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TWO LECTURES ABOUT MUMFORD–TATE GROUPS

Abstract. We define Hodge structures as representations, and introduce the Mumford–Tate group $MT(H)$ of a rational Hodge structure. We give a characterization of $MT(H)$ based on properties of reductive groups, and study it for elliptic curves and their powers. We define what it means for a Hodge structure to arise from an abelian variety (as is the case for K3 surfaces), and then focus on cases in which this is not true. The presentation includes a selection of exercises.

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Introduction

These notes correspond, more or less, to two lectures given during the school on Hodge theory that took place at the CIRM in Trento, Italy, from 31 August to 4 September 2009, for which I filled the role of a teaching assistant. Claire Voisin and Eduard Looijenga had both suggested that I review the basic theory of Mumford–Tate groups, together with one interesting application: Deligne's proof that the cohomology of a general hypersurface in $\mathbb{P}^3$ of degree at least 5 cannot be expressed in terms of abelian varieties.

The first lecture is mostly based on notes by Ben Moonen [4]; the second on Deligne's paper [1]. I have also included a set of exercises handed out during the school, and the solution to the one describing the Kuga–Satake construction, because of its relevance to the second lecture. I am very grateful to the organizers for the chance to participate.

1. First lecture

The object of this lecture is to introduce the Mumford–Tate group of a Hodge structure; this is an algebraic subgroup $MT(H)$ of $GL(H)$, naturally associated to any rational Hodge structure H. Roughly speaking, the knowledge of the Mumford–Tate group of H is the same as knowing all $(0,0)$-Hodge classes in any Hodge structure obtained from H by direct sums, duals, or tensor products. Of course, this means that finding the Mumford–Tate group of an arbitrary Hodge structure is impossible; on the other hand, if we do know $MT(H)$, we have a good chance of saying something nontrivial about Hodge classes. For example, the Hodge conjecture for arbitrary powers of an elliptic curve can be proved in this way.

1.1. Hodge structures as representations

Before defining the Mumford–Tate group, we briefly recall Deligne's way of viewing Hodge structures as representations of a certain algebraic group $\mathbb{S}$. To begin with, we can embed $\mathbb{C}^*$ into $GL_2(\mathbb{R})$ by the group homomorphism

$$\mathbb{C}^* \to GL_2(\mathbb{R}), \qquad a + ib \mapsto s(a,b) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

The image is the set of real points of an algebraic subgroup $\mathbb{S} \subseteq GL_2$. As with any scheme, we can describe an algebraic group by giving its set of points over arbitrary commutative rings A; in the case at hand, $\mathbb{S}(A)$ consists of those invertible matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A)$$

that satisfy $a - d = b + c = 0$. Then $\mathbb{S}(\mathbb{R})$ is isomorphic to $\mathbb{C}^*$ via the map above. Clearly, $\mathbb{S}$ is an algebraic subgroup of GL_2, defined over $\mathbb{Q}$, and abelian. We have $\mathbb{R}^* \hookrightarrow \mathbb{S}(\mathbb{R})$, embedded as the subgroup of scalar matrices.

Now let $H_\mathbb{R}$ be a finite-dimensional $\mathbb{R}$-vector space. A Hodge structure on $H_\mathbb{R}$ is a decomposition

$$H_\mathbb{C} = \mathbb{C} \otimes_\mathbb{R} H_\mathbb{R} = \bigoplus_{p,q} H^{p,q}$$

with $H^{q,p} = \overline{H^{p,q}}$. Here complex conjugation is defined via $\overline{\lambda \otimes h} = \bar{\lambda} \otimes h$ for $h \in H_\mathbb{R}$ and $\lambda \in \mathbb{C}$. The Hodge structure determines a representation

$$\rho \colon \mathbb{S}(\mathbb{R}) \to GL(H_\mathbb{R}),$$

by letting $z = a + ib$ act on $H^{p,q}$ as multiplication by $z^p \bar{z}^q = (a+ib)^p(a-ib)^q$. Since ρ is given by polynomials, this is an algebraic representation, and we can easily check that it is defined over $\mathbb{R}$. Note that the scalar matrix $a\,\mathrm{id} \in \mathbb{S}(\mathbb{R})$ acts as multiplication by a^{p+q} on $H^{p,q}$; if H is pure of weight k, then $\rho(a\,\mathrm{id}) = a^k\,\mathrm{id}$, which lets us read off the weight of the Hodge structure from the representation. Conversely, one can show that any algebraic representation of $\mathbb{S}$ that is defined over $\mathbb{R}$ determines an $\mathbb{R}$-Hodge structure.

LEMMA 1. *Let* $\rho\colon \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(H_{\mathbb{R}})$ *be an algebraic representation; then* ρ *comes from a Hodge structure as above. More precisely, the summands of the Hodge decomposition are given as eigenspaces*

$$H^{p,q} = \bigcap_{a^2+b^2 \neq 0} \ker\Big(\rho\big(s(a,b)\big) - (a+ib)^p(a-ib)^q\,\mathrm{id}\Big).$$

To get a Hodge structure defined over $\mathbb{Q}$, one should require in addition that $H_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Q}} H_{\mathbb{Q}}$ for a $\mathbb{Q}$-vector space $H_{\mathbb{Q}}$. A point to be careful about is that, nevertheless, the representation ρ is only defined over $\mathbb{R}$.

All the usual operations on Hodge structures, such as direct sums, tensor products, Hom, etc., can be performed in terms of the representations; one simply uses the standard definitions from representation theory. We also note that the Weil operator of the Hodge structure H is $\rho\big(s(0,1)\big)$; the reason is that the element $i \in \mathbb{C}^*$ is represented by the matrix

$$s(0,1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which, by definition, acts on the space $H^{p,q}$ as multiplication by i^{p-q}.

1.2. Mumford–Tate groups

We can now define the main object of these two lectures (I am following Ben Moonen's notes on Mumford–Tate groups [4] in this portion of the course).

DEFINITION 1. *Let* H *be a rational Hodge structure, and* $\rho\colon \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(H_{\mathbb{R}})$ *the corresponding algebraic representation. The* Mumford–Tate group *of* H *is the smallest algebraic subgroup of* $\mathrm{GL}(H)$, *defined over* $\mathbb{Q}$, *whose set of real points contains the image of* ρ. *It is usually denoted by* $\mathrm{MT}(H)$.

By definition, the Mumford–Tate group comes with a morphism of algebraic groups $\rho\colon \mathbb{S} \to \mathrm{MT}(H)$, defined over $\mathbb{R}$. In particular, any rational vector space V with an algebraic representation $\mathrm{MT}(H) \to \mathrm{GL}(V)$ naturally acquires a Hodge structure. For representations that are derived from the standard representation of $\mathrm{GL}(H)$ on H by operations such as direct sum or tensor product, the induced Hodge structure is of course the same as the usual one.

The main reason for introducing the Mumford–Tate group is its relationship with Hodge classes in Hodge structures derived from H. To make this precise, we

define for any pair of multi-indices $d, e \in \mathbb{N}^n$ the tensor space

$$T^{d,e}(H) = \bigoplus_{j=1}^{n} H^{\otimes d_j} \otimes (H^{\vee})^{\otimes e_j}.$$

It has a natural action by $GL(H)$, and therefore also by $MT(H)$; note that $T^{d,e}(H)$ is a direct sum of pure Hodge structures of weight $(d_j - e_j)k$, respectively, where k still denotes the weight of the pure Hodge structure H.

PROPOSITION 1. *Let $V \subseteq T^{d,e}(H)$ be any rational subspace. Then V is a sub-Hodge structure if and only if it is stable under the action of $MT(H)$. Similarly, a rational vector $t \in T^{d,e}(H)$ is a $(0,0)$-Hodge class if and only if it is invariant under $MT(H)$.*

Proof. If V is stable under the action of $MT(H)$, it becomes a representation of $MT(H)$, and therefore a $\mathbb{Q}$-Hodge structure. Conversely, suppose that $V \subseteq T^{d,e}(H)$ is a sub-Hodge structure defined over $\mathbb{Q}$. We can then look at the subgroup $G_V \subseteq GL(H)$ of those elements that preserve V. Clearly, this is an algebraic subgroup defined over $\mathbb{Q}$; moreover, its set of real points contains the image of ρ because V is a sub-Hodge structure. By definition, $MT(H) \subseteq G_V$, proving that V is preserved by $MT(H)$.

For the second assertion, suppose that t is invariant under $MT(H)$. Then t is also invariant under the induced action by $\mathbb{S}(\mathbb{R})$, and therefore has to be a Hodge class of type $(0,0)$. Conversely, let t be a $(0,0)$-Hodge class. As before, the stabilizer of t in $GL(H)$ is an algebraic subgroup, defined over $\mathbb{Q}$, and containing the image of ρ; consequently, it contains the entire Mumford–Tate group. $\qquad\square$

The morphisms of Hodge structure from H to itself are exactly the $(0,0)$-Hodge classes in $End(H)$; by Proposition 1,

(1) $$\mathrm{Hom}_{\mathbb{Q}\text{-HS}}(H, H) \simeq \mathrm{End}(H)^{MT(H)}.$$

Let us now look at some of the basic properties of Mumford–Tate groups.

PROPOSITION 2. *Let H be a Hodge structure of weight k, and $MT(H)$ its Mumford–Tate group.*

  (i) *If $k \neq 0$, then $MT(H)$ contains the center; if $k = 0$, then $MT(H) \subseteq SL(H)$.*

 (ii) *$MT(H)$ is always a connected group.*

(iii) *If H is polarizable, then $MT(H)$ is a reductive group.*

Proof. The first two assertions are simple consequences of the definition of $MT(H)$. To prove the third, we use the following criterion for being reductive: A connected group G over a field of characteristic zero is reductive iff it has a faithful and completely reducible representation. In the case of the Mumford–Tate group, this representation is

the tautological representation $\mathrm{MT}(\mathsf{H}) \hookrightarrow \mathrm{GL}(\mathsf{H})$. Its sub-representations are exactly the sub-Hodge structures of H; complete reducibility follows from the fact that the category of polarizable $\mathbb{Q}$-Hodge structures is semisimple. $\qquad\square$

The next lemma shows the relationship between the Mumford–Tate group of a direct sum of two Hodge structures and those of the two summands. (A similar, but slightly weaker result is true for tensor products.)

LEMMA 2. *Let H_1 and H_2 be two Hodge structures. Under the natural map* $\mathrm{GL}(\mathsf{H}_1) \times \mathrm{GL}(\mathsf{H}_2) \hookrightarrow \mathrm{GL}(\mathsf{H}_1 \oplus \mathsf{H}_2)$, *we have* $\mathrm{MT}(\mathsf{H}_1 \oplus \mathsf{H}_2) \subseteq \mathrm{MT}(\mathsf{H}_1) \times \mathrm{MT}(\mathsf{H}_2)$. *Moreover, the projection to either factor is surjective.*

Proof. This follows easily from the definition. $\qquad\square$

An alternative definition of the Mumford–Tate group is as the subgroup of $\mathrm{GL}(\mathsf{H})$ that fixes every $(0,0)$-Hodge class in any tensor space $\mathsf{T}^{d,e}(\mathsf{H})$. The two definitions agree because of the following fundamental result.

PROPOSITION 3. *Let $\mathsf{G} \subseteq \mathrm{GL}(\mathsf{H})$ be the subgroup of elements that fix every* $(0,0)$-*Hodge class in every tensor space* $\mathsf{T}^{d,e}(\mathsf{H})$. *Then* $\mathsf{G} = \mathrm{MT}(\mathsf{H})$.

Proof. By Proposition 1, we have at least the inclusion $\mathrm{MT}(\mathsf{H}) \subseteq \mathsf{G}$. The converse is a general fact about reductive groups, and is proved as follows (details can be found in 1.3 below). First, one shows by a rather formal argument that $\mathrm{MT}(\mathsf{H})$ is the stabilizer of a one-dimensional subspace L contained in one of the representations $\mathsf{T} = \mathsf{T}^{d,e}(\mathsf{H})$. Since $\mathrm{MT}(\mathsf{H})$ is reductive by Proposition 2, there is a decomposition $\mathsf{T} = \mathsf{L} \oplus \mathsf{T}'$ as representations, and $\mathrm{MT}(\mathsf{H})$ is the stabilizer of a generator of $\mathsf{L} \otimes \mathsf{L}^{\vee}$ inside $\mathsf{T} \otimes \mathsf{T}^{\vee}$. As we have seen in Proposition 1, such a generator has to be a Hodge class of type $(0,0)$, and therefore $\mathsf{G} \subseteq \mathrm{MT}(\mathsf{H})$. $\qquad\square$

1.3. A result about reductive groups

The alternative characterization of the Mumford–Tate group in Proposition 3 depends on a more general result about reductive groups. For the sake of completeness, we reproduce Deligne's proof from [3].

So let G be a reductive algebraic group, defined over a field k of characteristic zero. Accordingly, there are finite-dimensional representations $V_1, \ldots, V_n$ of G such that $V_1 \oplus \cdots \oplus V_n$ is faithful; in other words, such that the map

$$\mathsf{G} \to \prod_{i=1}^{n} \mathrm{GL}(V_i)$$

is injective. For any pair of multi-indices $d, e \in \mathbb{N}^n$, we again define the tensor space

$$\mathsf{T}^{d,e} = \bigoplus_{i=1}^{n} V_i^{\otimes d_i} \otimes (V_i^{\vee})^{\otimes e_i},$$

which is naturally a representation of G. Given any subgroup $H \subseteq G$, we let H' be the subgroup of G fixing all tensors (in any $T^{d,e}$) that are fixed by H. A priori, this is a bigger group than H; the question is whether $H' = H$.

PROPOSITION 4. *If* H *is itself a reductive group, then* $H' = H$.

Proof. According to Chevalley's theorem (see Lemma 4 below), H is the stabilizer of a line L in some representation V; by Lemma 3, we may furthermore assume that V is a direct sum of representations $T^{d,e}$. Since H is reductive, there exists a decomposition $V = L \oplus W$, with W another representation of H; it is then easy to see that H is the stabilizer of any generator of $L \otimes L^{\vee}$ inside $V \otimes V^{\vee}$. Since such a generator is a direct sum of tensors in various $T^{d,e}$, it follows that $H' \subseteq H$, and therefore $H' = H$. □

LEMMA 3. *Any finite-dimensional representation of* G *is contained in a direct sum of representations* $T^{d,e}$.

Proof. Let $k[G]$ be the ring of regular functions on the algebraic group G; as a representation of G, it is called the regular representation (and $(g \cdot f)(x) = f(g^{-1}x)$ for $f \in k[G]$ and $g \in G$). Given any representation W of G, let W_0 be the trivial representation with the same underlying vector space as W; thus $gw = w$ for any $w \in W_0$ and any $g \in G$. The multiplication map $G \times W \to W$ determines a G-equivariant embedding $W \hookrightarrow W_0 \otimes_k k[G]$, as can be seen by taking a basis; since $W_0 \otimes_k k[G]$ is isomorphic to a direct sum of copies of $k[G]$, it suffices to prove the lemma for the regular representation (which, of course, is not finite-dimensional).

Now let $V = V_1 \oplus \cdots \oplus V_n$ be the faithful representation of G from above, for which $G \to GL(V)$ is injective. The map $GL(V) \to End(V) \times End(V^{\vee})$ is a closed embedding, and therefore both $GL(V)$ and G are closed subvarieties of the product $End(V) \times End(V^{\vee})$. In general, the ring of regular functions on a vector space W is the symmetric algebra $Sym W^{\vee}$; from the closed embedding above, it follows that there is a G-equivariant surjection

$$Sym End(V^{\vee}) \otimes_k Sym End(V) \to k[G].$$

But now G is reductive, and so this surjection splits; therefore $k[G]$ is isomorphic to a direct summand of the left-hand side, which in turn is contained in an infinite sum of representations $T^{d,e}$. Being finite-dimensional, the original representation W then injects into the direct sum of finitely many of them. □

LEMMA 4. *Any subgroup* H *of* G *is the stabilizer of a line* L *in some finite-dimensional representation of* G.

Proof. Let $I_H \subseteq k[G]$ be the ideal of functions that vanish on H. In the regular representation, therefore, H is the stabilizer of I_H. Next, we claim that there is a finite-dimensional subspace $W \subseteq k[G]$ that is stable under G and contains a set of generators for the ideal I_H. In fact, since I_H is finitely generated, it suffices to show that the subspace $W(f)$ spanned by any $f \in k[G]$ and its translates under G has finite dimension. To see that this is the case, consider the multiplication map $\mu \colon G \times G \to G$,

and the induced map on regular functions $\mu^*\colon k[G] \to k[G] \otimes_k k[G]$. We can write $\mu^* f = \sum_{j=1}^m h_j \otimes f_j$ with regular functions $f_1, \ldots, f_m, h_1, \ldots, h_m$. For any $g, x \in G$, we then have

$$(g \cdot f)(x) = f(g^{-1}x) = (\mu^* f)(g^{-1}, x) = \sum_{j=1}^m h_j(g^{-1})f_j(x).$$

The formula shows that $W(f)$ is contained in the span of $f_1, \ldots, f_m$, and is therefore finite-dimensional.

Taking W as above, H is now the stabilizer of the subspace $W \cap I_H$ of W. Letting $d = \dim W \cap I_H$, we see that H is also the stabilizer of the one-dimensional subspace $(W \cap I_H)^{\wedge d}$ in $W^{\wedge d}$, proving the lemma. □

1.4. Examples of Mumford–Tate groups

First, consider the Hodge structure $\mathbb{Q}(n)$. Since $GL_1 = \mathbb{G}_m$ is the multiplicative group, we see directly from the definition that

$$MT\big(\mathbb{Q}(n)\big) = \begin{cases} \mathbb{G}_m & \text{if } n \neq 0, \\ 1 & \text{if } n = 0. \end{cases}$$

Next, we look at the Mumford–Tate group of elliptic curves. Let E be an elliptic curve, and $H_{\mathbb{Q}} = H^1(E, \mathbb{Q})$ the Hodge structure on its first cohomology. Let $D = \mathrm{End}(E) \otimes \mathbb{Q}$ be the rational endomorphism algebra of the elliptic curve; here $\mathrm{End}(E)$ consists of all morphisms $\phi\colon E \to E$ that fix the unit element of the group law. It is known that D is either $\mathbb{Q}$, or an imaginary quadratic field extension of $\mathbb{Q}$. In the second case, the curve E is said to have complex multiplication. For the elliptic curve

$$E_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau),$$

this happens precisely when τ belongs to an imaginary quadratic field; then $D \simeq \mathbb{Q}(\tau)$, and $\mathrm{End}(E_\tau)$ is isomorphic to an order in the ring of integers of D.

By Proposition 1 and (1), we know that

$$D \simeq \mathrm{End}(H_{\mathbb{Q}})^{MT(H)},$$

because D is naturally isomorphic to the space of $(0,0)$-Hodge classes in $\mathrm{End}(H)$ (due to the fact that E is an abelian variety). This means that $MT(H)$ has to be contained in the subgroup of D-linear automorphisms of H. That condition places enough restrictions on $MT(H)$ to let us determine the Mumford–Tate group.

There are two cases. First, let us consider the case $D \simeq \mathbb{Q}$. We know from Proposition 2 that $MT(H)$ is a connected and reductive subgroup of $GL(H) \simeq GL_2$ that contains the scalar matrices. Now the only connected reductive groups with $\mathbb{G}_m \cdot \mathrm{id} \subseteq G \subseteq GL_2$ are GL_2 itself, $\mathbb{G}_m \cdot \mathrm{id}$, or maximal tori of GL_2. The second and third possibility are ruled out, because the set of invariants in $\mathrm{End}(\mathbb{Q}^2)$ is bigger than just $\mathbb{Q} \cdot \mathrm{id}$. It follows that we have $MT(H) = GL(H)$.

Next, consider the case where the curve E has complex multiplication; then D is an imaginary quadratic field. In this case, $H_{\mathbb{Q}}$ is a free module of rank 1 over D. Since $MT(H)$ has to consist of D-linear automorphisms of H, it follows that we have $MT(H) \subseteq T_D$, where T_D is the algebraic torus whose set of points over any ring A is $T_D(A) = (A \otimes_{\mathbb{Q}} D)^*$. Thus we have $\mathbb{G}_m \cdot \mathrm{id} \subseteq MT(H) \subseteq T_D$. The possibility that $MT(H) = \mathbb{G}_m \cdot \mathrm{id}$ is again ruled out because the set of invariants in $\mathrm{End}(H_{\mathbb{Q}})$ would be too big; consequently, $MT(H) \simeq T_D$.

Once we know the Mumford–Tate group, we know in principle (because of Proposition 1) what all the $(0,0)$-Hodge classes in any tensor space $T^{d,e}(H)$ are. A nice application of our computation above is the following.

PROPOSITION 5. *Let E be an elliptic curve. Then the Hodge conjecture is true for any power $E^n = E \times \cdots \times E$.*

Proof. Let $X = E^n$. As for any smooth projective variety, we have the Hodge ring

$$B(X) = \bigoplus_{k \geqslant 0} \mathrm{Hom}_{\mathbb{Q}\text{-HS}}\big(\mathbb{Q}(0), H^{2k}(X, \mathbb{Q})(k)\big).$$

This is a graded ring, whose component in degree k consists of all (k,k)-Hodge classes in $H^{2k}(X, \mathbb{Q})$. Using our computation of the Mumford–Tate group, we can prove the stronger statement that $B(X)$ is generated in degree 2, that is, by the classes of divisors on X. Since any divisor class is represented by an algebraic cycle, this verifies the Hodge conjecture for X.

For simplicity, we will only treat the case where $D \simeq \mathbb{Q}$. As above, let $H_{\mathbb{Q}} = H^1(E, \mathbb{Q})$; according to our computation, $MT(H) = GL(H)$. Now $X = E^n$ is an abelian variety of dimension n; since $H^1(X, \mathbb{Q}) \simeq H^{\oplus n}$, the cohomology algebra of X is isomorphic to the wedge algebra of $H^{\oplus n}$. As direct sums of Hodge structures of weight 0, we therefore have

$$\bigoplus_{k \geqslant 0} H^{2k}(X, \mathbb{Q})(k) \simeq \bigoplus_{k \geqslant 0} \big((H^{\oplus n})^{\wedge 2k}\big)(k)$$

The assertion that $B(X)$ is generated by divisor classes is thus implied by the following result in invariant theory: Let V be a finite-dimensional $\mathbb{Q}$-vector space. As a $\mathbb{Q}$-algebra, the set of $GL(V)$-invariants in $\bigwedge\big(V^{\oplus n}\big)$ is generated in degree 2. $\square$

2. Second lecture

An excellent reference about Hodge theory and the topics in this lecture is the book by Voisin [5]. Let X be a K3-surface, and

$$H = H_0^2(X, \mathbb{Q}) = \ker\big(L \colon H^2(X, \mathbb{Q}) \to H^4(X, \mathbb{Q})\big)$$

its primitive second cohomology (where $L(\alpha) = \omega \wedge \alpha$ is the Lefschetz operator for a Kähler structure on X). The Kuga–Satake construction (see 3.1 in the Exercises)

produces an abelian variety A, together with an embedding of H into $\operatorname{End} H^1(A, \mathbb{Q})$. This means that the cohomology of K3-surfaces is closely related to the cohomology of abelian varieties, and is therefore very special. The goal of this lecture is to explain a result by Deligne, saying that something similar is not possible for a general surface in $\mathbb{P}^3$ of degree at least 5. The invariant that Deligne uses to distinguish Hodge structures that come from abelian varieties is the Mumford–Tate group.

2.1. The Mumford–Tate group in families

In the proof, we will need to look at families (such as Lefschetz pencils) of surfaces in $\mathbb{P}^3$. This makes it necessary to study the behavior of the Mumford–Tate group in a family of smooth projective varieties, or, more generally, in a family of polarized Hodge structures.

So let $f: \mathcal{X} \to B$ be a family of smooth complex projective varieties; that is to say, B is a complex manifold, and f a projective and submersive holomorphic map. Let $X_b = f^{-1}(b)$ be the fibers; they are complex projective manifolds. We consider the family of Hodge structures $H_b = H^k(X_b, \mathbb{Q})$, and their Mumford–Tate groups. Since f is projective, each H_b is naturally polarized. Let $b \in B$ be any point, and $\pi_1(B, b)$ the fundamental group. We have the monodromy action

$$\mu_b \colon \pi_1(B, b) \to \operatorname{Aut}\big(H_b, Q\big),$$

and the monodromy group (at b) is the image of μ_b.

DEFINITION 2. *In the setting above, we define the* algebraic monodromy group *as the smallest algebraic subgroup of* $\operatorname{GL}(H_b)$*, defined over* $\mathbb{Q}$*, that contains the image of* μ_b*.*

More generally, we can consider an arbitrary variation of polarized Hodge structure on B. Recall that such a variation is given by the following data. First, a local system $\mathcal{H}_{\mathbb{Q}}$ of $\mathbb{Q}$-vector spaces (in the example, $\mathcal{H}_{\mathbb{Q}} = R^k f_* \mathbb{Q}_{\mathcal{X}}$), or equivalently, a representation of the fundamental group

$$\mu_b \colon \pi_1(B, b) \to \operatorname{Aut}\big(H_{\mathbb{Q}, b}\big)$$

on any of its fibers $H_{\mathbb{Q}, b}$. Second, a filtration of the associated vector bundle $\mathcal{H}_{\mathcal{O}} = \mathcal{O}_B \otimes_{\mathbb{Q}} \mathcal{H}_{\mathbb{Q}}$ by holomorphic subbundles $F^p \mathcal{H}_{\mathcal{O}}$, such that Griffiths' transversality condition

$$\nabla\big(F^p \mathcal{H}_{\mathcal{O}}\big) \subseteq \Omega_B^1 \otimes_{\mathcal{O}_B} F^{p-1} \mathcal{H}_{\mathcal{O}}$$

is satisfied (here ∇ is the natural flat connection on $\mathcal{H}_{\mathcal{O}}$). Third, a flat pairing

$$Q \colon \mathcal{H}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{H}_{\mathbb{Q}} \to \mathbb{Q}_B$$

with the property that $Q\big(F^p \mathcal{H}_{\mathcal{O}}, F^q \mathcal{H}_{\mathcal{O}}\big) = 0$ if $p + q > k$. It follows that the monodromy representation μ_b has to preserve the pairing Q_b; just as above, we define the

algebraic monodromy group at a point $b \in B$ as the Zariski-closure of the image of μ_b in $\text{Aut}(H_{\mathbb{Q}, b}, Q_b)$.

Given a variation of polarized Hodge structure $\mathcal{H}$ on B, every fiber H_b is a polarized Hodge structure of weight k, and has its associated Mumford–Tate group $\text{MT}(H_b)$. Since $\text{MT}(H_b)$ is contained in $\text{GL}(H_{\mathbb{Q}, b})$, which is locally constant, it makes sense to ask how the Mumford–Tate group varies with the point $b \in B$. Even in simple examples (such as families of elliptic curves), $\text{MT}(H_b)$ is not locally constant on analytic subsets. This should not be too surprising: after all, Proposition 3 shows that $\text{MT}(H_b)$ is the stabilizer of every Hodge class in any tensor space $T^{d,e}(H_b)$, and we know that the set of Hodge classes can vary wildly as the point b moves around. On the other hand, there is the following result.

PROPOSITION 6. *Outside of a countable union of analytic subvarieties of* B*, the Mumford–Tate group* $\text{MT}(H_b)$ *is locally constant and contains a finite-index subgroup of the algebraic monodromy group.*

Proof. Recall the definition of the Hodge loci for a variation of Hodge structure on B. The underlying local system $\mathcal{H}_{\mathbb{Q}}$ defines an analytic covering space (non-connected) of B, whose points are pairs (h, b) with $h \in H_{\mathbb{Q}, b}$. Since the Hodge bundles are holomorphic, the subset of those points for which h is a Hodge class in H_b is a countable union of analytic subvarieties. The image of any irreducible component in B is one of the Hodge loci for $\mathcal{H}$.

Now consider the Hodge loci for the family of Hodge structures $T^{d,e}(H_b)$, as d, e range over all multi-indices of any length. Taken together, they form a countable union of analytic subsets of B; let B_0 be the complement of those that are not equal to all of B. It then follows from Proposition 3 that the Mumford–Tate group is locally constant on B_0. More precisely, we argue as follows. For every pair of multi-indices d, e, there is a sub-variation of Hodge structure contained in $T^{d,e}(\mathcal{H})$, entirely of type $(0, 0)$, whose fiber at any point $b \in B_0$ coincides with the set of Hodge classes in $T^{d,e}(H_b)$. The subgroup of $\text{GL}(H_b)$, consisting of those elements that fix every Hodge class in any tensor space $T^{d,e}(H_b)$, is therefore locally constant on B_0. But that subgroup is equal to $\text{MT}(H_b)$ by Proposition 3.

For the second assertion, fix any point $b \in B_0$. The proof of Proposition 3 shows that there is a Hodge class t in a finite direct sum $T^{d_1, e_1}(H_b) \oplus \cdots \oplus T^{d_k, e_k}(H_b)$ such that $\text{MT}(H_b)$ is the stabilizer of that class. Now let

$$\mathcal{T} = T^{d_1, e_1}(\mathcal{H}) \oplus \cdots \oplus T^{d_k, e_k}(\mathcal{H}),$$

and let $\mathcal{T}'$ be the sub-variation purely of type $(0, 0)$ from above; then $\mathcal{T}'_b$ contains the Hodge class t. Note that $\mathcal{T}'$ inherits a polarization from $\mathcal{T}$; according to one of the exercises, the monodromy action of $\pi_1(B, b)$ on $\mathcal{T}'_b$ is therefore of finite order. It follows that a subgroup of $\pi_1(B, b)$ of finite index stabilizes t, and is therefore contained in $\text{MT}(H_b)$. It is then easy to see that $\text{MT}(H_b)$ has to contain a finite-index subgroup of the algebraic monodromy group at b. $\qquad\square$

2.2. Hodge structures related to abelian varieties

After these preliminary remarks on Mumford–Tate groups in families, we now come to Deligne's result (see Section 7 in [1]) that the Hodge structures on the cohomology of varieties other than K3-surfaces are usually not related to those of abelian varieties. Following Deligne, we first define more carefully what we mean by the phrase, "related to the cohomology of abelian varieties."

DEFINITION 3. *We say that a rational Hodge structure* H *can be* expressed with the help of abelian varieties *if it belongs to the smallest category of rational Hodge structures that is stable under direct sums, tensor products, and passage to direct summands, and contains* $H^1(A, \mathbb{Q})$ *for any abelian variety* A, *as well as* $\mathbb{Q}(n)$ *for every* $n \in \mathbb{Z}$.

Note that any cohomology group $H^k(A, \mathbb{Q})$ of an abelian variety itself is in that category, since it is isomorphic to $H^1(A, \mathbb{Q})^{\wedge k}$, and the wedge product is a direct summand in $H^1(A, \mathbb{Q})^{\otimes k}$. Similarly, $H^1(X, \mathbb{Q})$ of any smooth projective variety X belongs to the category, because the Picard variety $\mathrm{Pic}^0(X)$ is an abelian variety with the same first cohomology. By virtue of the Kuga–Satake construction, the second cohomology of a K3-surface can also be expressed with the help of abelian varieties.

The next lemma gives a necessary condition, based on the Mumford–Tate group, for a Hodge structure to be expressible with the help of abelian varieties.

LEMMA 5. *If a Hodge structure* H *can be expressed with the help of abelian varieties, then the Hodge structure on the Lie algebra of its Mumford–Tate group is of type* $\{(-1, 1), (0, 0), (1, -1)\}$.

Proof. We note that $\mathrm{MT}(H)$ acts on its Lie algebra by conjugation; as pointed out before, the Lie algebra therefore has a natural Hodge structure. Now consider the category $\mathfrak{C}$, consisting of all rational Hodge structures H for which the Lie algebra $\mathrm{Lie}\,\mathrm{MT}(H)$ is of type $\{(-1, 1), (0, 0), (1, -1)\}$; to show that it contains all Hodge structures that can be expressed with the help of abelian varieties, it suffices to show that this category satisfies the conditions in Definition 3.

1. For $H = \mathbb{Q}(n)$, the Mumford–Tate group is either trivial or equal to $\mathbb{G}_m$, and so the Hodge structure on its Lie algebra is either 0 or $\mathbb{Q}(0)$. For $H = H^1(A, \mathbb{Q})$, the Lie algebra of $\mathrm{MT}(H)$ is contained in $\mathrm{End}(H)$, whose Hodge structure is of type $\{(-1, 1), (0, 0), (1, -1)\}$; it follows that the Hodge structure on $\mathrm{Lie}\,\mathrm{MT}(H)$ is of the same type.

2. Now suppose that $H = H_1 \oplus H_2$. Then $\mathrm{MT}(H) \hookrightarrow \mathrm{MT}(H_1) \times \mathrm{MT}(H_2)$, and the projection to either factor is surjective. If H_1 and H_2 belong to $\mathfrak{C}$, then it follows from the inclusion $\mathrm{Lie}\,\mathrm{MT}(H) \hookrightarrow \mathrm{Lie}\,\mathrm{MT}(H_1) \oplus \mathrm{Lie}\,\mathrm{MT}(H_2)$ that the Hodge structure on $\mathrm{Lie}\,\mathrm{MT}(H)$ has the correct type, which means that H also belongs to $\mathfrak{C}$. Conversely, suppose that H belongs to the category. The surjections $\mathrm{Lie}\,\mathrm{MT}(H) \twoheadrightarrow \mathrm{Lie}\,\mathrm{MT}(H_j)$ are morphisms of Hodge structure, and this implies

that either factor H_j is in $\mathfrak{C}$.

3. Finally, we consider a tensor product $H = H_1 \otimes H_2$ of two Hodge structures in $\mathfrak{C}$. Note that $MT(H)$ is contained in the image of the composition

$$MT(H_1) \times MT(H_2) \hookrightarrow GL(H_1) \times GL(H_2) \to GL(H),$$

and thus $\operatorname{Lie} MT(H)$ lies in the image of $\operatorname{Lie} MT(H_1) \oplus \operatorname{Lie} MT(H_2)$ in $End(H)$. Arguing as above, we see that the Hodge structure on $\operatorname{Lie} MT(H)$ is of the correct type, which proves that H belongs to the category $\mathfrak{C}$. $\qquad\square$

Let (H, Q) be a polarized Hodge structure of even weight. We write $End(H, Q)$ for the Lie algebra of the orthogonal group $O(H, Q) \subseteq GL(H)$. It consists of those $X \in End(H_{\mathbb{C}})$ that satisfy $Q(h, Xh) = 0$ for every $h \in H_{\mathbb{C}}$.

LEMMA 6. *Let (H, Q) be a polarized Hodge structure of type*

$$\{(0, 2), (1, 1), (2, 0)\},$$

and consider the induced Hodge structure on $End(H, Q)$. If $\dim H^{2,0} \leqslant 1$, then $End(H, Q)$ is of type $\{(-1, 1), (0, 0), (1, -1)\}$; otherwise, $End(H, Q)^{-2,2} \neq 0$.

Proof. In any case, the Hodge structure on the ambient Lie algebra $End(H)$ is of type

$$\{(-2, 2), (-1, 1), (0, 0), (1, -1), (2, -2)\}.$$

Elements $X \in End(H)^{-2,2}$ satisfy $XH^{2,0} \subseteq H^{0,2}$ and annihilate the other summands in the Hodge decomposition. Consequently, X belongs to $End(H, Q)^{-2,2}$ if and only if

$$\langle h, \overline{Xh} \rangle = -Q(h, Xh) = 0$$

for every $h \in H^{2,0}$. Now let $e_1, \ldots, e_n$ be an orthonormal basis for $H^{2,0}$, relative to the inner product defined by the polarization. Writing $Xe_j = \sum_{j=1}^{n} X_{jk} \overline{e_k}$, the condition above becomes

$$\sum_{j,k=1}^{n} \lambda_j X_{jk} \lambda_k = 0$$

for every $\lambda \in \mathbb{C}^n$. When $n = 1$, this cannot happen unless $X = 0$; but for $n > 1$, any anti-symmetric matrix gives an example. $\qquad\square$

2.3. Deligne's result

We now consider smooth surfaces in $\mathbb{P}^3$ and the Hodge structures on their primitive cohomology $H_0^2(X, \mathbb{Q})$.

LEMMA 7. *Let $X \subseteq \mathbb{P}^3$ be a smooth surface of degree at least 4, and $H = H_0^2(X, \mathbb{Q})$ the Hodge structure on its primitive cohomology, with the polarization Q given by the intersection pairing. Then the algebraic monodromy group is the full orthogonal group $O(H, Q)$.*

Proof. Let G be the algebraic monodromy group. We apply Lemma 8 to the vector space $H_{\mathbb{C}}$ and the bilinear form $-Q$; all the assumptions are met because of Lefschetz theory. It follows that G is either a finite group, or all of $O(H, Q)$. To conclude, we have to argue that the first possibility cannot happen if $d = \deg X$ is at least 4 (while it does happen for $d = 2, 3$). By Zariski's theorem, the action of G on H is irreducible; if G was finite, this would force the polarization Q to be definite. Using Griffiths' theory, we easily compute that

$$h^{2,0} = h^{0,2} = \binom{d-1}{3} \quad \text{while} \quad h^{1,1} = \binom{2d-1}{3} - 4\binom{d}{3}.$$

As long as $d \geqslant 4$, the Hodge structure of the surface has nonzero $H^{2,0}$ and $H^{1,1}$. By the Hodge-Riemann bilinear relations, Q has a different sign on the two subspaces, and is therefore not definite. This rules out the possibility that G is finite. □

In the proof, we used the following lemma of Deligne's (which is Lemma 4.4.2 in [2]). To apply it, we take Δ to be the collection of vanishing cycles in a Lefschetz pencil on $\mathbb{P}^3$ containing X. Since all vanishing cycles are conjugate under the action of the monodromy group – this is the proof of Zariski's theorem – Δ is a single orbit for the algebraic monodromy group G. Moreover, the self-intersection number of any vanishing cycle is $Q(\delta, \delta) = -2$, and the Picard–Lefschetz formula shows that G contains the transvection $h \mapsto h + Q(h, \delta)\delta$.

LEMMA 8. *Let* V *be a finite-dimensional complex vector space with a non-degenerate symmetric bilinear form* B*, and let* $G \subseteq O(H, B)$ *be an algebraic subgroup. Suppose that we have a subset* $\Delta \subseteq V$*, consisting of vectors* δ *with* $B(\delta, \delta) = 2$*, such that* Δ *spans* V*, and* Δ *is a single orbit under the action by* G*. If* G *is the smallest algebraic subgroup containing all the transvections* $v \mapsto v - B(v, \delta)\delta$*, then it is either a finite group, or all of* $O(H, B)$*.*

We are now in a position to prove Deligne's result about Hodge structures, by considering surfaces in $\mathbb{P}^3$ of degree at least 5.

THEOREM 1. *Let* $X \subseteq \mathbb{P}^3$ *be a very general surface of degree* $d \geqslant 5$*. Then the Hodge structure on* $H_0^2(X, \mathbb{Q})$ *cannot be expressed with the help of abelian varieties.*

Proof. Let $H = H_0^2(X, \mathbb{Q})$, with polarization Q coming from the intersection pairing. We know that $\mathrm{MT}(H) \subseteq O(H, Q)$. Since X is very general, the Mumford–Tate group contains a finite-index subgroup of the algebraic monodromy group by Proposition 6. Since $d \geqslant 4$, the algebraic monodromy group is the orthogonal group by Lemma 7; it follows that $\mathrm{MT}(H) = O(H, Q)$, and also that $\mathrm{Lie}\,\mathrm{MT}(H) = \mathrm{End}(H, Q)$. Now observe that, d being at least 5, we have $\dim H^{2,0} > 1$; by virtue of Lemma 6, the Hodge structure on $\mathrm{Lie}\,\mathrm{MT}(H)$ can therefore not be of type $\{(-1, 1), (0, 0), (1, -1)\}$. According to Lemma 5, this rules out the possibility of H being expressed with the help of abelian varieties. □

3. Exercises

Some of these exercises were kindly provided by Claire Voisin.

3.1. The Kuga–Satake construction

Let H be a rational vector space, and Q a nondegenerate bilinear form on H. Recall that the *Clifford algebra* $C(H, Q)$ is a $\mathbb{Q}$-algebra with a linear map $i \colon H \to C(H, Q)$, and the following universal property: Given any $\mathbb{Q}$-algebra A and linear map

$$f \colon H \to A$$

such that $f(h)^2 = Q(h, h)$ for every $h \in H$, there is a unique map of $\mathbb{Q}$-algebras $g \colon C(H, Q) \to A$ such that $f = g \circ i$. The universal property implies that

$$C(H, Q) = T(H)/I,$$

where I is the ideal of the tensor algebra generated by $h \otimes h - Q(h, h)$ for $h \in H$.

(a) As a warm-up, convince yourself that $C(H, Q)$ has dimension $2^{\dim H}$.

(b) Now let (H, Q) be a polarized Hodge structure of weight 2 with $\dim H^{2,0} = 1$. Let $P = \left(H^{2,0} \oplus H^{0,2} \right) \cap H_{\mathbb{R}}$. Show that P is an oriented 2-plane on which Q is negative definite.

(c) Let $C = C(H, Q)$ be the Clifford algebra. Let e_1, e_2 be an oriented basis for P with $Q(e_1, e_1) = Q(e_2, e_2) = -1$ and $Q(e_1, e_2) = 0$. Show that the element $e = e_2 e_1 \in C_{\mathbb{R}}$ does not depend on the choice of basis and satisfies $e^2 = -1$.

(d) Consider the complex structure on $C_{\mathbb{R}}$ given by left Clifford multiplication by e. It determines a Hodge structure of weight 1 on C, because a complex structure on a real vector space V is the same as a decomposition $V_{\mathbb{C}} = V^{1,0} \oplus \overline{V^{1,0}}$ of the complexification of V. Show that the map

$$H \to \operatorname{End} C,$$

given by left Clifford multiplication, is an injective morphism of Hodge structures of bidegree $(-1, -1)$. (The vector space on the right has a Hodge structure of weight 0, induced by the Hodge structure on C defined above.)

Solution

We first review the construction of the Clifford algebra. Let V be a rational vector space, and Q a nondegenerate bilinear form on V. The Clifford algebra $C(V, Q)$ is a $\mathbb{Q}$-algebra with a linear map $i \colon V \to C(V, Q)$, and the following universal property:

Given any $\mathbb{Q}$-algebra A and linear map $f\colon V \to A$ such that $f(v)^2 = Q(v,v)$ for every $v \in V$, there is a unique map of $\mathbb{Q}$-algebras $g\colon C(V,Q) \to A$ such that $f = g \circ i$. Thus

$$C(V,Q) = \bigoplus_{k \geqslant 0} V^{\otimes k}/I,$$

where I is the ideal generated by $v \otimes v - Q(v,v)$ for $v \in V$. To get an idea of the dimension of $C(V,Q)$, take an orthogonal basis $e_1, \ldots, e_n$ for V, meaning that $Q(e_i, e_j) = 0$ for $i \neq j$ (possible because Q is nondegenerate). Let $d_i = Q(e_i, e_i)$. Then I is generated by the elements $e_i \otimes e_i - d_i$ and $e_i \otimes e_j + e_j \otimes e_i$; therefore $C(V,Q)$ has dimension 2^n, and a basis is given by the vectors $e_{i_1} \otimes \cdots \otimes e_{i_k}$, for $1 \leqslant i_1 < \cdots < i_k \leqslant n$.

We write Q for the bilinear form given by the negative of the polarization. The orientation comes from the fact that P is naturally isomorphic to $H^{2,0}$, which has its standard orientation ($1 \wedge i > 0$); Q is positive definite on $H^{2,0} \oplus H^{0,2}$ by the bilinear relations. This proves (b).

In $C(H)$, we have $e_1^2 = e_2^2 = 1$ and $e_1 e_2 + e_2 e_1 = 0$. Let f_1, f_2 be another orthonormal basis; writing $f_1 = a e_1 + b e_2$ and $f_2 = c e_1 + d e_2$, we have

$$f_2 f_1 = (c e_1 + d e_2)(a e_1 + b e_2) = ac + bd + (ad - bc)e = e$$

since f_1, f_2 is oriented and orthonormal. Moreover, $e^2 = e_2 e_1 e_2 e_1 = -e_2 e_2 e_1 e_1 = -1$, and so (c) is proved.

Finally, we come to (d). Note that the map $H \to \operatorname{End} C(H)$ is injective, because H injects into $C(H)$, and Q is nondegenerate. Since $e^2 = -1$, left multiplication by e determines a complex structure on the real vector space $C(H_{\mathbb{R}})$. Since $C(H)$ is defined over $\mathbb{Q}$, this means that we have a rational Hodge structure of weight 1. We have

$$C(H)^{1,0} = \ker(e - i) \quad \text{and} \quad C(H)^{0,1} = \ker(e + i)$$

on $C(H_{\mathbb{C}})$. Let $\omega \in H^{2,0}$ be a generator, such that $\|\omega\|^2 = Q(\omega, \overline{\omega}) = \frac{1}{2}$. Then $e_1 = \omega + \overline{\omega}$ and $e_2 = i(\omega - \overline{\omega})$ form an oriented orthonormal basis. We have $\omega = \frac{1}{2}(e_1 - i e_2)$ and $\overline{\omega} = \frac{1}{2}(e_1 + i e_2)$. From the relations $e_1^2 = e_2^2 = 1$ and $e_1 e_2 + e_2 e_1 = 0$, we deduce that $\omega^2 = \overline{\omega}^2 = 0$. We then have

$$e = e_2 e_1 = i(\omega - \overline{\omega})(\omega + \overline{\omega}) = 2i\omega\overline{\omega}.$$

Now let $x \in C(H)^{1,0}$; then $ex = ix$, and so $2\omega\overline{\omega}x = x$, which means that $\omega x = 0$. On the other hand,

$$e\overline{\omega}x = -2i\overline{\omega}\omega\overline{\omega}x = -ix.$$

This shows that left multiplication by ω is of type $(1, -1)$, and left multiplication by $\overline{\omega}$ is of type $(-1, 1)$. Since the map from H to $\operatorname{End} C(H)$ is defined over $\mathbb{Q}$, it only remains to show that elements of $H^{1,1}$ give operators of type $(0,0)$. So let $h \in H^{1,1}$ be any element; since h is orthogonal to ω and $\overline{\omega}$ by type, it anti-commutes with e_1 and e_2, and therefore commutes with e, proving that it defines an endomorphism of type $(0,0)$.

3.2. Mumford–Tate groups of elliptic curves

Consider the elliptic curve

$$E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}),$$

with $\operatorname{Im}\tau > 0$, and let $H_\tau = H^1(E_\tau, \mathbb{Q})$.

(a) Find a formula for the representation $\rho\colon \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(\mathbb{R}^2)$ associated to H_τ.

(b) Compute the Mumford–Tate group of H_τ by finding the Zariski-closure of the image of ρ.

3.3. The big Mumford–Tate group

Let H be a $\mathbb{Q}$-Hodge structure of weight k. If one is interested in Hodge classes of arbitrary type (p, p), it is useful to define the so-called *big Mumford–Tate* group

$$\mathrm{MT}^\sharp(H) = \mathrm{MT}\bigl(H \oplus \mathbb{Q}(1)\bigr).$$

(a) We know that there is a map $\mathrm{MT}^\sharp(H) \to \mathrm{MT}(H) \times \mathbb{G}_m$, surjective onto either factor. Show that for $k = 0$, this map is an isomorphism.

(b) Let H be polarized and of weight 1. Show that in this case, $\mathrm{MT}^\sharp(H) \simeq \mathrm{MT}(H)$.

(c) Now suppose that $k \neq 0$. Show that the map $\mathrm{MT}^\sharp(H) \to \mathrm{MT}(H)$ is finite.

3.4. Mumford–Tate groups and Tannaka duality

This exercise discusses a third possible definition of the Mumford–Tate group, in terms of Tannaka duality. First, we list some definitions: A *tensor category* is an abelian category that has tensor products and an identity object 1 for $\otimes$. *Internal* Hom of X and Y, if it exists, is an object $\hom(X, Y)$ in the category with a natural isomorphism of functors $\mathrm{Hom}(- \otimes X, Y) \simeq \mathrm{Hom}\bigl(-, \hom(X, Y)\bigr)$. The tensor category is said to be *rigid* if $\hom(X, Y)$ exists for every two objects, is compatible with tensor products, and if $X \to (X^\vee)^\vee$ is an isomorphism for every X, duals being defined as $X^\vee = \hom(X, 1)$. Finally, a k-linear rigid tensor category is a *neutral Tannakian category over* k if there is a *fiber functor*, meaning a k-linear, faithful and exact functor to the category of finite-dimensional k-vector spaces that preserves tensor products.

The definitions are set up to make such a category look like the category of representations of some group, and in fact, the main theorem is that every neutral Tannakian category is equivalent to the category of representations of an algebraic group G over k. According to *Tannaka duality*, G is obtained as the group of tensor automorphisms of the fiber functor: For any k-algebra A, the A-valued points of G are given by a collection of automorphisms $\lambda_H \in \mathrm{Aut}\bigl(A \otimes_k H\bigr)$, one for every object H, such that $\lambda_{H_1 \otimes H_2} = \lambda_{H_1} \otimes \lambda_{H_2}$, $\lambda_1 = \mathrm{id}$, and $\lambda_{H_2} \circ (1 \otimes \phi) = (1 \otimes \phi) \circ \lambda_{H_1}$ for any morphism $\phi\colon H_1 \to H_2$.

(a) Convince yourself that the category $\mathbb{Q}$-HS of rational Hodge structures forms a neutral Tannakian category over $\mathbb{Q}$, with fiber functor taking a Hodge structure to the underlying $\mathbb{Q}$-vector space.

(b) Show that Deligne's group $\mathbb{S}$ is isomorphic to the group of tensor automorphisms of the forgetful functor from $\mathbb{R}$-HS to $\mathbb{R}$-vector spaces.

(c) Now let $\langle H \rangle$ be the full subcategory of $\mathbb{Q}$-HS whose objects are arbitrary subquotients of the tensor spaces $T^{d,e}(H)$ defined before. Show that $MT(H)$ is isomorphic to the group of tensor automorphisms of the forgetful functor from $\langle H \rangle$ to $\mathbb{Q}$-vector spaces. It follows that $\langle H \rangle$ is equivalent to the category of finite-dimensional representations of $MT(H)$.

3.5. Complex tori and abelian varieties

Let T and T' be two complex tori with $\dim T > \dim T' > 0$, and let

$$\phi \colon T \to T'$$

be a surjective morphism. Denote by T'' the connected component of the kernel of ϕ containing 0.

(a) Show that for a general deformation of the triple (T, T', ϕ), there is no splitting $T \sim T' \oplus T''$ up to isogeny.

(b) Show that if T is projective, such a splitting always exists.

(c) Show that the deformations of the triple (T, T', ϕ) for which T is projective are dense in the local universal family of deformations.

(d) Is it true that, in order for T to be projective, it suffices that T' and T'' are projective?

3.6. Hodge classes and monodromy

Let $\mathcal{H}$ be a polarized variation of Hodge structure on a complex manifold B, and suppose that for each $b \in B$, the Hodge structure H_b is entirely of type (k, k). Show that the image of the monodromy representation

$$\mu_b \colon \pi_1(B, b) \to \operatorname{Aut}\big(H_{\mathbb{Q}, b}, Q\big)$$

is necessarily a finite group.

References

[1] DELIGNE, P. La conjecture de Weil pour les surfaces K3. *Invent. Math. 15* (1972), 206–226.

[2] DELIGNE, P. La conjecture de Weil. II. *Inst. Hautes Études Sci. Publ. Math. 52* (1980), 137–252.

[3] DELIGNE, P., MILNE, J. S., OGUS, A., AND SHIH, K.-Y. *Hodge Cycles, Motives, and Shimura Varieties*. Lecture Notes in Mathematics 900. Springer Verlag, Berlin-New York, 1982.

[4] MOONEN, B. *An Introduction to Mumford-Tate Groups*. 2004. Available online from http://staff.science.uva.nl/~bmoonen/.

[5] VOISIN, C. *Hodge Theory and Complex Algebraic Geometry, I and II*. Translated from the French. Reprint of the English editions. Cambridge Studies in Advanced Mathematics 76 and 77. Cambridge University Press, Cambridge, 2007.

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