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ANALYTIC FUNCTIONS, CAUCHY KERNEL, AND CAUCHY INTEGRAL IN TUBES

Abstract. Analytic functions in tubes in association with ultradistributional boundary values are analyzed. Conditions are stated on the analytic functions satisfying a certain norm growth which force the functions to be in the Hardy space H^2. Properties of the Cauchy kernel and Cauchy integral are obtained which extend results obtained previously by the author and collaborators.

1. Introduction

The definitions of regular cone $C \subset \mathbb{R}^n$ and the corresponding dual cone C^* of C are given in [2, Chapter 1] where the notation used in this paper is also contained. The Cauchy and Poisson kernels corresponding to the tube $T^C = \mathbb{R}^n + iC \subset \mathbb{C}^n$ with $t \in \mathbb{R}^n$ are defined by

$$K(z-t) = \int_{C^*} \exp(2\pi i \langle z-t, u\rangle) du, \quad z \in T^C = \mathbb{R}^n + iC, \quad t \in \mathbb{R}^n,$$

and

$$Q(z;t) = \frac{|K(z-t)|^2}{K(2iy)}, \quad z = x+iy \in T^C = \mathbb{R}^n + iC, \quad t \in \mathbb{R}^n,$$

respectively; see [2, Chapter 1]. The sequences $M_p, p = 0,1,2,...,$ of positive integers with conditions $(M.1)$ through $(M.3')$ and the subsequently defined spaces of functions and ultradistributions of Beurling and Roumieu type $\mathcal{D}(*,L^s)$ and $\mathcal{D}'(*,L^s)$, where $*$ is either (M_p) of Beurling type or $\{M_p\}$ of Roumieu type, are given in [2, Chapter 2]. For sequences M_p which satisfy the conditions $(M.1)$ and $(M.3')$, the Cauchy kernel $K(z-t) \in \mathcal{D}(*,L^s), 1 < s \leq \infty$, [2, Theorem 4.1.1] as a function of $t \in \mathbb{R}^n$ for $z \in T^C$ where C is a regular cone in $\mathbb{R}^n$; and the Poisson kernel $Q(z;t) \in \mathcal{D}(*,L^s), 1 \leq s \leq \infty$, [2, Theorem 4.1.2] as a function of $t \in \mathbb{R}^n$ for $z \in T^C$. For $U \in \mathcal{D}'(*,L^s)$ the Cauchy and Poisson integrals are defined as $C(U;z) = \langle U_t, K(z-t)\rangle$ and $P(U;z) = \langle U_t, Q(z;t)\rangle$, respectively, for $z \in T^C$ and $t \in \mathbb{R}^n$ for appropriate values of s; see [2, Chapter 4].

In this paper we extend results in [2] concerning the norm growth of $C(U;z)$, $U \in \mathcal{D}'(*,L^s)$, to the values $1 < s < 2$. We obtain a new boundary value result for $C(U;z)$ and obtain a decomposition theorem for $U \in \mathcal{D}'(*.L^s), 1 < s < 2$. Considering functions analytic in the tube T^C which are known to have $\mathcal{D}'((M_p),L^2)$ boundary values, we impose conditions on the boundary value which force the analytic functions to be in the Hardy space $H^2(T^C)$.

2. Cauchy kernel and integral

Let the sequence M_p satisfy $(M.1)$ and $(M.3')$. For $U \in \mathcal{D}'(*, L^s), 1 < s < \infty, C(U;z)$ is an analytic function in $T^C = \mathbb{R}^n + iC$ [2, Theorem 4.2.1]; and we have a pointwise growth estimate on $C(U;z)$ ([1], [2, Theorem 4.2.2]). We have a norm growth estimate [2, Theorem 4.2.3] on $C(U;z)$ for $2 \le s < \infty$; we extend this to $1 < s < 2$ by obtaining a norm growth on $C(U;z)$ for these cases. We recall the associated function $M^*(\rho)$ given in [2, p. 15].

THEOREM 1. *Let C be a regular cone in $\mathbb{R}^n$ and let the sequence M_p satisfy properties $(M.1)$ and $(M.3')$.*

Let $U \in \mathcal{D}'((M_p), L^s), 1 < s < 2$. For $1/r + 1/s = 1$

$$\|C(U;z)\|_{L^r} \le A|y|^{-n} e^{M^*(T/|y|)}, \quad |y| \le 1. \tag{1}$$

If $n = 1$, (1) holds for $y \in C = (0,\infty)$ or $y \in C = (-\infty, 0)$ where A depends on r and s and $T > 0$ is a fixed constant. If $n \ge 2$, (1) holds for $y \in C$ in which case A depends on y, r, s, n, and C; and $T > 0$ is a fixed constant which depends on y. If $n \ge 2$, (1) also holds for $y \in C' \subset C$, for any compact subcone C' of C, in which case A depends on C, C', r, s, and n; and $T > 0$ is a fixed constant which depends on C'.

Let $U \in \mathcal{D}'(\{M_p\}, L^s), 1 < s < 2$, and $1/r + 1/s = 1$. If $n = 1$, (1) holds for $y \in C = (0,\infty)$ or $y \in C = (-\infty, 0)$ where A depends on r and s and $T > 0$ is arbitrary. If $n \ge 2$, (1) holds for $y \in C$ in which case A depends on y, r, s, n, and C; and $T > 0$ is arbitrary. If $n \ge 2$, (1) also holds for $y \in C' \subset C$, for any compact subcone C' of C, in which case A depends on C, C', r, s, and n; and $T > 0$ is arbitrary.

Proof. Both cases for $* = (M_p)$ or $* = \{M_p\}$ when the dimension $n = 1$ are proved by analysis similar to that contained in the proof of [2, Theorem 5.4.2, pp. 126–128]. By this proof we in fact have for $n = 1$

$$\|C(U;z)\|_{L^r} \le A e^{M^*(T/|y|)}$$

for $y \in (0,\infty)$ or $y \in (-\infty, 0)$; but the constant A depends on y in this case. By restricting $|y| \le 1$, (2.1) is obtained in both cases where A is independent of y.

We now prove (1) for dimension $n \ge 2$. Using [2, Theorems 2.3.1 and 2.3.2]

$$C(U;z) = \langle U_t, K(z-t) \rangle = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} F_\alpha(x,y) \tag{2}$$

where

$$F_\alpha(x,y) = \int_{\mathbb{R}^n} f_\alpha(t) D_t^\alpha K(z-t)\, dt$$

and the $f_\alpha \in L^r, 1/r + 1/s = 1$, satisfy the properties in [2, Theorems 2.3.1 and 2.3.2]. We note the estimate [3, (3.22)] on $D_t^\alpha K(z-t)$ which holds for $z = x + iy \in \mathbb{R}^n + iC$. In [3, (3.22)] the $\delta > 0$ depends on $y \in C$; whereas this δ depends on $C' \subset C$ if y is restricted to compact subcones $C' \subset C$. From this estimate [3, (3.22)] and restricting $|y| \leq 1$ we have a constant Q_δ, depending on δ, such that

$$|D_t^\alpha K(z-t)| \leq S(C^*)\Gamma(n)\pi^{-n-|\alpha|}|\alpha|^{|\alpha|}Q_\delta^{1+|\alpha|}|y|^{-n-|\alpha|}(\delta+|x-t|^2)^{-n+1};$$

and recall the other constants in this estimate from [3, (3.22)]. Using this estimate with $|y| \leq 1$,

$$|F_\alpha(x,y)| \leq S(C^*)\Gamma(n)\pi^{-n-|\alpha|}|\alpha|^{|\alpha|}Q_\delta^{1+|\alpha|}|y|^{-n-|\alpha|}\widetilde{F}_\alpha(x,y)$$

where

$$\widetilde{F}_\alpha(x,y) = \int_{\mathbb{R}^n} |f_\alpha(t)|(\delta+|x-t|^2)^{-n+1}dt$$

from which

$$|F_\alpha(x,y)| \leq S(C^*)\Gamma(n)\pi^{-n-|\alpha|}|\alpha|^{|\alpha|}Q_\delta^{1+|\alpha|}|y|^{-n-|\alpha|}Q'_{\delta,s,r} \times \left(\int_{\mathbb{R}^n} |f_\alpha(t)|^r(\delta+|x-t|^2)^{-1/2-r/4}dt\right)^{1/r}$$

follows using Hölder's inequality. Now using Fubini's theorem

$$\|F_\alpha(x,y)\|_{L^r} \leq S(C^*)\Gamma(n)\pi^{-n-|\alpha|}|\alpha|^{|\alpha|}Q_\delta^{1+|\alpha|}Q''_{\delta,s,r}|y|^{-n-|\alpha|}\|f_\alpha\|_{L^r}.$$

Using this estimate we return to (2) and obtain

$$\begin{aligned}\|C(U;z)\|_{L^r} &\leq \sum_{|\alpha|=0}^{\infty}\|F_\alpha(x,y)\|_{L^r}\\ &\leq S(C^*)\Gamma(n)\pi^{-n}Q'''_{\delta,s,r}|y|^{-n}\sum_{|\alpha|=0}^{\infty}\pi^{-|\alpha|}|\alpha|^{|\alpha|}(Q_\delta/|y|)^{|\alpha|}\|f_\alpha\|_{L^r}.\end{aligned}$$

From the proof of Stirling's formula

$$|\alpha|^{|\alpha|} \leq e^{|\alpha|}|\alpha|!, \qquad |\alpha| = 1,2,3,\ldots,$$

and we have the convention that $|\alpha|^{|\alpha|} = 1$ if $|\alpha| = 0$. Using these facts, the norm properties of f_α from [2, Theorems 2.3.1 and 2.3.2] and proceeding as in [2, (4.73) and (4.60)] the growth (1) follows where $T = 2eQ_\delta/k\pi$ for some $k > 0$ if $* = (M_p)$ Beurling and for all $k > 0$ if $* = \{M_p\}$ Roumieu. Throughout the analysis the constant Q_δ depends on $y \in C$ if y is not restricted to compact subcones $C' \subset C$. If $y \in C' \subset C$, the constant Q_δ, and hence the constants A and T, is not dependent on y but is dependent on the compact subcone $C' \subset C$. The proof of Theorem 1 is complete. □

In addition to completing the L^r norm growth properties for the considered Cauchy integral for all $s, 1 < s < \infty$, Theorem 1 shows that the Cauchy integral $C(U;z)$ studied there is an example of the type of analytic function with norm growth that we study in section 3 below in this paper.

We make a comment concerning the relation between Theorem 1 and [2, Theorem 5.4.2, p. 126]. For $y \in C$

$$|y|^{-n} e^{M^*(T/|y|)} \leq Q e^{M^*(T_1/|y|)}$$

where the constant Q does not depend on y for $T_1 > T$. The estimate obtained in the proof of [2, Theorem 5.4.2] is entirely correct, and the estimate obtained in Theorem 1 is a different one which is more precise.

The Fourier transform of a L^1 function ϕ will be symbolized by $\mathcal{F}[\phi(t);x]$ or by $\hat{\phi}(x)$ with $\mathcal{F}^{-1}[\phi(t);x]$ denoting the inverse Fourier transform. We have proved

$$\lim_{y \to 0, y \in C} \langle K(x+iy-t), \phi(x) \rangle = \mathcal{F}^{-1}[I_{C^*}(u)\hat{\phi}(u);t], \quad \phi \in \mathcal{D}(*, \mathbb{R}^n),$$

in $\mathcal{D}(*, L^s), 2 \leq s < \infty$, [2, Theorems 4.2.5 and 4.2.6]; here C is a regular cone, C^* is the dual cone, and $I_{C^*}(t)$ is the characteristic function of C^*. This result is used to obtain a boundary value result and a decomposition theorem for $U \in \mathcal{D}'(*, L^s), 2 \leq s < \infty$, [2, Corollary 4.2.1 and Theorem 4.2.7]. We extend the above limit property and subsequent results to $1 < s < 2$ for the cases that $C = (0, \infty)$ or $C = (-\infty, 0)$ in $\mathbb{R}^1$ or $C = C_\mu$ is a n-rant cone in $\mathbb{R}^n$ where

$$C_\mu = \{y \in \mathbb{R}^n : \mu_j y_j > 0, j = 1, ..., n\}, \quad \mu_j \in \{-1, 1\}, \quad j = l, ..., n.$$

THEOREM 2. *Let C_μ be any n-rant cone in $\mathbb{R}^n$, and let $I_{C_\mu^*}$ be the characteristic function of the dual cone $C_\mu^* = \overline{C}_\mu$. Let $\phi \in \mathcal{D}(*, \mathbb{R}^n)$ where the sequence M_p satisfies the properties $(M.1)$, $(M.2)$, and $(M.3')$. We have*

$$\lim_{y \to 0, y \in C_\mu} \langle K(x+iy-t), \phi(x) \rangle = \int_{\mathbb{R}^n} I_{C_\mu^*}(u)\hat{\phi}(u) e^{-2\pi i \langle t,u \rangle} du$$

in $\mathcal{D}(*, L^s), 1 < s < 2$.

Proof. Since the n-rant cone C_μ, its dual cone $C_\mu^* = \overline{C}_\mu$, and the corresponding Cauchy kernel function are products of one-dimensional half lines and the one-dimensional Cauchy kernel function, it is sufficient to prove the result in one dimension. We give an outline of the proof for the case that $C = (0, \infty)$. For $\phi \in \mathcal{D}(*, \mathbb{R}^n)$ we know

$$\mathcal{F}[D_x^\alpha \phi(x); u] = u^\alpha \mathcal{F}[\phi(x); u].$$

As noted in [2, p. 14], condition $(M.2)$ on the sequence M_p implies the existence of constants A and H larger than 1 such that

$$M_{p+q} \leq A H^{p+q} M_p M_q.$$

Using these facts and integration by parts techniques we prove the following for the cone $C = (0, \infty)$ with $1 < s < 2$:

$$\langle K(x+iy-t), \phi(x) \rangle \in \mathcal{D}(*, L^s), \quad t \in \mathbb{R}^1, \quad y \in C;$$

$$\int_0^\infty \hat{\phi}(u)e^{-2\pi itu}du \in \mathcal{D}(*,L^s), \quad t \in \mathbb{R}^1;$$

$$\left\| D_t^\alpha(\langle K(x+iy-t),\phi(x)\rangle - \int_0^\infty \hat{\phi}(u)e^{-2\pi itu}du)\right\|_{L^s} \le Nh^\alpha M_\alpha,$$

$\alpha = 0,1,2,\ldots,$ for every $h>0,(M_p)$ Beurling, or for some $h>0,\{M_p\}$ Roumieu, with $N>0$ independent of $y>0$ and α; and

$$\lim_{y\to 0,\, y\in(0,\infty)} \|D_t^\alpha(\langle K(x+iy-t),\phi(x)\rangle - \int_0^\infty \hat{\phi}(u)e^{-2\pi itu}du)\|_{L^s} = 0,$$

$\alpha = 0,1,2,\ldots,$ which proves the result. □

As noted above Theorem 2 extends [2, Theorems 4.2.5 and 4.2.6] to the cases $1<s<2$ for half line cones $C=(0,\infty)$ and $C=(-\infty,0)$ and for n-rant cones $C=C_\mu$.

The following result extends [2, Corollary 4.2.1] to the cases $1<s<2$ for the n-rant cones $C=C_\mu$ considered in Theorem 2.

THEOREM 3. *Let $U\in\mathcal{D}'(*,L^s),1<s<2,$ and $\phi\in\mathcal{D}(*,\mathbb{R}^n)$. Let the sequence M_p satisfy $(M.1),(M.2),$ and $(M.3')$. We have*

$$\lim_{y\to 0,\, y\in C_\mu} \langle C(U;x+iy),\phi(x)\rangle = \left\langle U, \int_{\mathbb{R}^n} I_{C_\mu^*}(u)\hat{\phi}(u)e^{-2\pi i\langle t,u\rangle}du\right\rangle.$$

Proof. Using the change of order of integration formula [2, Theorem 4.2.4], Theorem 2, and the continuity of $U\in\mathcal{D}'(*,L^s)$ we have

$$\begin{aligned}\lim_{y\to 0,\, y\in C_\mu} \langle C(U;x+iy),\phi(x)\rangle &= \lim_{y\to 0, y\in C_\mu} \langle U,\langle K(x+iy-t),\phi(x)\rangle\rangle \\ &= \left\langle U, \int_{\mathbb{R}^n} I_{C_\mu^*}(u)\hat{\phi}(u)e^{-2\pi i\langle t,u\rangle}du\right\rangle.\end{aligned}$$

□

Now we may obtain a decomposition result for $U\in\mathcal{D}'(*,L^s),1<s<2$, similar to that which we have obtained for $2\le s<\infty$ in [2, Theorem 4.2.7]. For each C_μ we form

$$f_\mu(z) = \left\langle U, \int_{C_\mu^*} \exp(2\pi i\langle z-t,u\rangle)du\right\rangle, \quad z\in T^{C_\mu},$$

and note that there are 2^n n-tuples μ. As in the proof of [2, Theorem 4.2.7] we use Theorem 3 here and obtain

$$\langle U,\phi\rangle = \langle U,\sum_\mu \int_{C_\mu^*} \hat{\phi}(u)e^{-2\pi i\langle t,u\rangle}du\rangle = \sum_\mu \lim_{y\to 0, y\in C_\mu} \langle f_\mu(x+iy),\phi(x)\rangle$$

for $U\in\mathcal{D}'(*,L^s),1<s<2,$ and $\phi\in\mathcal{D}(*,\mathbb{R}^n)$. This extends [2, Theorem 4.2.7] to $1<s<2$ for n-rant cones $C=C_\mu$.

3. Analytic functions

Let B denote a proper open subset of $\mathbb{R}^n$, and let $d(y)$ denote the distance from $y \in B$ to the complement of B in $\mathbb{R}^n$. In [2, Chapter 5] we have considered analytic functions in tubes $T^B = \mathbb{R}^n + iB$ satisfying

$$\|f(x+iy)\|_{L^r} \leq K(1+(d(y))^{-m})^q e^{M^*(T/|y|)}, \quad y \in B, \tag{3}$$

where $K > 0, T > 0, m \geq 0$, and $q \geq 0$ are all independent of $y \in B$ and $M^*(\rho)$ is the associated function of the sequence M_p defined in [2, p. 15].

For $B = C$, a regular cone in $\mathbb{R}^n$, we have shown in [2, section 5.2] that analytic functions $f(z), z \in T^C$, which satisfy (3) for $m = 0$ or $q = 0$ and $1 < r \leq 2$, obtain a boundary value $U \in \mathcal{D}'((M_p), L^1)$ as $y \to 0, y \in C$, [2, Theorem 5.2.1]. A converse result is proved in [2, Theorem 5.2.2]. In this converse result we can now easily prove as an additional conclusion that

$$f(z) = \langle U_t, K(z-t)\rangle, \quad z \in T^C,$$

using the proof of [2, Theorem 5.2.2]; that is, in [2, Theorem 5.2.2] we can add as a conclusion that the analytic function $f(z)$ constructed there can be recovered as the Cauchy integral of its boundary value.

Additionally we note that the result [2, Theorem 5.3.1], and hence the results [2, Theorems 5.3.2 and 5.3.3], can be stated and proved under the more general hypothesis that the set C is any open connected subset of $\mathbb{R}^n$ which is contained in or is any of the 2^n n-rants C_μ in $\mathbb{R}^n$. The only sacrifice in the conclusion is that the support of the constructed function $g(t)$ can not be determined under this more general hypothesis.

Let us recall the Hardy H^r functions in tubes $T^C = \mathbb{R}^n + iC$, for C being a regular cone, which have been studied extensively by Stein and Weiss [5]. An analytic function $f(z), z \in T^C$, is in the Hardy space $H^r = H^r(T^C), r > 0$, if

$$\|f(x+iy)\|_{L^r} \leq A, \quad y \in C,$$

where the constant $A > 0$ is independent of $y \in C$. In [4] we showed that if an analytic function $f(z), z \in T^C$, has a distributional boundary value in $\mathcal{S}'$ which is a $L^r, 1 \leq r \leq \infty$, function, the analytic function must be in H^r. Results of this type have applications in quantum field theory.

The Hardy spaces H^r are subspaces of the analytic functions in T^C which satisfy (3) for $m = 0$ or $q = 0$, which are the analytic functions we considered in [2, section 5.2] with respect to the existence of boundary values in $\mathcal{D}'((M_p), L^r)$. Thus for the values of r that we have considered in [2, section 5.2], $f(z) \in H^r$ will have an ultradistributional boundary value. We now obtain a result, like those in [4], in which we show for $r = 2$ that any analytic function $f(z), z \in T^C$, which satisfies (3) with $m = 0$ or $q = 0$ and with $r = 2$ and whose boundary value in $\mathcal{D}'((M_p), L^2)$, which exists by [2, Corollary 5.2.3], is a bounded L^2 function in $\mathcal{D}'((M_p), L^2)$ must be a H^2 function.

THEOREM 4. *Let $f(z)$ be analytic in T^C, C being a regular cone, and satisfy*

$$\|f(x+iy)\|_{L^2} \leq Ke^{M^*(T/|y|)}, y \in C. \tag{4}$$

Let the $\mathcal{D}'((M_p),L^2)$ boundary value of $f(z)$ be a bounded function $h \in \mathcal{D}'((M_p),L^2)$. We have $f(z) \in H^2(T^C)$ and

$$f(z) = \int_{\mathbb{R}^n} h(t)K(z-t)dt = \int_{\mathbb{R}^n} h(t)Q(z;t)dt, \quad z \in T^C.$$

Proof. From [2, Corollary 5.2.3] and its proof we have

$$f(z) = \int_{\mathbb{R}^n} h(t)K(z-t)dt = \int_{\mathbb{R}^n} g(t)e^{2\pi i\langle z,t\rangle}dt, \quad z \in T^C, \tag{5}$$

where $\text{supp}(g) \subseteq C^*$ almost everywhere and $h = \mathcal{F}^{-1}[\tilde{g}]$ with this inverse Fourier transform being an element in $\mathcal{D}'((M_p),L^2)$ [2, (2.52), p. 27]. Now let $w = u+iv \in T^C$ be arbitrary but fixed and consider $K(z+w)f(z), z \in T^C$, where

$$K(z+w) = \int_{C^*} \exp(2\pi i\langle z+w,u\rangle)du.$$

Using [4, Lemma 3.2] we have that $K(z+w)$ is analytic in $z \in T^C$ and

$$|K(z+w)| \leq M_v < \infty, \quad z \in T^C,$$

where $M_v > 0$ is a constant that depends only on $v = \text{Im}(w)$. Thus $K(z+w)f(z)$ is analytic in $z \in T^C$ and satisfies

$$\|K(x+iy+w)f(x+iy)\|_{L^2} \leq KM_v e^{M^*(T/|y|)}, \quad y \in C,$$

with M_v being independent of $z \in T^C$. We have $K(x+iy+w)f(x+iy) \to K(x+w)h(x)$ in $\mathcal{D}'((M_p).L^2)$ as $y \to 0, y \in C$; and $K(x+w)h(x) \in \mathcal{D}'((M_p),L^2)$ since $K(x+w)$ is bounded in $x \in \mathbb{R}^n$. By the proof of [2, Corollary 5.2.3] applied to $K(z+w)f(z), z \in T^C$, we have

$$K(z+w)f(z) = \int_{\mathbb{R}^n} K(t+w)h(t)K(z-t)dt, z \in T^C, \tag{6}$$

for any fixed $w \in T^C$. Now corresponding to $z = x+iy \in T^C$ choose $w = -x+iy \in T^C$ and obtain

$$K(t+w)K(z-t) = |K(z-t)|^2$$

and

$$K(z+w) = K(2iy).$$

With this choice of $w = -x+iy \in T^C$, (6) becomes

$$f(z) = \int_{\mathbb{R}^n} h(t)Q(z;t)dt, \quad z \in T^C, \tag{7}$$

where $Q(z;t)$ is the Poisson kernel for $z \in T^C$ and $t \in \mathbb{R}^n$. From (7) and the proof of [4, Lemma 3.5] we have

$$\|f(x+iy)\|_{L^2} \leq \|h\|_{L^2} < \infty, \quad y \in C;$$

and $f(z) \in H^2(T^C)$. □

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