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DIFFERENTIAL SANDWICH-TYPE RESULTS AND CRITERIA FOR STARLIKENESS

Abstract. We find certain sufficient conditions for starlikeness of normalized analytic functions. We show that the subordination theorem proved here unifies a number of known results and improves certain criteria of starlikeness. Some sandwich-type results regarding starlike functions are also given. Mathematica 7.0 is used to plot the images of the unit disk under certain functions.

1. Introduction

A function f is said to be analytic at a point z in a domain $\mathbb{D}$ if it is differentiable not only at z but also in some neighborhood of the point z. A function f is said to be analytic in a domain $\mathbb{D}$ if it is analytic at each point of $\mathbb{D}$. Let $\mathscr{H}$ be the class of functions analytic in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$ and for $a \in \mathbb{C}$ (the complex plane) and $n \in \mathbb{N}$ (set of natural numbers), let $\mathscr{H}[a,n]$ be the subclass of $\mathscr{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. Let $\mathscr{A}$ be the class of all functions f which are analytic in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$ and normalized by the conditions that $f(0) = f'(0) - 1 = 0$. Thus, $f \in \mathscr{A}$ has a Taylor series

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

A function f is said to be *univalent* in a domain $\mathbb{D}$ in the extended complex plane if and only if it is regular (analytic) in $\mathbb{D}$ except for at most one simple pole, and $f(z_1) \neq f(z_2)$ for $z_1 \neq z_2$ $(z_1, z_2 \in \mathbb{D})$. In this case, the equation $f(z) = w$ has at most one root in $\mathbb{D}$ for any complex number w. Such functions map $\mathbb{D}$ conformally onto a domain in the w-plane. Let $\mathscr{S}$ denote the class of all analytic univalent functions f defined on the unit disk $\mathbb{E}$ which are normalized by the conditions $f(0) = f'(0) - 1 = 0$.

A domain $\mathbb{D}$ in $\mathbb{C}$ is said to be *starlike* (with respect to the origin) if $z = 0 \in \mathbb{D}$ and the linear segment joining 0 to any other point of $\mathbb{D}$ lies entirely in $\mathbb{D}$, i.e. for all $z \in \mathbb{D}$, $\lambda z \in \mathbb{D}$ where $0 \leq \lambda \leq 1$. A function $f \in \mathscr{A}$ is said to be starlike in the open unit disk $\mathbb{E}$ if it is univalent in $\mathbb{E}$ and $f(\mathbb{E})$ is a starlike domain. Denote by $\mathscr{S}^*(\alpha)$, the class of starlike functions of order α which is analytically defined as follows:

$$\mathscr{S}^*(\alpha) = \left\{ f \in \mathscr{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ z \in \mathbb{E} \right\},$$

where α is a real number such that $0 \leq \alpha < 1$. Write $\mathscr{S}^* = \mathscr{S}^*(0)$, the class of univalent starlike functions with respect to the origin.

For two functions f and g analytic in the open unit disk $\mathbb{E}$, we say that f is *subordinate to g in* $\mathbb{E}$ and write as $f \prec g$ if there exists a Schwarz function w analytic in $\mathbb{E}$ with $w(0) = 0$ and $|w(z)| < 1,\ z \in \mathbb{E}$ such that $f(z) = g(w(z)),\ z \in \mathbb{E}$. In case the function g is univalent, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, p be an analytic function in $\mathbb{E}$ such that $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy a *first order differential subordination* if

$$\Phi(p(z), zp'(z); z) \prec h(z),\ \Phi(p(0), 0; 0) = h(0). \tag{1}$$

A univalent function q is called a *dominant* of the differential subordination (1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1), is said to be the *best dominant* of (1).

Let $\Psi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be analytic and univalent in domain $\mathbb{C}^2 \times \mathbb{E}$, h be analytic in $\mathbb{E}$, p be analytic and univalent in $\mathbb{E}$, with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$. Then p is called a *solution* of the first order differential superordination if

$$h(z) \prec \Psi(p(z), zp'(z); z),\ h(0) = \Psi(p(0), 0; 0). \tag{2}$$

An analytic function q is called a *subordinant* of the differential superordination (2), if $q \prec p$ for all p satisfying (2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (2), is said to be the *best subordinant* of (2).

The expressions $\dfrac{zf'(z)}{f(z)}$ and $1 + \dfrac{zf''(z)}{f'(z)}$ play an important role in the theory of univalent functions. Several new classes have been introduced and studied by various researchers by combining these expressions in different manners. For example, in 1973, Miller, Mocanu and Reade [5] investigated the class $\mathscr{M}_\alpha$ (known as the class of α-convex functions) defined by

$$\mathscr{M}_\alpha = \left\{ f \in \mathscr{A} : \Re\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > 0,\ z \in \mathbb{E} \right\},$$

where α is any real number. They proved that the members of $\mathscr{M}_\alpha$ are starlike in $\mathbb{E}$. In 1976, Lewandowski et al. [3] showed that the functions $f \in \mathscr{A}$ which satisfy

$$\Re\left[\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > 0, \quad z \in \mathbb{E},$$

are starlike in $\mathbb{E}$.

Singh et al. [12] proved a generalized criterion for starlikeness whose particular case when $\alpha = 1,\ \beta = \lambda = 0$ improves the above result of Lewandowski et al. [3] by giving the following result.

$$f \in \mathscr{A} \text{ satisfying } \Re\left[\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > -\tfrac{1}{2},\ z \in \mathbb{E} \ \Rightarrow\ f \in \mathscr{S}^*. \tag{3}$$

In 1999, Silverman [11] defined the class $\mathscr{G}_b$ by

$$\mathscr{G}_b = \left\{ f \in \mathscr{A} : \left| \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < b,\ z \in \mathbb{E} \right\},$$

and proved the sharp inclusion $\mathscr{G}_b \subset S^*(2/(1+\sqrt{1+8b})),\ 0 < b \le 1$. The class $\mathscr{G}_b$ was, later on, studied extensively by Tuneski [7, 15]. In 2006, Obradovič et al. [6] proved that

$$f \in \mathscr{A} \text{ satisfying } \left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| < 1,\ z \in \mathbb{E} \ \Rightarrow \ f \in \mathscr{S}^*. \tag{4}$$

In 2007, Tuneski [16] proved the following result:

Let $f \in \mathscr{A}$ and $\dfrac{1+|A|}{3+|A|} \le \alpha \le 1,\ -1 \le B < A \le 1$. Then

$$\frac{1-\alpha+\alpha z f''(z)/f'(z)}{zf'(z)/f(z)} \prec \alpha + (1-2\alpha)\frac{1+Bz}{1+Az} + \frac{\alpha(A-B)z}{(1+Az)^2} \ \Rightarrow \ f \in \mathscr{S}^*[A,B]. \tag{5}$$

In the literature of univalent function theory, a number of criteria of starlikeness have been developed by various researchers by using the combinations of the expressions $\dfrac{zf'(z)}{f(z)}$ and $1+\dfrac{zf''(z)}{f'(z)}$ in different manners, and it has always been a matter of interest for the researchers either to find a new criterion for starlikeness of analytic functions or to generalize or improve certain known ones. With this in mind, we establish here a generalized criterion for starlikeness in terms of differential subordination which unifies a number of known results of starlikeness and gives some new ones. We refer to [6, 7, 8, 9, 10, 13, 15, 14, 16, 17]. We also establish the corresponding superordination theorem which can be applied to obtain corresponding results in superordination form. In addition, we give some sandwich-type results for starlikeness.

To state and prove our main results, we shall use the following definition and lemmas.

DEFINITION 1 ([4, p. 21, Definition 2.2b]). *We denote by Q the set of functions p that are analytic and injective on* $\overline{\mathbb{E}} \setminus \mathbb{B}(p)$, *where*

$$\mathbb{B}(p) = \left\{\zeta \in \partial\mathbb{E} : \lim_{z\to\zeta} p(z) = \infty\right\},$$

and are such that $p'(\zeta) \ne 0$ *for* $\zeta \in \partial\mathbb{E} \setminus \mathbb{B}(p)$.

LEMMA 1 ([4, p. 132, Theorem 3.4h]). *Let q be univalent in* $\mathbb{E}$ *and let* θ *and* ϕ *be analytic in a domain* $\mathbb{D}$ *containing* $q(\mathbb{E})$, *with* $\phi(w) \ne 0$, *when* $w \in q(\mathbb{E})$. *Set* $Q_1(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q_1(z)$ *and suppose that either*

(i) *h is convex, or*

(ii) Q_1 *is starlike.*

In addition, assume that

(iii) $\Re \frac{zh'(z)}{Q_1(z)} > 0,\ z \in \mathbb{E}$.

If p is analytic in $\mathbb{E}$, *with* $p(0) = q(0), p(\mathbb{E}) \subset \mathbb{D}$ *and*

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)],$$

then $p(z) \prec q(z)$ *and q is the best dominant.*

LEMMA 2 ([1]). *Let q be univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q_1(z)$ and suppose that*

(i) *Q_1 is starlike in $\mathbb{E}$, and*

(ii) $\Re\, \frac{\theta'(q(z))}{\phi(q(z))} > 0,\ z \in \mathbb{E}$.

If $p \in \mathscr{H}[q(0),1] \cap Q$, with $p(\mathbb{E}) \subset \mathbb{D}$ and $\theta[p(z)] + zp'(z)\phi[p(z)]$ is univalent in $\mathbb{E}$ and

$$\theta[q(z)] + zq'(z)\phi[q(z)] \prec \theta[p(z)] + zp'(z)\phi[p(z)],$$

then $q(z) \prec p(z)$ and q is the best subordinant.

2. Main results

In what follows, all the powers taken are the principal ones.

THEOREM 1. *Let $q,\ q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that*

(i) $\Re\left[1 + \dfrac{zq''(z)}{q'(z)} + \left(\dfrac{\delta}{\mu} - 2\right)\dfrac{zq'(z)}{q(z)}\right] > 0$, *and*

(ii) $\Re\left[1 + \dfrac{zq''(z)}{q'(z)} + \left(\dfrac{\delta}{\mu} - 2\right)\dfrac{zq'(z)}{q(z)} + \dfrac{\beta\delta}{\alpha\mu}q(z) + \dfrac{\gamma}{\alpha}\left(\dfrac{\delta}{\mu} - 1\right)\right] > 0.$

If $f \in \mathscr{A},\ \dfrac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, satisfies the differential subordination

$$\left(\frac{zf'(z)}{f(z)}\right)^{\delta}\left[\beta + \gamma\frac{f(z)}{zf'(z)} + \alpha\frac{f(z)}{zf'(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right]^{\mu} \prec (q(z))^{\delta}\left(\beta + \frac{\gamma}{q(z)} + \alpha\frac{zq'(z)}{(q(z))^2}\right)^{\mu}, \tag{6}$$

where $\alpha, \beta, \gamma, \delta$ and μ are complex numbers with $\alpha, \mu \neq 0$, then $\dfrac{zf'(z)}{f(z)} \prec q(z)$ and q is the best dominant.

Proof. On writing $p(z) = \dfrac{zf'(z)}{f(z)}$, the subordination (6) can be rewritten as:

$$(p(z))^{\frac{\delta}{\mu}}\left(\beta + \frac{\gamma}{p(z)} + \alpha\frac{zp'(z)}{(p(z))^2}\right) \prec (q(z))^{\frac{\delta}{\mu}}\left(\beta + \frac{\gamma}{q(z)} + \alpha\frac{zq'(z)}{(q(z))^2}\right), \tag{7}$$

Define the functions θ and ϕ as under:

$$\theta(w) = (\beta w + \gamma)w^{\frac{\delta}{\mu}-1} \text{ and } \phi(w) = \alpha w^{\frac{\delta}{\mu}-2}.$$

Obviously, the functions θ and ϕ are analytic in domain $\mathbb{D} = \mathbb{C} \setminus \{0\}$ and $\phi(w) \neq 0,\ w \in \mathbb{D}$. Setting the functions Q_1 and h as follows:

$$Q_1(z) = zq'(z)\phi(q(z)) = \alpha zq'(z)(q(z))^{\frac{\delta}{\mu}-2},$$

and

$$h(z) = \theta(q(z)) + Q_1(z) = (q(z))^{\frac{\delta}{\mu}} \left(\beta + \frac{\gamma}{q(z)} + \alpha \frac{zq'(z)}{(q(z))^2}\right).$$

A little calculation yields

$$\frac{zQ_1'(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\delta}{\mu} - 2\right)\frac{zq'(z)}{q(z)},$$

and

$$\frac{zh'(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\delta}{\mu} - 2\right)\frac{zq'(z)}{q(z)} + \frac{\beta\delta}{\alpha\mu}q(z) + \frac{\gamma}{\alpha}\left(\frac{\delta}{\mu} - 1\right).$$

In view of conditions (i) and (ii), we get

(1) Q_1 is starlike in $\mathbb{E}$ and

(2) $\Re \dfrac{zh'(z)}{Q_1(z)} > 0,\ z \in \mathbb{E}$.

Thus conditions (ii) and (iii) of Lemma 1, are satisfied. In view of (7), we have

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)].$$

Therefore, the proof follows from Lemma 1. □

THEOREM 2. *Let $q,\ q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that*

(i) $\Re\left[1 + \dfrac{zq''(z)}{q'(z)} + \left(\dfrac{\delta}{\mu} - 2\right)\dfrac{zq'(z)}{q(z)}\right] > 0$ *and*

(ii) $\Re\left[\dfrac{\beta\delta}{\alpha\mu}q(z) + \dfrac{\gamma}{\alpha}\left(\dfrac{\delta}{\mu} - 1\right)\right] > 0.$

If $f \in \mathscr{A}$, $\dfrac{zf'(z)}{f(z)} \in \mathscr{H}[q(0),1] \cap Q$ with $\dfrac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, satisfies the differential superordination

$$(8)\qquad \begin{aligned} &(q(z))^{\delta}\left(\beta + \frac{\gamma}{q(z)} + \alpha\frac{zq'(z)}{(q(z))^2}\right)^{\mu} \\ &\prec \left(\frac{zf'(z)}{f(z)}\right)^{\delta}\left[\beta + \gamma\frac{f(z)}{zf'(z)} + \alpha\frac{f(z)}{zf'(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right]^{\mu} = h(z),\end{aligned}$$

where $\alpha, \beta, \gamma, \delta$ and μ are complex numbers with $\alpha, \mu \neq 0$ and h is univalent in $\mathbb{E}$, then $q(z) \prec \dfrac{zf'(z)}{f(z)}$ and q is the best subordinant.

Proof. Setting $p(z) = \dfrac{zf'(z)}{f(z)}$, the superordination (8) can be rewritten as:

$$(9)\qquad (q(z))^{\frac{\delta}{\mu}}\left(\beta + \frac{\gamma}{q(z)} + \alpha\frac{zq'(z)}{(q(z))^2}\right) \prec (p(z))^{\frac{\delta}{\mu}}\left(\beta + \frac{\gamma}{p(z)} + \alpha\frac{zp'(z)}{(p(z))^2}\right),$$

By defining the functions θ, ϕ and Q_1 same as in case of Theorem 1 and observing that

$$\frac{\theta'(q(z))}{\phi(q(z))} = \frac{\beta\delta}{\alpha\mu}q(z) + \frac{\gamma}{\alpha}\left(\frac{\delta}{\mu} - 1\right).$$

The use of Lemma 2 along with (9) completes the proof on the same lines as in case of Theorem 1. $\square$

On combining Theorem 1 and Theorem 2, we obtain the following sandwich-type theorem.

THEOREM 3. *Suppose* $\alpha,\beta,\gamma,\delta$ *and* μ *are complex numbers with* $\alpha,\mu \neq 0$ *and suppose that* q_1, q_2 $(q_1(z) \neq 0, q_2(z) \neq 0,\ z \in \mathbb{E})$ *are univalent functions in* $\mathbb{E}$ *such that* q_1 *satisfies the conditions (i) and (ii) of Theorem 2 and* q_2 *follows the conditions (i) and (ii) of Theorem 1. If* $f \in \mathscr{A}$, $\dfrac{zf'(z)}{f(z)} \in \mathscr{H}[q_1(0),1] \cap Q$ *with* $\dfrac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, *satisfies the sandwich-type condition*

$$\begin{aligned}(q_1(z))^{\delta}&\left(\beta + \frac{\gamma}{q_1(z)} + \alpha\frac{zq_1'(z)}{(q_1(z))^2}\right)^{\mu} \\ &\prec h(z) = \left(\frac{zf'(z)}{f(z)}\right)^{\delta}\left[\beta + \gamma\frac{f(z)}{zf'(z)} + \alpha\frac{f(z)}{zf'(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right]^{\mu} \\ &\prec (q_2(z))^{\delta}\left(\beta + \frac{\gamma}{q_2(z)} + \alpha\frac{zq_2'(z)}{(q_2(z))^2}\right)^{\mu},\end{aligned}$$

where h *is univalent in* $\mathbb{E}$, *then* $q_1(z) \prec \dfrac{zf'(z)}{f(z)} \prec q_2(z)$. *Moreover* q_1 *and* q_2 *are the best subordinant and the best dominant respectively.*

3. Conditions for starlikeness

In this section, we use Theorem 1 to derive certain criteria for starlikeness by specifying the function q and selecting particular values of $\alpha,\beta,\gamma,\delta$ and μ. When we select the dominant $q(z) = \dfrac{1+(1-2\lambda)z}{1-z}$, $0 \leq \lambda < 1$ in Theorem 1, it is easy to check that this dominant satisfies the conditions of Theorem 1 in following particular cases. Consequently, we derive the following results giving starlikeness of order λ. Setting $\alpha = \delta = \mu = 1$ and $\beta = \gamma = 0$ in Theorem 1, we get:

COROLLARY 1. *If* $f \in \mathscr{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, *satisfies*

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec \frac{2(1-\lambda)z}{(1-z)[1+(1-2\lambda)z]}, \quad z \in \mathbb{E},$$

then $f \in \mathscr{S}^*(\lambda),\ 0 \leq \lambda < 1$.

REMARK 1. For $\lambda = 0$, the above corollary gives the following result.
If $f \in \mathscr{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0$, $z \in \mathbb{E}$, satisfies

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec \frac{2z}{1-z^2} = F(z),$$

then $f \in \mathscr{S}^*$. Clearly the function F is the conformal mapping of the unit disk $\mathbb{E}$ with $F(0) = 0$ and

$$F(\mathbb{E}) = \mathbb{C} \setminus \{w \in \mathbb{C} : \Re(w) = 0,\ |\Im(w)| \geq 1\}.$$

On comparing the above result with that of Obradovič et al. [6] stated in (4), we see that our result extends the region of variability of the differential operator

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}$$

to get the conclusion of starlikeness i.e. $f \in \mathscr{S}^*$. According to the result of Obradovič et al. [6], the above differential operator can take values in the open unit disk to ensure that $f \in \mathscr{S}^*$ whereas our result shows that the same differential operator can take values in $\mathbb{C}$ minus the two slits $\{w \in \mathbb{C} : \Re(w) = 0,\ |\Im(w)| \geq 1\}$ to have the same conclusion.

Writing $\beta = \alpha$, $\gamma = 1 - \alpha$, $\delta = 2$ and $\mu = 1$ in Theorem 1, we get:

COROLLARY 2. *Let* α *and* λ *be real numbers such that* $0 < \alpha \leq 1$, $0 \leq \lambda < 1$ *and let* $f \in \mathscr{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0$, $z \in \mathbb{E}$, *satisfy*

$$\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)} + 1\right) \prec (1-\alpha)\frac{1+(1-2\lambda)z}{1-z} + \alpha\frac{[1+(1-2\lambda)z]^2}{(1-z)^2} + \frac{2\alpha(1-\lambda)z}{(1-z)^2}.$$

Then $f \in \mathscr{S}^*(\lambda)$.

REMARK 2. In particular when $\alpha = 1$, $\lambda = 0$ in above corollary, we get:
For $f \in \mathscr{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0$, $z \in \mathbb{E}$,

$$\frac{zf'(z)}{f(z)}\left(\frac{zf''(z)}{f'(z)} + 1\right) \prec \frac{1+z}{1-z} + \frac{2z}{(1-z)^2} = G(z) \quad \Rightarrow \quad f \in \mathscr{S}^*.$$

We claim that the above result extends the region of variability of the differential operator $\dfrac{zf'(z)}{f(z)}\left(\dfrac{zf''(z)}{f'(z)} + 1\right)$ over the result of Singh et al. [12] stated in (3).

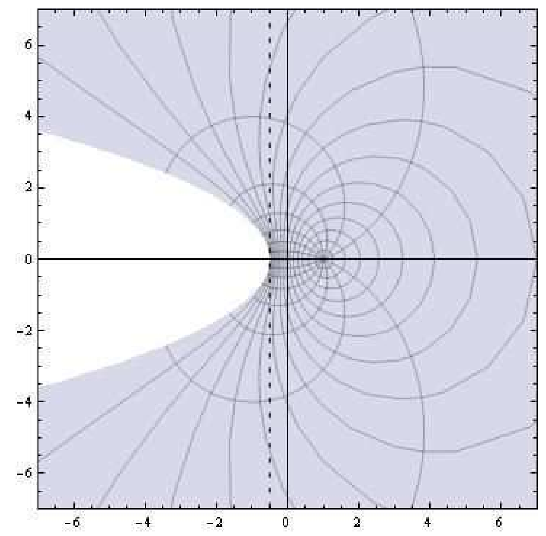

Figure 1

In Figure 1, we plot the dashed line $\Re w = -\frac{1}{2}$ and the image of the unit disk $\mathbb{E}$ under the above function G. According to the result in (3), we see that $f \in \mathscr{S}^*$ if $\frac{zf'(z)}{f(z)}\left(\frac{zf''(z)}{f'(z)}+1\right)$ takes values in the complex plane to the right of the dashed line whereas, according to the above result, the same conclusion remains true even if the operator $\frac{zf'(z)}{f(z)}\left(\frac{zf''(z)}{f'(z)}+1\right)$ takes values in the shaded region left of the dashed line. This justifies our claim.

By taking the dominant $q(z) = \frac{1+z}{1-z}$ in Theorem 1 and on writing $\beta = \delta = 0$, $\gamma = 1-2\alpha$ and $\mu = 1$ in Theorem 1, we get:

COROLLARY 3. *Let* α *be a non-zero complex number with* $\Re\left(\frac{2\alpha-1}{\alpha}\right) \geq 0$. *For all* z *in* $\mathbb{E}$, *if* $f \in \mathscr{A}$, $\frac{zf'(z)}{f(z)} \neq 0$, $z \in \mathbb{E}$, *satisfies*

$$\frac{1-\alpha+\alpha z f''(z)/f'(z)}{zf'(z)/f(z)} \prec \alpha + (1-2\alpha)\frac{1-z}{1+z} + \frac{2\alpha z}{(1+z)^2} \quad \Rightarrow \quad f \in \mathscr{S}^*. \tag{10}$$

REMARK 3. It is worthwhile to compare the result in the above corollary with the result of Tuneski [16] stated in (5) in case where $A = 1$ and $B = -1$. In this particular case (5) and (10) are same, but we notice that the range of α in case of the result of

Tuneski [16] is restricted to $1/2 \le \alpha \le 1$ where as in case of our result (Corollary 3), if we consider α as a real number, then the result holds for $\alpha \in (-\infty, 0) \cup [1/2, \infty)$. Thus, in case of our result, the range of α is extended largely to get the same conclusion .

REMARK 4. For $\alpha = a,\ \beta = a+b,\ \gamma = 0$ and $\delta = \mu = 1$ and dominant $q(z) = \dfrac{1+z}{1-z}$ in Theorem 1, we have

$$\Re\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right)=\Re\left(\frac{1+z^2}{1-z^2}\right)>0$$
$$\Re\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{a+b}{a}q(z)\right)=\Re\left(\frac{1+z^2}{1-z^2}+\frac{a+b}{a}\frac{1+z}{1-z}\right)>0,$$

if $\dfrac{2a+b}{a} > 0$, because at $z = 0$, $\dfrac{1+z^2}{1-z^2}+\dfrac{a+b}{a}\dfrac{1+z}{1-z} = \dfrac{2a+b}{a}$. In view of this remark, we obtain the following result.

COROLLARY 4. *Suppose a, b are real numbers such that $a \neq 0,\ \dfrac{2a+b}{a} > 0$. If $f \in \mathscr{A},\ \dfrac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, satisfies*

$$a\left(1+\frac{zf''(z)}{f'(z)}\right)+b\frac{zf'(z)}{f(z)} \prec (a+b)\frac{1+z}{1-z}+\frac{2az}{1-z^2}=H(z),$$

then $\dfrac{zf'(z)}{f(z)} \prec \dfrac{1+z}{1-z}$, i.e. $f \in \mathscr{S}^*$.

The above corollary, in turn, gives the following result.

COROLLARY 5. *Let $a\ (\neq 0)$ and b be real numbers such that either $a > 0$, $3a+2b \ge 0$, or $a < 0$, $3a+2b \le 0$, and let $f \in \mathscr{A},\ \dfrac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$a\left(1+\frac{zf''(z)}{f'(z)}\right)+b\frac{zf'(z)}{f(z)} \prec H(z).$$

Then $f \in \mathscr{S}^*$, where H is the conformal mapping of the unit disk $\mathbb{E}$ with $H(0) = a+b$ and

$$H(\mathbb{E}) = \mathbb{C} \setminus \left\{w \in \mathbb{C} : \Re\, w = 0,\ |\Im\, w| \ge \sqrt{a(3a+2b)}\right\}.$$

REMARK 5. It is worthwhile to compare the result in the above corollary with the main result (Theorem 2.4(ii)) of Tuneski [17]. We note that the result of Tuneski [17] is proved for $a > 0$ whereas our result in above corollary also holds for $a < 0$.

REMARK 6. By specifying the function q and selecting the particular values of $\alpha,\ \beta,\ \gamma,\ \delta$ and μ, we can derive a number of known results. Some of them are given below.

1. Considering $\alpha=\beta=\mu=1,\ \gamma=-1,\ \delta=0$ and $q(z)=\dfrac{1+z}{1-z}$ in Theorem 1, we obtain the result of Bulboacă and Tuneski [2].
2. Selecting $\alpha=\mu=1,\ \beta=\gamma=\delta=0$ and $q(z)=\dfrac{1+z}{1-z}$ in Theorem 1, we obtain Theorem 3 of Obradovič and Tuneski [7].
3. Letting $\beta=\alpha,\ \gamma=1-\alpha,\ \delta=2$ and $\mu=1$ in Theorem 1, we get the main result of Ravichamdran [8].
4. On writing $\alpha=\beta=\mu=1,\ \gamma=-1,\ \delta=0$ and $q(z)=\dfrac{1-z}{1+z}$ in Theorem 1, we get Theorem 1 of Tuneski [13].
5. Taking $\alpha=\mu=1,\ \beta=\gamma=\delta=0$ and $q(z)=\dfrac{1+Az}{1+Bz},\ -1\leq B<A\leq 1$, in Theorem 1, we obtain the main result of Tuneski [15].
6. Setting $\alpha=\mu=1,\ \beta=\delta=0$ and replacing γ by $-\gamma$ and $q(z)=\dfrac{1-z}{1+z}$ in Theorem 1, we get the main result (Theorem 1) of Tuneski [14].

REMARK 7. Similar to the above remark, Theorem 2 can be applied to obtain some results in superordination form giving the best subordinant of $\dfrac{zf'(z)}{f(z)}$.

4. Sandwich-type results

In this section, we apply Theorem 3 to find certain sandwich-type results which give the best subordinant and the best dominant for the starlike operator $\dfrac{zf'(z)}{f(z)}$. By selecting the subordinant $q_1(z)=1+az$ and the dominant $q_2(z)=1+bz,\ 0<a<b$, in Theorem 3, we deduce, below, some criteria for starlikeness.

Writing $\alpha=\beta=\mu=1,\ \gamma=-1$ and $\delta=2$ in Theorem 3, we get:

COROLLARY 6. *Let $a,\ b$ be real numbers for which $0<a<b<\frac{1}{2}$. If $f\in\mathscr{A}$ is such that $\dfrac{zf'(z)}{f(z)}\in\mathscr{H}[1,1]\cap Q$, with $\dfrac{zf'(z)}{f(z)}\neq 0$ and $\dfrac{z^2f''(z)}{f(z)}$ is univalent in $\mathbb{E}$. Then*

$$2az+a^2z^2\prec\frac{z^2f''(z)}{f(z)}\prec 2bz+b^2z^2$$

implies

$$1+az\prec\frac{zf'(z)}{f(z)}\prec 1+bz,\quad z\in\mathbb{E}.$$

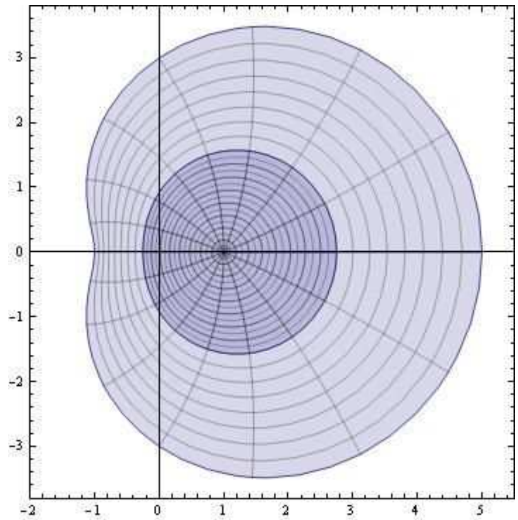

Figure 2

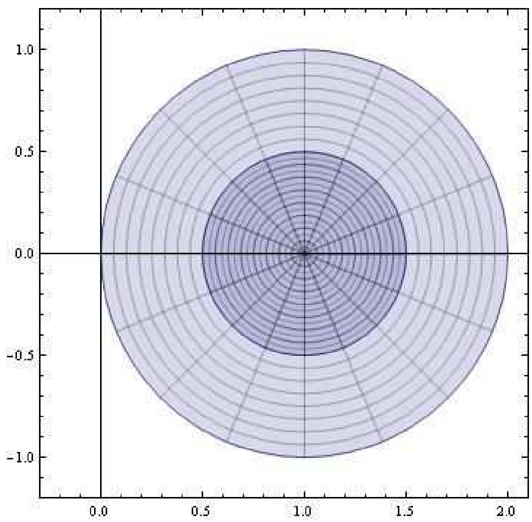

Figure 3

EXAMPLE 1. Taking $a = 1/4,\ b = 47/100$, and f to be the same as in the above corollary, we obtain:

$$\frac{1}{2}z + \frac{1}{16}z^2 \prec \frac{z^2 f''(z)}{f(z)} \prec \frac{47}{50}z + \frac{2209}{10000}z^2,\ z \in \mathbb{E},$$

$$\Rightarrow\ 1 + \frac{1}{4}z \prec \frac{zf'(z)}{f(z)} \prec 1 + \frac{47}{100}z.$$

Writing $\beta = \alpha,\ \gamma = 1 - \alpha,\ \delta = 2$ and $\mu = 1$ in Theorem 3, we get:

COROLLARY 7. *Let* $\alpha > 0$ *and* $a,\ b,\ (a < b)$, *be real numbers for which* $0 < a < \dfrac{1}{2} + \dfrac{1}{2\alpha}$ *and* $0 < b < 1 + \dfrac{1}{2\alpha}$. *If* $f \in \mathscr{A}$ *is such that* $\dfrac{zf'(z)}{f(z)} \in \mathscr{H}[1,1] \cap Q$, *with* $\dfrac{zf'(z)}{f(z)} \neq 0$ *and* $\dfrac{zf'(z)}{f(z)}\left(1 + \alpha\dfrac{zf''(z)}{f'(z)}\right)$ *is univalent in* $\mathbb{E}$ *and in addition* f *satisfies*

$$1 + (1 + 2\alpha)az + \alpha a^2 z^2 \prec \frac{zf'(z)}{f(z)}\left(1 + \alpha\frac{zf''(z)}{f'(z)}\right) \prec 1 + (1 + 2\alpha)bz + \alpha b^2 z^2,$$

then

$$1 + az \prec \frac{zf'(z)}{f(z)} \prec 1 + bz.$$

EXAMPLE 2. Writing $\alpha = 1,\ a = 1/2,\ b = 1$ and f same as in above corollary, we get that

$$1 + \frac{3}{2}z + \frac{1}{4}z^2 \prec \frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec 1 + 3z + z^2 \tag{11}$$

implies

$$1 + \frac{1}{2}z \prec \frac{zf'(z)}{f(z)} \prec 1 + z, \quad z \in \mathbb{E}. \tag{12}$$

Mathematica 7.0 is used to plot the images of the unit disk under the functions on the left-hand side and on the right-hand side (11) in Figure 2 and those of (12) in Figure 3, respectively. It follows that if $\dfrac{zf'(z)}{f(z)}\left(1 + \dfrac{zf''(z)}{f'(z)}\right)$ takes values in the light shaded portion of Figure 2, then $\dfrac{zf'(z)}{f(z)}$ will take values in the light shaded portion of Figure 3. Hence f is starlike in $\mathbb{E}$.

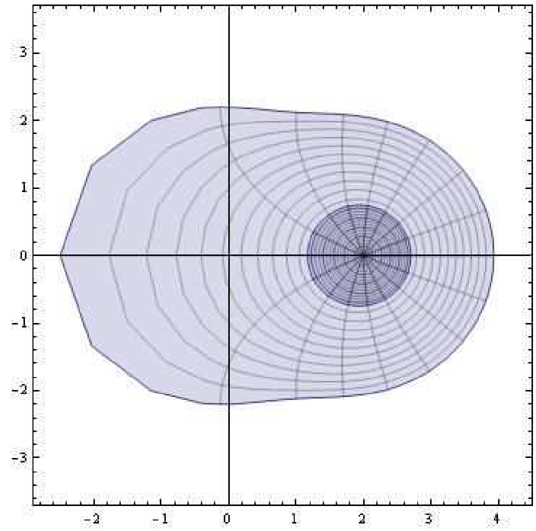

Figure 4

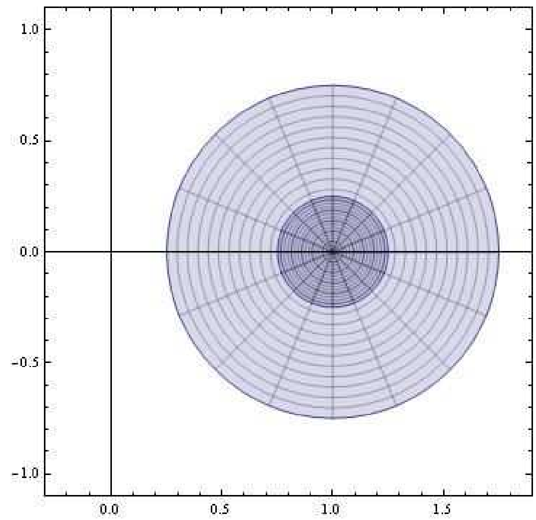

Figure 5

Writing $\alpha=\delta=\mu=1,\ \beta=2$ and $\gamma=0$ in Theorem 3, we get:

COROLLARY 8. *Let $a,\ b$ be real numbers for which $0<a<b<1$. If $f\in\mathscr{A}$ is such that $\frac{zf'(z)}{f(z)}\in\mathscr{H}[1,1]\cap Q$, with $\frac{zf'(z)}{f(z)}\neq 0$ and $1+\frac{zf''(z)}{f'(z)}+\frac{zf'(z)}{f(z)}$ is univalent in $\mathbb{E}$, and satisfies*

$$2+2az+\frac{az}{1+az}\prec 1+\frac{zf''(z)}{f'(z)}+\frac{zf'(z)}{f(z)}\prec 2+2bz+\frac{bz}{1+bz},$$

then $1+az\prec\frac{zf'(z)}{f(z)}\prec 1+bz$.

EXAMPLE 3. For $a=1/4,\ b=3/4$ and f same as in above corollary, we obtain:

$$2+\frac{1}{2}z+\frac{z}{4+z}\prec 1+\frac{zf''(z)}{f'(z)}+\frac{zf'(z)}{f(z)}\prec 2+\frac{3}{4}z+\frac{3z}{4+3z}, \tag{13}$$

then

$$1+\frac{1}{4}z\prec\frac{zf'(z)}{f(z)}\prec 1+\frac{3}{4}z. \tag{14}$$

Using Mathematica 7.0, we plot the images of the unit disk under the functions on the left-hand side and on the right-hand side (13) in Figure 4 and those of (14) in Figure 5, respectively. It follows that if $1+\frac{zf''(z)}{f'(z)}+\frac{zf'(z)}{f(z)}$ takes values in the light shaded portion of Figure 4, then $\frac{zf'(z)}{f(z)}$ will take values in the light shaded portion of Figure 5. Hence f is starlike in $\mathbb{E}$.

References

[1] BULBOACĂ T. Classes of first-order differential superordinations. *Demonstratio Math. 35*, 2 (2002), 287–292.

[2] BULBOACĂ T. AND TUNESKI N. New criteria for starlikeness and strongly starlikeness. *Mathematica 43(66)*, 1 (2001), 11–22. International Conference on Complex Analysis and the 8th Romanian-Finnish Seminar (Iaşi, 1999).

[3] LEWANDOWSKI Z., MILLER S. S. AND ZLOTKIEWICZ E. Generating functions for some classes of univalent functions. *Proc. Amer. Math. Soc. 56* (1976), 111–117.

[4] MILLER S. S. AND MOCANU P. T. *Differential Suordinations: Theory and Applications*. Monographs and Textbooks in Pure and Applied Mathematics, 225. Marcel Dekker, New York, 2000.

[5] MILLER S. S., MOCANU P. T. AND READE M. O. All α-convex functions are univalent and starlike. *Proc. Amer. Math. Soc. 37* (1973), 553–554.

[6] OBRADOVIČ M., PONNUSAMY S., SINGH V. AND VASUNDHRA P. Differential inequalities and criteria for starlike and univalent functions. *Rocky Mountain J. Math. 36*, 1 (2006), 303–317.

[7] OBRADOVIČ M. AND TUNESKI N. On the starlike criteria defined by silverman. *Zeszyty Nauk. Politech. Rzeszowskiej. Mat. 181*, 24 (2000), 59–64.

[8] RAVICHANDRAN V. Certain applications of first order differential subordination. *Far East J. Math. Sci. 12*, 1 (2004), 41–51.

[9] RAVICHANDRAN V., MAGESH N. AND RAJALAKSHMI R. On certain applications of differential subordinations for ϕ-like functions. *Tamkang J. Math. 36*, 2 (2005), 137–142.

[10] ROSIHAN M. A., RAVICHANDRAN V., KHAN M. H. AND SUBRAMANIAN K. G. Differential sandwich theorems for certain analytic functions. *Far East J. Math. Sci. 15*, 1 (2004), 87–94.

[11] SILVERMAN H. Convex and starlike criteria. *Int. J. Math. Math. Sci. 22*, 1 (1999), 75–79.

[12] SINGH S., GUPTA S. AND SINGH S. Starlikeness of analytic maps satisfying a differential inequality. *General Mathematics 18*, 3 (2010), 51–58.

[13] TUNESKI N. On certain sufficient conditions for starlikeness. *Int. J. Math. Math. Sci. 23*, 8 (2000), 521–527.

[14] TUNESKI N. On a criteria for starlikeness of analytic functions. *Integral Transforms Spec. Funct. 14*, 3 (2003), 263–270.

[15] TUNESKI N. On the quotient of the representations of convexity and starlikeness. *Math. Nachr. 248-249*, 1 (2003), 200–203.

[16] TUNESKI N. Some results on starlike and convex functions. *Applicable Analysis and Discrete Math. 1* (2007), 293–298.

[17] TUNESKI N. On starlikeness of an analytic function. *Southeast Asian Bull. Math. 34*, 2 (2010), 365–370.

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