e_i,e_j)) - N(e_i) - N(e_j)\Big) + N(x_i e_i) + N(y_j e_j)\Big] \\ &\geq \inf_{i,j\in\mathbb{N}}\Big[2v(\varphi(e_i,e_j)) - N(e_i) - N(e_j)\Big] + N(x) + N(y) \end{aligned}$$

and hence

$$2v(\varphi(x,y)) - N(x) - N(y) \geq \inf_{i,j\in\mathbb{N}}\Big[2v(\varphi(e_i,e_j)) - N(e_i) - N(e_j)\Big].$$

One completes the proof by combining the first and the latest inequalities. □

DEFINITION 6. *A bounded bilinear form* $\varphi : c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma) \times c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma) \to \mathbb{K}$ *is said to be representable whether there exists a bounded linear operator* $A : c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma) \to c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ *such that*

$$\varphi(x,y) = \langle Ax, y\rangle, \qquad \forall x,y \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma).$$

THEOREM 2. *Let* $\varphi : c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)\times c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma) \to \mathbb{K}$ *be a non-degenerate bounded bilinear form on* $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)\times c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$*. Then* φ *is representable whenever* (3) *holds. In that case, if A denotes the linear operator associated with* φ*, then the adjoint* A^* *of A exists.*

Proof. Define the linear operator A on $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ associated with φ by

$$Ax := \sum_{i,j\in\mathbb{N}} \left[\frac{\varphi(e_j,e_i)}{\omega_i}\right] e'_j(x)\, e_i,$$

for each $x \in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$.

We first check that the linear operator A given above is well-defined on the space $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$. For that, it suffices to show that, for all $j\in\mathbb{N}$,

$$\lim_{i\to\infty} N\left(\frac{\varphi(e_j,e_i)}{\omega_i}e_i\right) = \infty.$$

This in fact follows from

$$N\left(\frac{\varphi(e_j,e_i)}{\omega_i}e_i\right) = 2v(\varphi(e_j,e_i)) - N(e_i) \to \infty, \quad \text{as } i\to\infty,$$

by using (3). It is also routine to see that $\varphi(x,y) = \langle Ax,y\rangle$ for all $x,y\in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$. Of course, the linear operator A given above is bounded. Moreover, it is unique since φ is non-degenerate. It remains to show that A^*, the adjoint of A exists. Indeed, using Proposition 3 we have

$$\begin{aligned}\lim_{i\to\infty}\left[2v\left(\frac{\varphi(e_i,e_j)}{\omega_j}\right) - N(e_i)\right] &= \lim_{i\to\infty}\left[2v\left(\varphi(e_i,e_j)\right) - 2v(\omega_j)) - N(e_i)\right]\\ &= \lim_{i\to\infty}\Big(2v\left(\varphi(e_i,e_j)\right) - N(e_i)\Big) - 2N(e_j)\\ &= \lim_{i\to\infty}\Big(2v\left(\varphi(e_i,e_j)\right) - N(e_i)\Big)\\ &= \infty \ \text{ for all } j\in\mathbb{N},\end{aligned}$$

using assumption (3), and hence the adjoint A^* of A exists. ☐

REMARK 1. One should mention that in Theorem 2, if we suppose that $\|\varphi\|$ exists, then $\|A\|$ exists and $\|A\| = \|\varphi\|$. Indeed, using Proposition 2,

$$\begin{aligned}\|A\| &:= \inf_{i,j\in\mathbb{N}}\left[2v\left(\frac{\varphi(e_j,e_i)}{\omega_i}\right) + N(e_i) - N(e_j)\right]\\ &= \inf_{i,j\in\mathbb{N}}\Big[2v(\varphi(e_j,e_i)) - 2v(\omega_i) + N(e_i) - N(e_j)\Big]\\ &= \inf_{i,j\in\mathbb{N}}\Big(2v(\varphi(e_j,e_i)) - N(e_i) - N(e_j)\Big)\\ &= \|\varphi\|.\end{aligned}$$

EXAMPLE 1. Let $\mathbb{K} = \mathbb{Q}_p$ (p being a prime number such that $p > 2$) and let the group Γ be $(\mathbb{Z}, +, \leq)$. Define the Krull valuation $v : \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$ by $v(0) = \infty$ and for $0 \neq x \in \mathbb{Q}_p$,

$$v(x) = \max\Big\{r \in \mathbb{Z} : p^r \text{ divides } x\Big\}.$$

Let $\omega_i = p^{-i}$ for each $i \in \mathbb{N}$. Let $N_0 \in \mathbb{N}$ with $N_0 \geq 1$ (fixed) and set

$$\pi_{ij}^{N_0} = 1 + \frac{1}{\omega_j} + \frac{1}{\omega_i^2 \omega_j^2} + \cdots + \frac{1}{\omega_i^{N_0} \omega_j^{N_0}},$$

for all $i, j \in \mathbb{N}$.

For all $x = (x_i)_{i\in\mathbb{N}},\ y = (y_i)_{i\in\mathbb{N}} \in c_0^{\mathbb{Q}_p}(\mathbb{N}, \omega, \mathbb{Z})$, define the bilinear form

$$\varphi(x, y) = \sum_{i,j=0}^{\infty} \pi_{ij}^{N_0}\, x_i y_j.$$

Now,

$$\begin{aligned}
\|\varphi\| &:= \inf_{i,j\in\mathbb{N}} \Big[2v\Big(\pi_{ij}^{N_0}\Big) - N(e_i) - N(e_j)\Big] \\
&= \inf_{i,j\in\mathbb{N}} \Big[2v\Big(\pi_{ij}^{N_0}\Big) - v(\omega_i) - v(\omega_j)\Big] \\
&= \inf_{i,j\in\mathbb{N}} (i + j) \\
&= 0,
\end{aligned}$$

and hence φ is bounded.

Therefore, the only bounded linear operator on $c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma)$ associated with φ is the one defined by

$$Au = \sum_{i,j\in\mathbb{N}} \left[\frac{\pi_{ji}^{N_0}}{\omega_i}\right] e'_j(u) e_i$$

for each $u \in c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma)$ with $\|A\| = \inf_{i,j\in\mathbb{N}} \Big(N(Ae_i) - N(e_j)\Big) = \|\varphi\| = 0.$

4.2. Stable unbounded bilinear forms

In this subsection we provide a representation theorem for some unbounded bilinear forms. More precisely, we consider those unbounded bilinear forms whose domains contain all elements of the canonical basis $(e_i)_{i\in\mathbb{N}}$ of $c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma)$, as such a basis plays a key role in the present setting. The subclass of all those types of unbounded bilinear forms will be called stable and denoted $\Sigma_S(c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma) \times c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma))$.

Similarly, the subclass of all unbounded bilinear forms whose domains do not contain elements of the above-mentioned canonical basis will be called unstable and denoted $\Sigma_U(c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma) \times c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma))$. Note that a representation theorem similar to Theorem 3 for elements of $\Sigma_U(c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma) \times c_0^{\mathbb{K}}(\mathbb{N}, \omega, \Gamma))$ will be left as an open question.

DEFINITION 7. *A mapping* $\varphi : c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)\times c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)\supset D(\varphi)\times D(\varphi)\to\mathbb{K}$ *is said to be a stable unbounded bilinear form if* $u\mapsto\varphi(u,v)$ *is linear for each* $v\in D(\varphi)$, $v\mapsto\varphi(u,v)$ *linear for each* $u\in D(\varphi)$, *where* $D(\varphi)$ *is a vector subspace of* $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ *that contains the basis* $(e_i)_{i\in\mathbb{N}}$, *and consists of all* $x=(x_i)_{i\in\mathbb{N}}\in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ *such that*

$$\lim_{i,j\to\infty}\Big(v(\Omega_{ij})+2v(x_i)\Big)=\lim_{i,j\to\infty}\Big(v(\Omega_{ji})+2v(x_j)\Big)=\infty$$

and

$$\varphi(x,y)=\sum_{i,j=0}^{\infty}\Omega_{ij}\,x_iy_j,\ \ \textit{for all}\ \ x,y\in D(\varphi)$$

where $\Omega_{ij}=\varphi(e_i,e_j)$.

The space $D(\varphi)$ defined above is called the *domain* of the bilinear form φ.

DEFINITION 8. *A bilinear form* $\varphi : D(\varphi)\times D(\varphi)\to\mathbb{K}$ *(*$D(\varphi)$ *being its domain) is said to be representable whenever there exists a (possibly unbounded) linear operator* $A : D(A)\to c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ *(*$D(A)$ *being the domain of A) such that*

$$\varphi(x,y)=\langle Ax,y\rangle,\qquad \forall x\in D(A),\ v\in D(\varphi).$$

THEOREM 3. *Suppose that* $\omega=(\omega_j)_{j\in\mathbb{N}}$ *is chosen so that* $N(e_j)\to\gamma$ *as* $j\to\infty$ *where* $\gamma\in\Gamma$. *Let* $\varphi : D(\varphi)\times D(\varphi)\to\mathbb{K}$ *be a non-degenerate stable unbounded bilinear form. Then* φ *is representable whenever assumption* (3) *holds. In that case, if A denotes the linear operator associated with* φ, *then the adjoint* A^* *of A exists.*

Proof. For all $x=(x_i)_{i\in\mathbb{N}},y=(y_j)_{j\in\mathbb{N}}\in D(\varphi)$, write $\varphi(x,y)=\sum_{i,j=0}^{\infty}\Omega_{ij}\,x_iy_j$ and define the linear operator A on $c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ associated to it as follows:

$$\begin{cases} D(A):=\Big\{x=(x_i)_{i\in\mathbb{N}}\in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma):\ \lim\limits_{i\to\infty}N(x_iAe_i)=\infty\Big\},\\[2ex] Ax=\sum\limits_{i,j\in\mathbb{N}}\left[\dfrac{\varphi(e_j,e_i)}{\omega_i}\right]e'_j(x)e_i\ \text{ for each }\ x=(x_i)_{i\in\mathbb{N}}\in D(A).\end{cases}$$

One can prove as in the proof of Theorem 2 that A is well-defined.

Now

$$Ax=\sum_{i\in\mathbb{N}}\frac{1}{\omega_i}\left(\sum_{j\in\mathbb{N}}x_j\varphi(e_j,e_i)\right)e_i\ \text{ for each }\ x=(x_i)_{i\in\mathbb{N}}\in D(A),$$

and hence

$$\langle Ae_i,e_j\rangle=\varphi(e_i,e_j)\ \text{ for all }\ i,j\in\mathbb{N}.$$

Moreover, $D(A) \subset D(\varphi)$. That is, $\lim_{i\to\infty} N(x_i A e_i) = \infty$ yields

$$\lim_{i,j\to\infty}\Big(v(\Omega_{ij}) + 2v(x_i)\Big) = \lim_{i,j\to\infty}\Big(v(\Omega_{ji}) + 2v(x_i)\Big) = \infty.$$

Indeed, if $x = (x_i)_{i\in\mathbb{N}} \in D(A)$, then using the Cauchy-Schwarz inequality, it follows that, for all $i, j \in \mathbb{N}$,

$$2v(\varphi(e_i, e_j)) \geq N(Ae_i) + N(e_j),$$

and hence

$$\begin{aligned} 2\Big[v(\varphi(e_i,e_j)) + 2v(x_i)\Big] &= 2v(\varphi(e_i,e_j)) + 4v(x_i) \\ &\geq N(Ae_i) + N(e_j) + 4v(x_i) \\ &= N(x_i A e_i) + N(e_j) + 2v(x_i) \\ &= N(x_i A e_i) + N(x_i e_i) + N(e_j) - N(e_i) \\ &\to \infty \ \text{ as } i, j \to \infty. \end{aligned}$$

Similarly, using the fact $(e_j)_{j\in\mathbb{N}} \subset D(A)$, i.e., $N(Ae_j) \to \infty$ as $j \to \infty$, we obtain

$$\begin{aligned} 2\Big[v(\varphi(e_j,e_i)) + 2v(x_i)\Big] &= 2v(\varphi(e_j,e_i)) + 4v(x_i) \\ &\geq N(Ae_j) + N(e_i) + 4v(x_i) \\ &= N(Ae_j) + 2N(x_i e_i) - N(e_i) \\ &\to \infty \ \text{ as } i, j \to \infty. \end{aligned}$$

Note that $v(x_i y_k \varphi(e_i, e_k)) \to \infty$ as $i, k \to \infty$, using the fact that $x \in D(A) \subset D(\varphi)$ and $y \in D(\varphi)$, as

$$2v\Big(x_i y_k \varphi(e_i,e_k)\Big) = 2v(x_i) + v\Big(\varphi(e_i,e_k)\Big) + v\Big(\varphi(e_i,e_k)\Big) + 2v(y_k) \to \infty, \ \text{ as } \ i,k \to \infty.$$

Hence

$$\sum_{k\in\mathbb{N}}\sum_{i\in\mathbb{N}} x_i y_k \varphi(e_i,e_k) = \sum_{i\in\mathbb{N}}\sum_{k\in\mathbb{N}} x_i y_k \varphi(e_i,e_k).$$

Consequently, the following sequence of equalities is justified:

$$\begin{aligned} \langle Ax, y\rangle &= \sum_{k\in\mathbb{N}} \omega_k y_k \frac{1}{\omega_k}\left(\sum_{i\in\mathbb{N}} x_i \varphi(e_i,e_k)\right) \\ &= \sum_{k\in\mathbb{N}} y_k \left(\sum_{i\in\mathbb{N}} x_i \varphi(e_i,e_k)\right) \\ &= \sum_{i,k\in\mathbb{N}} \varphi(e_i,e_k) x_i y_k \\ &= \varphi(x,y), \end{aligned}$$

for all $x=(x_i)_{i\in\mathbb{N}}\in D(A)$ and $y=(y_i)_{i\in\mathbb{N}}\in D(\varphi)$.

Furthermore, the uniqueness of A is guaranteed by the fact that the form φ is non-degenerate. It remains to show that A^*, the adjoint of A, exists. Though this can be done as in the bounded case. □

EXAMPLE 2. This is a generalization of Example 1 to the case in which φ is possibly unbounded.

Consider the bilinear form defined by

$$\varphi(x,y)=\sum_{i,j\in\mathbb{N}}\pi_{ij}\,x_iy_j,\quad \forall x=(x_i)_{i\in\mathbb{N}},\ y=(y_i)_{i\in\mathbb{N}}\in D(\varphi),$$

where $(\pi_{ij})_{i,j\in\mathbb{N}}\subset\mathbb{K}$ is an arbitrary sequence, and the domain $D(\varphi)$ of φ is defined by all $x=(x_i)_{i\in\mathbb{N}}\in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma)$ such that

$$\lim_{i,j\to\infty}\Big(v(\pi_{ij})+2v(x_i)\Big)=\lim_{i,j\to\infty}\Big(v(\pi_{ji})+2v(x_i)\Big)=\infty.$$

Note that $\varphi(e_i,e_j)=\pi_{ij}$ for all $i,j\in\mathbb{N}$ and therefore an equivalent version of assumption (3) is

$$\lim_{i\to\infty}\Big(v(\pi_{ij})-N(e_i)\Big)=\lim_{i\to\infty}\Big(v(\pi_{ji})-N(e_i)\Big)=\infty. \tag{9}$$

Furthermore, if ω_i, $i\in\mathbb{N}$ are chosen such that $N(e_i)\to\gamma\in\Gamma$ as $i\to\infty$ then upon making assumption (9), the unique (possibly unbounded) linear operator associated with φ is given by

$$Ax=\sum_{i,j\in\mathbb{N}}\frac{\pi_{ji}}{\omega_i}e'_j(x)\,e_i,\quad \forall x=(x_i)_{i\in\mathbb{N}}\in D(A)$$

where $D(A)=\Big\{x=(x_i)_{i\in\mathbb{N}}\in c_0^{\mathbb{K}}(\mathbb{N},\omega,\Gamma):\lim_{i\to\infty}N(x_iAe_i)=\infty\Big\}$.

In addition to the above, the adjoint A^* of A does exist under assumption (9).

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