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A. Peterson and G.K. Sankaran
with an appendix by V. Gritsenko

ON SOME LATTICE COMPUTATIONS RELATED TO MODULI PROBLEMS

Abstract. The method used in [6] to prove that most moduli spaces of K3 surfaces are of general type leads to a combinatorial problem about the possible number of roots orthogonal to a vector of given length in E_8. A similar problem arises for E_7 in [8]. Both were solved partly by computer methods. We use an improved computation and find one further case, omitted from [6]: the moduli space $\mathcal{F}_{2d}$ of K3 surfaces with polarisation of degree $2d$ is also of general type for $d = 52$. We also apply this method to some related problems. In Appendix A, V. Gritsenko shows how to arrive at the case $d = 52$ and some others directly.

Introduction

Many moduli spaces in algebraic geometry can be described as locally symmetric varieties, i.e. quotients of a Hermitian symmetric domain $\mathcal{D}$ by an arithmetic group Γ. One method of understanding the birational geometry of such quotients is to use modular forms for Γ to give information about differential forms on $\Gamma\backslash\mathcal{D}$. In [6] this method was used to prove that the moduli space $\mathcal{F}_{2d}$ of polarised K3 surfaces of degree $2d$ is of general type in all but a few cases. The method works if there exists a modular form of sufficiently low weight with sufficiently large divisor. In [6], and again in [8] where a similar method was applied to certain moduli of polarised hyperkähler manifolds, the required modular form is constructed by quasi-pullback of the Borcherds form Φ_{12}.

A suitable quasi-pullback exists if a combinatorial condition is satisfied: there should exist a vector l in the root lattice E_8 (or E_7 in the hyperkähler case) of square $2d$, orthogonal to very few roots. This is evidently the case if d is large, but for small d the search for such an l invites the use of a computer. This was done in both [6] and [8] by a randomised search, relying on the large Weyl group to ensure that in practice no cases would be missed.

Here we present an exhaustive search carried out by the first author. For the hyperkähler case the exhaustive search confirmed the results of the earlier randomised search, but in the K3 case one previously overlooked value of d with a suitable vector was found, namely $d = 52$. In fact it turned out that the randomised search had indeed found this value, and the omission of the case $d = 52$ from [6] happened because the output had been interpreted incorrectly (by GKS).

Nevertheless the following result is true and has not previously appeared in the literature.

THEOREM 1. *The moduli space $\mathcal{F}_{2\cdot 52}$ of* K3 *surfaces with polarisation of degree* 104 *is of general type.*

combinatorial problem is and how it arises, and give some more general combinatorial problems of the same nature. In Section 2 we describe the theoretical and computational methods used to solve it, along with some other results obtained in the same way. In Appendix A, Valery Gritsenko explains how the case $d = 52$ could have been foreseen without the help of a computer. Some of the relevant computer code is given in Appendix B.

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1. Combinatorial problems and moduli

In this section we first give a list of combinatorial questions and then explain the geometry that originally motivated them. First we fix some terminology. We say that L is a *lattice of signature* (a,b) if $L \cong \mathbb{Z}^{a+b}$ and we fix a bilinear form $(\,,\,)\colon L \times L \to \mathbb{Z}$ of signature (a,b). If $x \in L$ we refer to (x,x) as x^2 and call it the *length* of x. If the length of x is 2 then x is called a *root*. If the roots of L generate L as an abelian group then L is called a *root lattice*. A lattice L is *unimodular* if it is equal to its dual $L^\vee = \mathrm{Hom}(L,\mathbb{Z}) \supseteq L$. We do not assume that L is always unimodular but for simplicity we do assume that L is *even*, i.e. that x^2 is always an even integer.

E_8 denotes the unique even unimodular positive-definite lattice of rank 8, i.e. with signature $(8,0)$: this is the sign convention of [3] and is also used in [6]. If $n \in 2\mathbb{Z}$ then $\langle n \rangle$ is the rank 1 lattice spanned by a vector of length n, and U denotes the integral hyperbolic plane $\mathbb{Z}e + \mathbb{Z}f$ with $e^2 = f^2 = 0$ and $(e,f) = 1$. The symbol $\oplus$ denotes the orthogonal direct sum of lattices. If Λ is a lattice and $n \in \mathbb{Z}$, then $\Lambda(n)$ denotes the same lattice with the quadratic form multiplied by n. In particular, $E_8(-1)$ is the negative-definite even unimodular lattice of rank 8.

1.1. Combinatorial problems

Let Λ be a root lattice (usually it will be E_8 or E_7) and denote by $R(\Lambda)$ the set of its roots, i.e. $R(\Lambda) = \{r \in \Lambda \mid r^2 = 2\}$. The combinatorial questions arising in [6] and [8] are special cases of the following.

QUESTION 1. Given integers $p > q \geq 0$, what are the values of d for which every vector of length $2d$ that is orthogonal to at least $2q$ roots is orthogonal to at least $2p$ roots?

More generally we may ask about all possibilities.

QUESTION 2. Given an even natural number $2d$, what are the possible numbers of roots orthogonal to a vector of length $2d$?

If $l \in \Lambda$ we denote by $R(l^\perp)$ the system of roots of Λ orthogonal to l. We denote the answer to Question 2 by $P(\Lambda, d)$: that is

$$(1) \qquad P(\Lambda, d) := \{m \in \mathbb{Z} \mid \exists l \in \Lambda \; l^2 = 2d, \; \#R(l^\perp) = m\}.$$

Thus $P(\Lambda, d)$ is a finite set of even non-negative integers. We call this the *root type* of the non-negative even integer $2d$ for the lattice Λ

There are some immediate restrictions on what the root type can be: for example, if $\Lambda = E_8$ then the largest m that can occur is 126, when $R(l^\perp) \cong E_7$; but in that case $l \in (E_7)^\perp_{E_8} \cong A_1$, so d must be a square.

Especially for $\Lambda = E_8$, the value of $m_0(d) = \min P(E_8, d)$ is of interest as it determines the lowest weight of modular form obtained by quasi-pullback (see Equation (2) below). If $m_0(d) = 0$ then this form will not be a cusp form, so the value of $m_1(d) = \min P(E_8, d) \cap \mathbb{N}$ is also significant. We should also like to know whether this form is unique. So we also have the following questions.

QUESTION 3. For given d and Λ, how can we compute $m_0(d)$?

QUESTION 4. For given m, what is the smallest value $d(m)$ of d for which $m_1(d) \leq m$?

If in Question 4 we replace m_1 by m_0, then the case $m = 0$ asks for the length of shortest vectors in the interior of a Weyl chamber: these are the Weyl vectors, which are well known.

If $m \in P(\Lambda, d)$ there is a further natural refinement.

QUESTION 5. How many Weyl group orbits of vectors l with $l^2 = 2d$ and $\#R(l^\perp) = m$ are there?

Some values of m are of particular interest for geometric reasons: for instance, if $14 \in P(E_8, d)$ then quasi-pullback of Φ_{12} gives a canonical form on $\mathcal{F}_{2d}$ (see Section 1.2 below). This leads us to the following variant of Question 1.

QUESTION 6. For given m and Λ, what are the values of d such that $m \in P(\Lambda, 2d)$?

We can compute the answers to some cases of these questions by the methods described in Section 2.

1.2. Moduli

The following construction describes several moduli spaces in algebraic geometry, including the moduli of polarised K3 surfaces.

Let L be an even lattice of signature $(2, n)$. The Hermitian symmetric domain associated with L is $\mathcal{D}_L$, one of the two connected components of

$$\mathcal{D}_L \cup \overline{\mathcal{D}}_L = \{[w] \in \mathbb{P}(L \otimes \mathbb{C}) \mid w^2 = 0,\ (w, \overline{w}) > 0\}.$$

The group $\mathrm{O}(L)$ of isometries of L acts on this union and we denote by $\mathrm{O}^+(L)$ the index 2 subgroup preserving $\mathcal{D}_L$. The action is discontinuous, with finite stabilisers, so if Γ is any finite index subgroup of $\mathrm{O}^+(L)$ then

$$\mathcal{F}_L(\Gamma) := \Gamma\backslash\mathcal{D}_L$$

is a complex analytic space. In fact it is a quasi-projective variety, having a minimal projective compactification, the Baily-Borel compactification $\mathcal{F}_L(\Gamma)^*$, obtained by adding finitely many curves (called 1-dimensional cusps) meeting at finitely many points (0-dimensional cusps). It is often preferable to work with a toroidal compactification $\overline{\mathcal{F}_L(\Gamma)}$, which is a modification of $\mathcal{F}_L(\Gamma)^*$ depending on some combinatorial choices at the 0-dimensional cusps.

A modular form for Γ of weight k and character $\chi\colon \Gamma \to \mathbb{C}^*$ is a holomorphic function F on the affine cone $\mathcal{D}_L^\bullet \subset L \otimes \mathbb{C}$ such that

$$F(tZ) = t^{-k}F(Z)\ \ \forall t \in \mathbb{C}^* \quad \text{and} \quad F(gZ) = \chi(g)F(Z)\ \ \forall g \in \Gamma.$$

F is a cusp form if it vanishes at every cusp. For the cases we shall consider the only possible characters are 1 and $\det(g)$, and the order of vanishing at a cusp is an integer: see [7].

The aim of [6] is to show that the moduli space $\mathcal{F}_{2d}$ of polarised K3 surfaces of degree $2d$ is of general type for most values of $d \in \mathbb{N}$. Using the Torelli theorem for K3 surfaces one can show that

$$\mathcal{F}_{2d} = \mathcal{F}_{L_{2d}}(\widetilde{\mathrm{O}}^+(L_{2d})),$$

where $\widetilde{\mathrm{O}}^+(L)$ is the finite index subgroup of $\mathrm{O}^+(L)$ that acts trivially on the discriminant group $L^\vee/L$ and

$$L_{2d} := 2U \oplus 2E_8(-1) \oplus \langle -2d \rangle.$$

Modular forms of suitable weight can be interpreted as differential forms on the moduli space provided that they have sufficiently large divisor. Therefore, to prove that the moduli space is of general type it is enough to give a sufficient supply of such modular forms. There are several technical difficulties here, one of which is the presence of singularities. A sufficient condition, however, was given in [6].

THEOREM 2. *Suppose that* $n \geq 9$ *and that there exists a nonzero cusp form* F_a *of weight* $a < n$ *and character* $\chi \equiv 1$ *or* $\chi(g) = \det(g)$, *vanishing along any divisor* $\mathcal{H} \subset \mathcal{D}_L$ *fixed by reflections in* Γ. *Then* $\mathcal{F}_L(\Gamma)$ *is of general type.*

The form F_a is then used to give many forms of high weight with sufficiently large divisor, of the form $F = F_a^k F_{(n-a)k}$, and these in turn give pluricanonical forms on a smooth model of $\overline{\mathcal{F}_L(\Gamma)}$.

To apply this in specific cases such as $\mathcal{F}_{2d}$ one must therefore construct F_a. The method used in [6] to do this is quasi-pullback of the Borcherds form Φ_{12}. This construction first appeared in [2]. The Borcherds form itself was constructed in [1] by means of a product expansion, whereby its divisor is evident. It is a modular form (not a cusp form) of weight 12 and character det for the group $\mathrm{O}^+(II_{2,26})$. The lattice $II_{2,26}$ of signature $(2,26)$ is $2U \oplus N(-1)$, where N is any one of the 24 Niemeier lattices, positive definite unimodular lattices of rank 24: see [4]. For our purposes the correct choice of N is $3E_8$. A choice of a (not necessarily primitive) vector $l \in E_8$ of length $2d$ gives an embedding

$$L_{2d} = 2U \oplus 2E_8(-1) \oplus \langle -2d \rangle \hookrightarrow II_{2,26} = 2U \oplus 3E_8(-1)$$

which in turn gives an embedding

$$\mathcal{D}^\bullet_{L_{2d}} \hookrightarrow \mathcal{D}^\bullet_{II_{2,26}}.$$

Denote the images of these embeddings by $L_{2d}[l]$ and $\mathcal{D}^\bullet[l]$ respectively.

If $r \in L$ is a root it determines a Heegner divisor $\mathcal{H}^\bullet_r \subset \mathcal{D}^\bullet_L$, given by the equation $(Z,r) = 0$. The Borcherds form vanishes (to order 1) along all the Heegner divisors for $L = II_{2,26}$ and in particular its restriction to $\mathcal{D}^\bullet[l]$ vanishes, as needed to apply Theorem 2. However, $\Phi_{12}|_{\mathcal{D}^\bullet[l]}$ may well be zero, since if r is a root of $II_{2,26}$ orthogonal to $L_{2d}[l]$ then $\mathcal{D}^\bullet[l] \subset \mathcal{H}^\bullet_r$.

Instead we take the quasi-pullback, simply dividing by the equation of each such $\mathcal{H}^\bullet_r$, noting that $\mathcal{H}^\bullet_{-r} = \mathcal{H}^\bullet_r$. We put

$$R_l = \{r \in R(II_{2,26}) \mid (r, L_{2d}[l]) = 0\} \cong \{r \in R(E_8) \mid (r,l) = 0\}$$

and define the quasi-pullback to be

$$F[l] = \left. \frac{\Phi_{12}}{\prod_{\pm r \in R_l} (r,Z)} \right|_{\mathcal{D}^\bullet[l]}. \tag{2}$$

This is a nonzero modular form, and one can show that it is a cusp form provided $R_l \neq \emptyset$. It vanishes along all the Heegner divisors fixed by reflections in $\mathrm{O}^+(L_{2d})$.

The weight, however, goes up by 1 every time we divide, so the weight of $F[l]$ is $12 + \frac{1}{2}\#R_l$. We can therefore show that $\mathcal{F}_{2d}$ is of general type if we can find an $l \in E_8$ of length $2d$ with $2 \leq \#R_l < 2(n-12) = 14$. Moreover, if we can find a cusp form of weight precisely $n = 19$ then, by a result of Freitag [5], $\mathcal{F}_{2d}$ has $p_g > 0$ and in particular is not uniruled.

This leads us to Question 1, with $q = 1$ and $p = 7$ or $p = 8$, for $\Lambda = E_8$. In [8], similar considerations about the moduli of some hyperkähler manifolds with a certain type of polarisation lead to Question 1 with $q = 1$ and $p = 6$ or $p = 7$, for $\Lambda = E_7$.

2. Solving the combinatorial problems

The specific combinatorial problems encountered in [6] and [8] can be solved in principle by first bounding d. It is clear that for sufficiently large d an l will exist orthogonal to a number of roots in the required range: indeed, for sufficiently large d we can find l orthogonal to exactly two roots. An explicit bound, followed by a finite calculation, will solve the problem. Neither is entirely straightforward, though. In [6] a counting argument is used to show that an $l \in E_8$ with $l^2 = 2d$, orthogonal to at least two and at most 12 roots, exists (and therefore $\mathcal{F}_{2d}$ is of general type) unless

$$28N_{E_6}(2d) + 63N_{D_6}(2d) \geq 4N_{E_7}(2d), \tag{3}$$

where $N_L(2d)$ is the number of ways of representing $2d$ by the quadratic form L. The inequality (3) certainly fails for large d, but to obtain an effective bound on d one must bound $N_{E_6}(2d)$ and $N_{D_6}(2d)$ from above and $N_{E_7}(2d)$ from below by explicit functions. This is a non-trivial problem in analytic number theory but it can be done, and after some refinements it gives a reasonable bound of around $d = 150$. It would be possible to resort to direct computation at that point, but there is no need yet. Some integers in that range are excluded from the list of possibly non-general type polarisations because the inequality (3) (or another similar inequality) in fact fails. Others can be excluded by inspection, actually producing a vector l by guessing the root system $R(l^{\perp}_{E_8})$. The root systems used in this way in [6] were $4A_1$, $2A_1 \oplus A_2$, A_3 and $A_1 \oplus A_2$. The root systems $3A_1 \oplus A_2$ and $2A_2$ were not tried: see Appendix A.

In [8] a similar procedure was used, although there is an extra difficulty caused by the opposite parity of the rank: working in E_7, one needs to estimate $N_R(2d)$ from above for some odd-rank root systems R, and this problem is not so well studied as in the even rank case.

In either case, eventually one is left with a residual list of values of d for which the problem has not been settled. In [6] it consists of most integers between 15 and 60 (for very small d the moduli space is known to be unirational). The residual problem in the hyperkähler case considered in [8] is much smaller.

Now, if we want to be (reasonably) sure that no cases have been missed, we do need a computer. Moreover, the methods we now use to solve this problem can also be used to give answers to question such as those posed in Section 1.1.

2.1. Algorithms

We begin by representing E_8 in the usual way, as the set of points $l = (l_1, \ldots, l_8) \in \mathbb{R}^8$ such that the l_i are either all integers or all strict half-integers (i.e. either $l_i \in \mathbb{Z}$ for all i or $2l_i$ is an odd integer for all i) and $\sum l_i \in 2\mathbb{Z}$, with the standard Euclidean quadratic form on $\mathbb{R}^8$.

We need a very rough upper bound on $N_{E_8}(2d)$, because we want to know whether $N_{E_8}(2d)$ is small enough to allow a brute-force search for $l \in E_8$ with $l^2 = 2d$ having $2 \leq \#R(l^{\perp}) \leq 12$. We can easily find such a bound by noting that if $l^2 = 2d$ then each of the 8 components l_i of l must have $l_i^2 \leq 2d$, so $-\sqrt{2d} \leq l_i \leq \sqrt{2d}$, and must be a half-integer: that gives

$$N_{E_8}(2d) \leq (2\lfloor 2\sqrt{2d} \rfloor + 1)^8 \tag{4}$$

For $d = 52$, this bound is about $8 \cdot 10^{12}$.

If we are a bit more precise, and note that the components of l are either all integers, or all proper (i.e. non-integer) half-integers, we save a factor 2^7, giving a bound of about $5 \cdot 10^{10}$. This is within reach of a brute-force search, but it is still high, especially considering that we have to do some substantial work for each candidate (compute the inner product with 240 different vectors[1]).

Thus an exhaustive search of all vectors in E_8 of length ≤ 60 is not computationally impossible but it would be cumbersome and would not extend to even slightly larger problems such as other cases of Question 1. The Weyl group $W(E_8)$ has order $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7 = 696729600$ and should be used to reduce the size of the problem. There are two approaches to doing this.

(A) Randomised search. This is what was actually done in [6] and [8]. Since the non-existence of a vector l gives no information about the moduli space, we are willing to accept a very small probability of failing to detect such a vector. We therefore choose a large number of vectors of length less than $2 \cdot 61$ at random and expect that, as the Weyl group orbits are large, every orbit will be represented.

This approach worked very fast, using only a laptop computer and immediately available software (Maple). A search of twenty thousand randomly chosen vectors found all the pairs $(d, \#R(l^\perp))$ in the ranges wanted within the first two thousand iterations, in approximately two minutes. That is fairly convincing practical evidence that there are no more. Unfortunately the output was then mistranscribed, leading to the omission of the case $d = 52$ and the erroneous (but not really misleading) statement in [6] that "an extensive computer search for vectors orthogonal to at least 2 and at most 14 roots for other d has not found any".

It is noteworthy that a similar search in the case $\Lambda = E_7$ did find some cases not discovered analytically, and for which a constructive method of finding l is still not known. In other words, some cases of the main theorem of [8] still have only a computer proof, although once l has been found it is easy enough to verify its properties by hand.

It is not so easy to estimate the probability *a priori* that a Weyl orbit might be missed. The Weyl group of $R(l^\perp)$, which is a subgroup of the Weyl group of E_8, obviously stabilises l and has order no more than 24 if $\#R(l^\perp) \leq 12$, but in principle the stabiliser of l in $W(E_8)$ could be much larger. In that case the Weyl group orbit would be small and more easily missed. In practice the randomised method seems to find all the orbits.

(B) Exhaustive search. The first author organised an exhaustive search, exploiting the Weyl group by searching a fundamental domain for the subgroup $H < W(E_8)$ generated by permutations of the eight components l_i and sign changes of an even number of components. This subgroup H has size $2^7 \cdot 8!$, so index 135 in $W(E_8)$: it gives us most of the symmetries, with very little effort.

[1] We can be a lot more efficient than that, and skip most of these inner products, but even then we still have to compute dozens of inner products per candidate vector.

We say that $l \in E_8$ is in *normal form* if its components are all nonnegative (except possibly the first, l_1) and the squares of the components are nondecreasing from low index to high index. By acting with an element of H, we can translate any $l \in E_8$ to one in normal form: first permute the components, so their squares are in order; then make them all (but l_1) nonnegative, by changing the sign of every negative component (except l_1), and flipping the sign of l_1 once for every such change.

It is straightforward to enumerate the elements of length $2d$ in E_8 that are in normal form. For brevity, we will describe this only for the ones having integer components (one can get the ones with proper half-integer components in a very similar manner).

Step 1. For every index $i \neq 1$, in descending order, we consider all the possible values of l_i: we require l_i to be a non-negative integer such that

- its square, added to the sum of the squares of the coordinates that have been chosen (i.e. the l_j^2 with $j > i$), does not exceed $2d$ (otherwise $l^2 > 2d$, for any further choice of coordinates); and
- (unless $i = 8$) it is not greater than l_{i+1} (otherwise l would not be in normal form).

In other words, we let l_i take any value $s \in \mathbb{Z}$ such that

$$0 \leq s \leq \min\left\{ l_{i+1}, \sqrt{2d - \sum_{j>i} l_j^2} \right\}. \tag{5}$$

Step 2. See if $2d - \sum_{j=2}^{8} l_j^2$ is a perfect square m^2. If so, let l_1 take values $-m$ and m; if not, discard this choice of coordinates.

Step 3. Check whether the l so obtained are in E_8, i.e. whether $\sum_{j=1}^{8} l_j \in 2\mathbb{Z}$. Discard any that are not in E_8.

We must then filter these enumerated $l \in E_8$ to find the ones with $\#R(l^\perp)$ in the required range ($2 \leq \#R(l^\perp) \leq 12$ for the case considered in [6]): this part of the procedure is exactly the same as for the randomised version. Since the roots come in pairs $\pm r$ it is enough to take inner products with a prepared list of positive roots (120 or them), and of course we can stop examining l as soon as we find a seventh pair of roots orthogonal to it.

The first author implemented this search in a high-level programming language (Haskell). Without spending much time optimising, this runs fast enough (a second or so on commercial hardware, for each of the low values of d we are interested in, namely $d \leq 60$). The partial use of the symmetries of E_8 is crucial, though: to go through all the vectors of given length $2d$ would have taken weeks or months for a single value of d.

This program discovered the lost case $d = 52$ and therefore Theorem 1. A variant of it for E_7 reconfirmed the results obtained by the randomised method in [8]. The code used for the E_8 case is given in Appendix B.

2.2. Further results

The exhaustive algorithm (B) from Section 2.1 can be modified to compute, in reasonable time, answers to some of the questions from Section 1.1 for small values of the parameters. We investigated Question 2 and Question 6 for small m and d with $\Lambda = E_7$ and $\Lambda = E_8$. For $\Lambda = E_8$ we also investigated Question 5 for the particular case $m = 14$, corresponding to canonical forms on $\mathcal{F}_{2d}$.

Specifically, we have so far computed the root type $P(\Lambda, 2d)$ for $\Lambda = E_7$ and $\Lambda = E_8$ and $d \leq 150$, and the first part of the root type (whether $m \in P(\Lambda, 2d)$ for $2 \leq m \leq 20$, say) for larger d, up to about 300 (further for some values of d). This part of the computation is fairly fast and only minor changes to the program are needed.

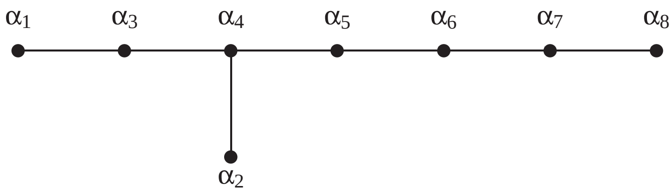
A little more work, and more computer time, is needed for Question 5. We must work now with $W(E_8)$, not with H, and we first compute a transversal for $W(E_8) : H$ (representatives for each of the 135 left cosets of H) and then reduce each of the 135 translates of each l to standard form before comparing them.

The outcome counts the number of ways of obtaining a canonical form on $\mathcal{F}_{2d}$ by quasi-pullback of Φ_{12}. There is no assurance either that the forms so obtained are linearly independent or that there are not more canonical forms that do not arise this way. The results are nevertheless intriguingly unpredictable. There are no such vectors for $d < 40$. There is such a vector for $d = 40$, and also for $d = 42,\ 43,\ 48$ (two orbits), $49,\ 51$–$54,\ 55$ and 56 (two orbits each), 57 and 59. There is no such vector for $d = 60$, but for 61 there are three orbits and thereafter the number of orbits drifts upwards irregularly. Without further comment, we tabulate below the number ν_{14} of $W(E_8)$ orbits of length $2d$ vectors in E_8 orthogonal to exactly 14 roots for $61 \leq d \leq 150$.

d	ν_{14}	d	ν_{14}	d	ν_{14}	d	ν_{14}	d	ν_{14}	d	ν_{14}
61	3	76	1	91	5	106	2	121	4	136	8
62	1	77	2	92	3	107	6	122	5	137	7
63	2	78	1	93	2	108	3	124	5	138	5
64	2	79	4	94	4	109	6	124	3	139	11
65	0	80	2	95	3	110	0	125	6	140	5
66	2	81	2	96	4	111	6	126	8	141	6
67	1	82	2	97	2	112	6	127	6	142	8
68	2	83	3	98	3	113	5	128	6	143	3
69	2	84	5	99	2	114	3	129	7	144	8
70	1	85	4	100	4	115	7	130	4	145	8
71	2	86	4	101	5	116	6	131	9	146	7
72	2	87	3	102	5	117	2	132	2	147	11
73	1	88	2	103	5	118	6	133	8	148	5
74	3	89	3	104	4	119	9	134	9	149	10
75	3	90	2	105	4	120	8	135	5	150	6

Appendix A. ***d* = *46, 50, 52, 54, 57*, by V. Gritsenko**

In this appendix we find a vector $l \in E_8$ of square $2d$ orthogonal to exactly 12 roots in E_8, where d is as in the title of the appendix. (See [6] and [8] for the general context of this question.) We use below the combinatorics of the Dynkin diagram of E_8. We take the Coxeter basis of simple roots in E_8 as in [3]:



where $(e_1,\dots,e_8)$ is a Euclidean basis in the lattice $\mathbb{Z}^8$ and

$$\alpha_1 = \frac{1}{2}(e_1+e_8) - \frac{1}{2}(e_2+e_3+e_4+e_5+e_6+e_7),$$
$$\alpha_2 = e_1+e_2, \quad \alpha_k = e_{k-1}-e_{k-2} \ \ (3 \le k \le 8).$$

The lattice E_8 contains 240 roots. We recall that any root is a sum of simple roots with integral coefficients of the same sign. The fundamental weights ω_j of E_8 form the dual basis in $E_8 = E_8^\vee$, so $(\alpha_i, \omega_j) = \delta_{ij}$. The formulae for the weights are given in [3, Tabl. VII]. The Cartan matrix of the dual basis is

$$((\omega_i,\omega_j)) = \begin{pmatrix} 4 & 5 & 7 & 10 & 8 & 6 & 4 & 2 \\ 5 & 8 & 10 & 15 & 12 & 9 & 6 & 3 \\ 7 & 10 & 14 & 20 & 16 & 12 & 8 & 4 \\ 10 & 15 & 20 & 30 & 24 & 18 & 12 & 6 \\ 8 & 12 & 16 & 24 & 20 & 15 & 10 & 5 \\ 6 & 9 & 12 & 18 & 15 & 12 & 8 & 4 \\ 4 & 6 & 8 & 12 & 10 & 8 & 6 & 3 \\ 2 & 3 & 4 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}. \tag{6}$$

We consider the two following cases when the orthogonal complement of a vector l in E_8 contains exactly 12 roots: $R(l^\perp_{E_8}) = A_2 \oplus 3A_1$ or $A_2 \oplus A_2$. (We note that $\#R(A_1) = 2$ and $\#R(A_2) = 6$.)

The cases *d* = *46, 50, 54, 57*. There are four possible choices of the subsystem $A_2 \oplus 3A_1$ inside the Dynkin diagram of E_8 according to the choices of simple roots of A_2, namely $A_2^{(1,3)} = \langle\alpha_1,\alpha_3\rangle$, $A_2^{(2,4)} = \langle\alpha_2,\alpha_4\rangle$, $A_2^{(5,6)} = \langle\alpha_5,\alpha_6\rangle$ or $A_2^{(7,8)} = \langle\alpha_7,\alpha_8\rangle$. If A_2 is fixed then the three pairwise orthogonal copies of A_1 in the Dynkin diagram are defined automatically.

First, we consider $A_2^{(5,6)} = \langle\alpha_5,\alpha_6\rangle$. Then $3A_1^{(5,6)} = \langle\alpha_2\rangle \oplus \langle\alpha_3\rangle \oplus \langle\alpha_8\rangle$. Moreover $A_2^{(5,6)} \oplus 3A_1^{(5,6)}$ is the root system of the orthogonal complement of the vector $l_{5,6} = \omega_1+\omega_4+\omega_7 \in E_8$. In fact, if $r = \sum_{i=1}^8 x_i\alpha_i$ is a positive root ($x_i \ge 0$) then $(r, l_{5,6}) = x_1+x_4+x_7 = 0$. Therefore $x_1 = x_4 = x_7 = 0$ and r belongs to $A_2^{(5,6)} \oplus 3A_1^{(5,6)}$.

Using the Cartan matrix (6) we obtain that $l_{5,6}^2 = 2\cdot 46$. Doing similar calculations with the other three copies of A_2 given above we find

$$l_{1,3} = \omega_4 + \omega_6 + \omega_8, \quad l_{2,4} = \omega_3 + \omega_5 + \omega_7, \quad l_{7,8} = \omega_1 + \omega_4 + \omega_6$$

with $l_{1,3}^2 = 2\cdot 50$, $l_{2,4}^2 = 2\cdot 54$ and $l_{7,8}^2 = 2\cdot 57$.

The case *d = 52*. We consider the sublattice $M = A_2 \oplus A_2 = \langle \alpha_3, \alpha_4\rangle \oplus \langle \alpha_6, \alpha_7\rangle$ in E_8. Then M is the root system of the orthogonal complement of the vector $l_M = \omega_1 + \omega_2 + \omega_5 + \omega_8$ with $l_M^2 = 2\cdot 52$.

Appendix B. The computer code

Below is the code used to check the combinatorial problem from [6], and thus to find Theorem 1. The programs were written in the functional programming language Haskell (`http://www.haskell.org`). The web page

`http://people.bath.ac.uk/masgks/Rootcounts`

contains links to further code and output.

```
{-# LANGUAGE TypeSynonymInstances,NoImplicitPrelude #-}
module E8 where

import qualified Algebra.Ring
import           Control.Applicative      ((<$>),(<*>))
import qualified Data.Vector          as V
import           Data.List                (intercalate,nubBy)
import qualified Data.MemoCombinators as Memo
import           Data.Ratio
  (Ratio,numerator,denominator,(%))
import qualified Data.Set             as Set
import           Data.Typeable            (Typeable)
import           Math.Combinatorics.Species
  (ksubsets,set,ofSize,enumerate,Set(getSet,Set),Prod(Prod))
import           MyPrelude hiding (numerator,denominator,(%))
import qualified Prelude
import           System.Environment       (getArgs)
import qualified Algebra.Additive

 -- Some types and helper functions for dealing with
 -- "vectors" (implemented as arrays of rational numbers).

type Coordinate
  = Ratio Int
```

```
type Vector
  = V.Vector Coordinate

-- Inner product.
inp :: Vector -> Vector -> Coordinate
inp a b = V.sum (V.zipWith (*) a b)

half :: Coordinate
half = 1 % 2

-- Product of scalar with vector.
l :: Coordinate -> Vector -> Vector
l = V.map . (*)

instance Algebra.Additive.C Vector where
  (+) = V.zipWith (+)
  (-) = V.zipWith (-)
  negate = l (-1)
  zero = V.fromList [0,0,0,0,0,0,0,0]

-- Some data regarding E_8

delta :: (Eq a,Algebra.Ring.C b) => a -> a -> b
delta i j = if i == j then 1 else 0

-- 'e i' gives the i'th standard basis vector of R_8.
e :: Int -> Vector
e i = V.fromList $ map (delta i) [1 .. 8]

-- This is the usual integral basis of the lattice E_8.
basis :: [Vector]
basis =
  [
    l half $ (e 1 + e 8) - (sum $ map e [2 .. 7])
  , e 1 + e 2
  ] ++ map (\ i -> e (i - 1) - e (i - 2)) [3 .. 8]

roots :: [Vector]
roots = d8 ++ x118 where
  d8 = concatMap ((\ [a,b] ->
    [a + b,a - b,b - a,negate a - b]) . map e . getSet) $
      enumerate (ksubsets 2) [1 .. 8]
  x118 = map (\ (Prod (Set neg) (Set pos)) ->
    l half $ sum (map (negate . e) neg) + sum (map e pos)) $
      enumerate ((set `ofSize` even) * set) [1 .. 8]
```

```
-- 'posRoots' contains exactly one of every pair
-- (a,-a) of roots.
posRoots :: [Vector]
posRoots = nubBy (\ a b -> a == b || a == negate b) roots

-- Generate elements l of the E_8 lattice with the property
-- that l^2 = 2 d.  We need only one element of each orbit
-- under the action of the Weyl group.  In particular, we
-- may assume that all coordinates but one (say, the first)
-- are nonnegative, and that the successive coordinates are
-- nondecreasing.  We generate exactly one element of each
-- H-orbit, where H is the subgroup of permutations and even
-- sign changes.

gen :: Int -> [Vector]
gen d = genInt d ++ genHalfInt d

genInt :: Int -> [Vector]
genInt d = map (V.fromList . map fromIntegral) $ go [] 0 where
  -- Given the length of a partial vector, compute the maximal
  -- new coordinate which does not increase the length of the
  -- vector beyond 2 d.
  maxCoord :: Int -> Int
  maxCoord s = floor (sqrt (fromIntegral $ dD - s) :: Double)

  dD :: Int
  dD = 2 * d

  -- We maintain a list of coordinates chosen so far, every
  -- one together with the sum of squares of the coordinates
  -- up to and including that coordinate.
  -- The generated vectors are elements of E_8, because the
  -- sum of the squares of their components is even, hence
  -- the sum of the components as well.
  go :: [(Int,Int)] -> Int -> [[Int]]
  -- We have fixed all eight coordinates.
  go fixed@((_,sq) : ps) 8
    -- The vector has the right length; add the relevant
    -- solutions (using 'vary'), and continue searching.
    | sq == dD  = vary (map fst fixed) ++ lower ps 7
    -- The vector has the wrong length, continue searching.
    | otherwise = lower ps 7
  go fixed               n = let
    (m,s) = case fixed of
```

```
      []        -> (maxCoord 0,0)
      (c,s) : _ -> (Prelude.min (maxCoord s) c,s)
   in
    go ((m,s + m ^ 2) : fixed) (n + 1)

  -- Lexicographically decrease the given vector, and continue
  -- the generation from there.
  lower :: [(Int,Int)] -> Int -> [[Int]]
  lower []           _ = []
  lower ((x,s) : ps) n
    | x == 0    = lower ps (n - 1)
    | otherwise = go ((x - 1,s + 1 - 2 * x) : ps) n

  vary :: [Int] -> [[Int]]
  vary (x : xs) = if x == 0
    then [0 : xs]
    else [x : xs,negate x : xs]

-- For vectors with all coordinates half-integers, we work
-- with the doubles of the coordinates.
genHalfInt :: Int -> [Vector]
genHalfInt d = map (V.fromList . map (% 2)) $ go [] 0 where
  maxCoord :: Int -> Int
  maxCoord = Memo.integral m where
    m s = f $ floor (sqrt (fromIntegral $ dE - s) :: Double)
    f k = if odd k then k else k - 1

  dE :: Int
  dE = 8 * d

  go :: [(Int,Int)] -> Int -> [[Int]]
  go fixed@((_,sq) : ps) 8
    | sq == dE  = filter e8 (vary $ map fst fixed)
               ++ lower ps 7
    | otherwise = lower ps 7
  go fixed                   n = let
    (m,s) = case fixed of
      []        -> (maxCoord 0,0)
      (c,s) : _ -> (Prelude.min (maxCoord s) c,s)
   in
    go ((m,s + m ^ 2) : fixed) (n + 1)

  -- Decides whether a given vector is an element of E_8
  e8 :: [Int] -> Bool
  e8 = (== 0) . flip rem 4 . sum
```

```
lower :: [(Int,Int)] -> Int -> [[Int]]
lower []           _ = []
lower ((x,s) : ps) n
  | x == 1    = lower ps (n - 1)
  | otherwise = go ((x - 2,s + 4 - 4 * x) : ps) n

vary :: [Int] -> [[Int]]
vary (x : xs) = [x : xs,negate x : xs]
```

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Arie PETERSON,
Korteweg-de Vries Institute for Mathematics, University of Amsterdam,
P.O. Box 9424, 1090 GE Amsterdam, THE NETHERLANDS
e-mail: `ariep@xs4all.nl`

Gregory SANKARAN,
Department of Mathematical Sciences, University of Bath,
Bath, BA2 7AY, UK
e-mail: `gks@maths.bath.ac.uk`

Valery GRITSENKO,
Université Lille 1, Laboratoire Paul Painlevé,
59655 Villeneuve d'Ascq Cedex, FRANCE
e-mail: valery.gritsenko@math.univ-lille1.fr

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