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ON THE FORM OF INSTANTON-TYPE SOLUTIONS FOR EQUATIONS OF THE FIRST PAINLEVÉ HIERARCHY BY MULTIPLE-SCALE ANALYSIS

Abstract. We construct, using multiple-scale analysis, a formal solution containing sufficiently many free parameters for the first Painlevé hierarchy $(P_{\mathrm{I}})_m$ with a large parameter. This note is a short summary of our forthcoming paper [3].

1. Introduction

Aoki, Kawai and Takei, in 1990's, investigated the traditional Painlevé equations with a large parameter η from a viewpoint of the exact WKB analysis and local structure of formal solutions near turning points. In the papers [4, 8, 9, 10, 12], they constructed the formal solutions with 2-parameters called *instanton-type solutions* and established the connection formula among these solutions.

Several Painlevé hierarchies have recently been found in various areas of mathematics and it is also expected to establish the connection formula of instanton-type solutions for these hierarchies with a large parameter. For that purpose, we need to construct instanton-type solutions with sufficiently many free parameters so that Stokes phenomena are correctly caught.

In this note, we consider the first Painlevé hierarchy $(P_{\mathrm{I}})_m$ $(m = 1, 2, \dots)$ with a large parameter η and construct its instanton-type solutions. For the second member $(P_{\mathrm{I}})_2$ of the hierarchy, Y. Takei [13] had constructed instanton-type solutions by using singular perturbative reduction of a Hamiltonian system to its Birkhoff normal form. The first author [2] also constructed them by multiple-scale analysis. We follow the latter method and construct instanton-type solutions for a general member $(P_{\mathrm{I}})_m$. Detailed construction will be given in our forthcoming article [3].

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2. Instanton-type solutions and multiple-scale analysis

2.1. The first Painlevé hierarchy with a large parameter

Let w_j $(j = 1, 2, \dots)$ be the polynomial of variables u_k and v_l $(1 \le k,\, l \le j)$ defined by the recurrence relation

$$w_j := \frac{1}{2}\sum_{k=1}^{j} u_k u_{j+1-k} + \sum_{k=1}^{j-1} u_k w_{j-k} - \frac{1}{2}\sum_{k=1}^{j-1} v_k v_{j-k} + c_j + \delta_{jm} t. \tag{1}$$

Here c_j is a constant and δ_{jm} stands for the Kronecker delta. Then the first Painlevé hierarchy $(P_{\mathrm{I}})_m$ with a large parameter η $(m = 1, 2, \dots)$ is the system of non-linear equations

$$\begin{cases} \eta^{-1}\dfrac{du_j}{dt} = 2v_j, & j = 1,\, 2,\, \dots,\, m, \\[2ex] \eta^{-1}\dfrac{dv_j}{dt} = 2(u_{j+1} + u_1 u_j + w_j), & j = 1,\, 2,\, \dots,\, m, \end{cases} \tag{2}$$

where u_j and v_j are unknown functions of t with the additional condition $u_{m+1} = 0$.

Note that the first member $(P_{\mathrm{I}})_1$ gives the traditional first Painlevé equation P_{I} with a large parameter η.

As the definition of the system is very complicated, we rewrite the system into the simpler form with the generating functions defined by

$$\begin{aligned} &U(\theta) := \sum_{k=1}^{\infty} u_k \theta^k, \quad V(\theta) := \sum_{k=1}^{\infty} v_k \theta^k, \quad W(\theta) := \sum_{k=1}^{\infty} w_k \theta^{k+1} \\ &C(\theta) := \sum_{k=1}^{\infty} (c_k + \delta_{km} t)\theta^{k+1}. \end{aligned} \tag{3}$$

Here θ denotes an independent variable. Then the system (2) becomes

$$\eta^{-1}\frac{d}{dt}\begin{pmatrix} U\theta \\ V\theta \end{pmatrix} \equiv \begin{pmatrix} 2V\theta \\ -(1 + 2u_1\theta)(1 - U) + \dfrac{1 + 2C - \theta V^2}{1 - U} \end{pmatrix} \tag{4}$$

with the condition that the coefficients of θ^{m+1} of U and V are zero. Here $A \equiv B$ implies that $A - B$ is equal to zero modulo θ^{m+2}.

2.2. 0-parameter solutions of $(P_{\mathrm{I}})_m$

For the construction of instanton-type solutions, we first construct a special kind of the solution of $(P_{\mathrm{I}})_m$ called a 0-*parameter solution*. We rewrite the result [7] on the

0-parameter solution of $(P_{\mathrm{I}})_m$ by using generating functions. Let us consider formal series in η^{-1} of the form

$$\bar{u}_j(t) := \sum_{k=0}^{\infty} \eta^{-k} \hat{u}_{j,k}(t), \quad \bar{v}_j(t) := \sum_{k=0}^{\infty} \eta^{-k} \hat{v}_{j,k}(t), \quad j = 1, \dots, m, \tag{5}$$

and let us define the generating functions with respect to the leading terms $\hat{u}_{j,0}$ and $\hat{v}_{j,0}$ of $\bar{u}_j$ and $\bar{v}_j$ by

$$\hat{u}_0(\theta) := \sum_{j=1}^{\infty} \hat{u}_{j,0}\theta^j \quad \text{and} \quad \hat{v}_0(\theta) := \sum_{j=1}^{\infty} \hat{v}_{j,0}\theta^j, \tag{6}$$

respectively. Then, putting (5) into (2), we find the following equations for the generating functions:

$$\hat{v}_0 = 0, \qquad (1 + 2\hat{u}_{1,0}\theta) = \frac{1+2C}{(1-\hat{u}_0)^2}. \tag{7}$$

The equations can be easily solved and we have

$$\hat{u}_0 = 1 - \sqrt{\frac{1+2C}{1+2\hat{u}_{1,0}\theta}}. \tag{8}$$

Note that the $\hat{u}_{1,0}$ in the right-hand side of (8) is taken so that the coefficient $\hat{u}_{m+1,0}$ of θ^{m+1} in $\hat{u}_0$ is zero.

2.3. Instanton-type solutions of $(P_{\mathrm{I}})_m$

Let $\alpha = -\frac{1}{2}$, and we fix it in what follows. We first introduce several notations to define instanton-type solutions.

Let $u_{k,j\alpha}$ and $v_{k,j\alpha}$ $(k = 1,2,\dots,\ j = 0,1,2,\dots)$ be unknown functions of the variable t. We define

$$u := \sum_{j=0}^{\infty}\sum_{k=1}^{\infty} u_{k,j\alpha}(t)\,\theta^k\,\eta^{j\alpha}, \qquad v := \sum_{j=0}^{\infty}\sum_{k=1}^{\infty} v_{k,j\alpha}(t)\,\theta^k\,\eta^{j\alpha}, \tag{9}$$

and denote by $\sigma_k^{\theta}(u)$ (resp. $\sigma_k^{\theta}(v)$) the coefficient of θ^k in u (resp. v).

Let Θ be the set of formal power series of θ without constant terms, and let $Q : (\Theta\theta)^2 \longrightarrow \Theta^2$ be the map defined by the relation

$$Q\begin{pmatrix} x\theta \\ y\theta \end{pmatrix} := 2\begin{pmatrix} y\theta \\ (1+2\hat{u}_{1,0}\theta)\,x - \sigma_1^{\theta}(x)\theta \end{pmatrix} \tag{10}$$

for $x = \sum_{j=1}^{\infty} x_j\theta^j$, $y = \sum_{j=1}^{\infty} y_j\theta^j \in \Theta$.

Then, by the change of unknown functions in (4),

$$U = \hat{u}_0 + \eta^{\alpha}(1-\hat{u}_0)u, \qquad V = \hat{v}_0 + \eta^{\alpha}(1-\hat{u}_0)v, \tag{11}$$

we obtain the system of unknown functions (u, v) in the form

$$\begin{aligned}\left(\eta^{-1}\frac{d}{dt} - Q\right)\begin{pmatrix} u\theta \\ v\theta \end{pmatrix} \equiv \eta^{\alpha}&\left(\begin{pmatrix} h\theta \\ S(u,v) \end{pmatrix} - uQ\begin{pmatrix} u\theta \\ v\theta \end{pmatrix}\right) \\ &- \eta^{2\alpha}\left(u\begin{pmatrix} h \\ 2\sigma_1^{\theta}(u)u \end{pmatrix} + h\begin{pmatrix} u \\ v \end{pmatrix}\right)\theta \\ &+ \eta^{3\alpha}u\left(h + \frac{d}{dt}\right)\begin{pmatrix} u \\ v \end{pmatrix}\theta,\end{aligned} \tag{12}$$

with

$$S(u,v) := \frac{1}{2}(-v,u)Q\begin{pmatrix} u\theta \\ v\theta \end{pmatrix} + 3\sigma_1^{\theta}(u)u\theta \ \text{ and } \ h := \frac{d}{dt}(\log(1-\hat{u}_0)). \tag{13}$$

As the form of the above system suggests, the map Q plays an important role in the study of $(P_{\mathrm{I}})_m$ and its eigenvector $A(\lambda)$ corresponding to an eigenvalue λ in the sense of $Q(A(\lambda)\theta) = \lambda A(\lambda)\theta$ has the special form $\begin{pmatrix} a(\lambda) \\ \lambda\, a(\lambda)/2 \end{pmatrix}$ with

$$a(\lambda) := \frac{\theta}{1-g(\lambda)\theta} = \sum_{k=0}^{\infty} g(\lambda)^k \theta^{k+1}, \quad g(\lambda) := \frac{\lambda^2 - 8\hat{u}_{1,0}}{4}. \tag{14}$$

Since the coefficients of θ^{m+1} in U and V are zero, the coefficient $(1-\hat{u}_0)a(\lambda)$ of θ^{m+1} must be zero. Hence the eigenvalue λ of Q is a root of the algebraic equation

$$\Lambda(\lambda, t) := g(\lambda)^m - \sum_{k=1}^{m} \hat{u}_{k,0}\, g(\lambda)^{m-k} = 0, \tag{15}$$

where $\hat{u}_{k,0}$ is given by (5). Note that $\Lambda(\lambda, t)$ is an even function of λ.

Let $\nu_{\pm 1}(t), \ldots, \nu_{\pm m}(t)$ be the roots of the algebraic equation of λ where we set $\nu_k = -\nu_{-k}$, and let Ω be an open subset in $\mathbb{C}_t$. We always assume the following two conditions from now on.

(A1) The roots $\nu_i(t)$'s $(1 \le |i| \le m)$ are mutually distinct for each $t \in \Omega$.

(A2) The function $p_1\nu_1(t) + \cdots + p_m\nu_m(t)$ does not vanish identically on Ω for any $(p_1, \ldots, p_m) \in \mathbb{Z}^m \setminus \{0\}$.

Let $\tau := (\tau_1, \dots, \tau_m)$ be m-independent variables, and let us define the rings

$$(16)\qquad \begin{aligned} \mathcal{A}_\alpha(\Omega) &:= \mathcal{M}(\Omega)\left[\left[\eta^\alpha e^{\tau_1}, \dots, \eta^\alpha e^{\tau_m}, \eta^\alpha e^{-\tau_1}, \dots, \eta^\alpha e^{-\tau_m}\right]\right], \\ \mathcal{A}^{O}_\alpha(\Omega) &:= \ O(\Omega)\left[\left[\eta^\alpha e^{\tau_1}, \dots, \eta^\alpha e^{\tau_m}, \eta^\alpha e^{-\tau_1}, \dots, \eta^\alpha e^{-\tau_m}\right]\right], \end{aligned}$$

where $\mathcal{M}(\Omega)$ (resp. $O(\Omega)$) denotes the set of formal power series in θ with coefficients in multi-valued holomorphic functions with a finite number of branching points and poles (resp. holomorphic functions) on Ω. We also denote by $\hat{\mathcal{A}}_\alpha(\Omega)$ (resp. $\hat{\mathcal{A}}^{O}_\alpha(\Omega)$) the subset in $\mathcal{A}_\alpha(\Omega)$ (resp. $\mathcal{A}^{O}_\alpha(\Omega)$) consisting of a formal power series of order less than or equal to α with respect to η. For $\varphi(\tau_1, \dots, \tau_m, t, \theta, \eta) \in \mathcal{A}_\alpha(\Omega)$, we define the morphism ι by

$$(17)\qquad \iota(\varphi) = \varphi\left(\eta \int^t \nu_1(s)ds, \dots, \eta \int^t \nu_m(s)ds, t, \theta, \eta\right).$$

By replacing $\dfrac{d}{dt}$ in (12) with

$$(18)\qquad \frac{\partial}{\partial t} + \eta\nu_1 \frac{\partial}{\partial \tau_1} + \eta\nu_2 \frac{\partial}{\partial \tau_2} + \cdots + \eta\nu_m \frac{\partial}{\partial \tau_m},$$

we obtain the partial differential equation associated with (12) of the form

$$(19)\qquad \begin{aligned} P\begin{pmatrix} u\theta \\ v\theta \end{pmatrix} \equiv \eta^\alpha &\left(\begin{pmatrix} h\theta \\ S(u,v) \end{pmatrix} + u\,P\begin{pmatrix} u\theta \\ v\theta \end{pmatrix}\right) \\ &- \eta^{2\alpha}\left(u\begin{pmatrix} h \\ 2\sigma_1^\theta(u)u \end{pmatrix} + \left(h + \frac{\partial}{\partial t}\right)\begin{pmatrix} u \\ v \end{pmatrix}\right)\theta \\ &+ \eta^{3\alpha} u\left(h + \frac{\partial}{\partial t}\right)\begin{pmatrix} u \\ v \end{pmatrix}\theta. \end{aligned}$$

Here the operator P is defined by

$$(20)\qquad P := \chi_\tau - Q, \qquad \chi_\tau := \nu_1 \frac{\partial}{\partial \tau_1} + \cdots + \nu_m \frac{\partial}{\partial \tau_m}.$$

Then, for a solution $(u, v) \in \mathcal{A}^2_\alpha(\Omega) := (\mathcal{A}_\alpha(\Omega))^2$ of the system (19), the $(\iota(u), \iota(v))$ becomes a formal solution of the system (12).

DEFINITION 1. *We say that a formal solution (U, V) on Ω of the system (4) is of instanton-type if (U, V) has the form $(\hat{u}_0, \hat{v}_0) + \eta^\alpha(1 - \hat{u}_0)(\iota(u), \iota(v))$ for which $(u, v) \in \mathcal{A}^2_\alpha(\Omega)$ is a solution of* (19).

2.4. Existence of instanton-type solutions for $(P_{\mathrm{I}})_m$

Now we state our main theorem whose proof is given in [3].

THEOREM 1. *Let Ω be an open subset in $\mathbb{C}_t$ and we assume the conditions* (A1) *and* (A2). *Then we have instanton-type solutions of equations of $(P_{\mathrm{I}})_m$ with free $2m$-parameters $(\beta_{-m}, \dots, \beta_m) \in \mathbb{C}^{2m}[[\eta^{-1}]]$. In particular, we can construct the solution (u, v) in $\mathcal{A}^2_\alpha(\Omega)$ for* (19) *of the form*

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{1 \le |k| \le m} f_k(\tau, t; \eta) A(\nu_k), \tag{21}$$

with

$$f_k(\tau, t; \eta) = \sum_{j=0,\, \ell=0}^{\infty} \eta^{(j+2\ell)\alpha} \left(\sum_{p \in \mathbb{Z}^m,\, |p|=j} f_{k,p,\ell}(t) e^{p\cdot\tau} \right),$$

where $|p| := |p_1| + \cdots + |p_m|$.

We can give the more precise form of f_k appearing in the above theorem. The leading term $f_{k,0}$ and the subleading term $f_{k,\alpha}$ of f_k, for example, are described by the following Lemmas 1 and 2.

LEMMA 1. *We have*

$$f_{k,0} = \omega_k e^{\tau_k} \quad (1 \le |k| \le m), \tag{22}$$

where ω_k, ω_{-k} $(1 \le k \le m)$ are multi-valued holomorphic functions on Ω of the form

$$\begin{aligned} \omega_k &= \beta_k^{(0)} \exp\left(\int^t \left(\frac{1}{\nu_k} \sum_{j=1}^{m} \phi(k,\, j) \beta_j^{(0)} \beta_{-j}^{(0)} \exp\left(-2 \int^t h_j dt \right) - h_k \right) dt \right), \\ \omega_{-k} &= \beta_{-k}^{(0)} \exp\left(\int^t \left(-\frac{1}{\nu_k} \sum_{j=1}^{m} \phi(k,\, j) \beta_j^{(0)} \beta_{-j}^{(0)} \exp\left(-2 \int^t h_j dt \right) - h_k \right) dt \right) \end{aligned} \tag{23}$$

with free $2m$-parameters $(\beta_{-m}^{(0)}, \dots, \beta_m^{(0)}) \in \mathbb{C}^{2m}$. Here $\phi(k,\, j)$ are rational functions of the variables ν_k's and h_k are holomorphic functions in Ω with the conditions

$$\phi(k,\, j) = \phi(-k,\, j) \quad (1 \le j \le m), \quad h_k = h_{-k}. \tag{24}$$

For the explicit forms of $\phi(k,\, j)$ and h_k, see [3]. Furthermore the subleading term of the solution is given by the following.

LEMMA 2. *For any k $(1 \le |k| \le m)$, the $f_{k,\alpha}$ is given by*

$$\begin{aligned} f_{k,\alpha} \quad &= \sum_{\substack{1 \le |j| \le m, \\ j \ne -k}} \frac{2}{(\nu_k + \nu_j)\nu_k \nu_j} \left((2\nu_k + \nu_j) \omega_k \omega_j e^{\tau_k + \tau_j} - \nu_j \omega_{-k} \omega_{-j} e^{-\tau_k - \tau_j} \right) \\ &\quad - \left(\sum_{j=1}^{m} \frac{\nu_j^2}{\nu_k} h_{j,k} \omega_j \omega_{-j} + \frac{6}{\nu_k} \omega_k \omega_{-k} + \frac{1}{2} \gamma_k \right) \times \frac{1}{\nu_k}. \end{aligned} \tag{25}$$

Here γ_k *are holomorphic functions in* Ω *with* $\gamma_k = \gamma_{-k}$ *and* $h_{k,j}$ *are defined by*

$$h_{k,j} := \frac{4 \prod\limits_{\substack{1 \le l \le m, \\ l \neq k,j}} \left(\nu_k^2 - \nu_l^2\right)}{\prod\limits_{\substack{1 \le l \le m, \\ l \neq j}} \left(\nu_j^2 - \nu_l^2\right)} \quad (j \neq k), \qquad h_{k,k} := \sum_{\substack{l=1, \\ l \neq k}}^{m} \frac{4}{\nu_k^2 - \nu_l^2} \tag{26}$$

with the convention $h_{k,j} := h_{|k|,|j|}$.

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