

M. Codegone

COMPENSATED COMPACTNESS IN HOMOGENIZATION THEORY

Abstract. Homogenization theory studies the macroscopic behavior of a non-homogeneous medium with a fine periodic structure. If the size of the period goes to zero the solution of the problem, with some suitable hypotheses, converges to the solution of a so-called homogenized problem. The limit problem is achieved through a limit process of a product of weakly convergent function sequences. In order to identify the limit product one can sometimes make use of appropriate test functions, but in general the right tool is compensated compactness. This tool makes it possible to pass to the limit in a product of weakly convergent functions under hypotheses where the curl of one factor is bounded and the divergence of the other is compact. In this way homogenization theory becomes an interesting example where compensated compactness has an important role.

1. Introduction

In the paper [7] A. Negro presents a time periodic boundary value problem for quasi-stationary Maxwell equations in a non-homogeneous multiply connected domain Ω (see figure 1). In this formulation some links are linear, but the characteristic relating the magnetic field H to the magnetic induction B is non-linear. The non-homogeneity of the medium has a natural approximation with a periodic fine structure of period εY and discloses homogenization theory as a worthwhile tool. In the papers [4] and [3] conductivity $\sigma(y)$ is linear, strictly positive and bounded in the simply connected region $Y_1 \subseteq Y$ and is zero in $Y_2 \subseteq Y$, where $Y = Y_1 \cup Y_2$ is the reference period, see, in figure 2, the two-dimensional section of the reference period Y and of the periodical reproduction of εY . The non-linear magnetic characteristic $\mathcal{X}^\varepsilon(x, H) = \mathcal{X}(\frac{x}{\varepsilon}, H)$ is supposed to be measurable, Lipschitz continuous and strictly monotone in H .

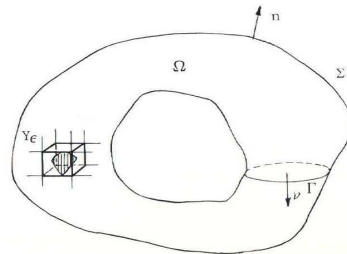


Figure 1: The domain Ω

The problem formulation then presents a linear part related to conductivity σ^ε and a non-linear part related to the magnetic characteristic \mathcal{X}^ε . The parameter εY is the

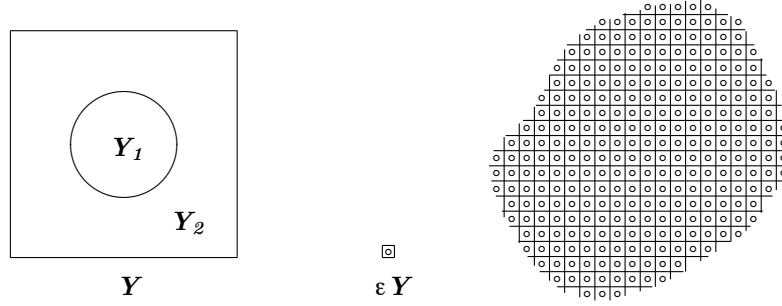


Figure 2: Two-dimensional section of the reference period \mathbf{Y} and of the periodical reproduction of $\varepsilon \mathbf{Y}$.

period of the non-homogeneity of the medium and then the limit, when ε goes to zero, is taken in the linear part σ^ε and in the non-linear part \mathcal{X}^ε . In order to emphasize the use of compensated compactness, we simplify the problem taking into account only the non linear characteristic in a more simple contest.

2. Variational formulation

We denote by

$$\chi : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

a function such that, for every $H \in \mathbb{R}^3$, $\chi(\cdot, H)$ is Lebesgue measurable and εY -periodic and there exist two constants l and L with $0 < l \leq L < +\infty$ such that

$$\begin{aligned} (\chi(x, H) - \chi(x, H'), H - H') &\geq l |H - H'|^2 \\ |\chi(x, H) - \chi(x, H')| &\leq L |H - H'| \end{aligned}$$

for a.e. $x \in \mathbb{R}^n$ and for every $H, H' \in \mathbb{R}^n$ and $\chi(x, 0) = 0$ for a.e. $x \in \mathbb{R}^n$.

Let us consider the following Dirichlet boundary value problem on the bounded open subset Ω of \mathbb{R}^n

$$(1) \quad \begin{cases} -\operatorname{div}(\chi(\frac{x}{\varepsilon}, \nabla u^\varepsilon)) = f_\varepsilon & \text{on } \Omega \\ u^\varepsilon \in H_0^1(\Omega) \end{cases}$$

with f_ε strongly converging in $H^{-1}(\Omega)$ to f . We can see that

$$(2) \quad \begin{cases} \forall \varphi \in C_0^\infty(\Omega) & \text{e} & \varphi \geq 0 \\ \int_\Omega ((\chi(x/\varepsilon, H) - \chi(x/\varepsilon, H')) \cdot (H - H')) \varphi(x) dx \geq 0 \end{cases}$$

Now we recall some results in homogenization theory.

3. Homogenization results

In order to obtain the limit process (see [8], [9] and [1]) of equation (1) we have to solve a problem in the reference period Y . Find a function $w^\lambda(y)$ such that

$$(3) \quad \begin{cases} -\operatorname{div}(\chi(y, \lambda + \nabla w^\lambda(y))) = 0 & \text{in } D'(\mathbb{R}^n) \\ w^\lambda(y) \in H^1(Y), Y\text{-periodic and with mean value zero.} \end{cases}$$

In homogenization theory it is proved that the problem (3) admits a unique solution. We note

$$(4) \quad w_\varepsilon^\lambda(x) = \lambda \cdot x + \varepsilon w^\lambda\left(\frac{x}{\varepsilon}\right)$$

then, replacing H and H' by ∇u^ε and $\nabla w_\varepsilon^\lambda$, in equation (2), we get

$$(5) \quad \int_{\Omega} \left((\chi(x/\varepsilon, \nabla u^\varepsilon) - \chi(x/\varepsilon, \nabla w_\varepsilon^\lambda)) \cdot (\nabla u^\varepsilon - \nabla w_\varepsilon^\lambda) \right) \varphi(x) dx \geq 0.$$

We remark that any term in the equation 5 weakly converges:

$$(6) \quad \begin{cases} \chi(x/\varepsilon, \nabla u^\varepsilon) & \rightharpoonup \chi^0 \\ \chi(x/\varepsilon, \nabla w_\varepsilon^\lambda) & \rightharpoonup \frac{1}{|Y|} \int_Y \chi(y, \lambda + \nabla w^\lambda(y)) dy \\ \nabla u^\varepsilon & \rightharpoonup \nabla u^0 \\ \nabla w_\varepsilon^\lambda & \rightharpoonup \lambda \end{cases}$$

and then any factor weakly converges:

$$(7) \quad \begin{cases} (\chi(x/\varepsilon, \nabla u^\varepsilon) - \chi(x/\varepsilon, \nabla w_\varepsilon^\lambda)) & \rightharpoonup \left(\chi^0 - \frac{1}{|Y|} \int_Y \chi(y, \lambda + \nabla w^\lambda(y)) dy \right) \\ (\nabla u^\varepsilon - \nabla w_\varepsilon^\lambda) & \rightharpoonup (\nabla u^0 - \lambda). \end{cases}$$

In order to pass to the limit in (5) we need the result of compensated compactness.

4. Compensated compactness

The product of the two brackets in (5) is only weakly convergent and it is known that the product of two weakly convergent sequences does not converge, in general, to the product of the limits, and this is the main difficulty to characterize χ^0 in terms of u^0 . Compensated compactness (see [6], [10]) shows that under some additional assumptions, the product of the two weak convergent sequences converges in the sense of distribution to the product of the limits. We present a general lemma and a particular version of the result which interests the problem of homogenization.

LEMMA 1. *Let V^ε and U^ε be two sequences converging weakly in $L^2(\Omega)$ to V^0 and U^0 respectively. Moreover suppose that:*

$$\begin{cases} \operatorname{div} V^\varepsilon & \text{converges strongly in } H^{-1}(\Omega) \\ \operatorname{curl} U^\varepsilon & \text{is bounded in } L^2(\Omega) \end{cases}$$

then one has

$$\int_{\Omega} (V^\varepsilon U^\varepsilon) \varphi \, dx \quad \longrightarrow \quad \int_{\Omega} (V^0 U^0) \varphi \, dx$$

for every $\varphi \in C_0^\infty(\Omega)$.

It is possible to see a proof of the Lemma 1 in [5] and [2].

PROPOSITION 1. *In the hypothesis of equations (1) and (3) we obtain that expression (5) converges:*

$$(8) \quad \begin{aligned} & \int_{\Omega} \left((\chi(x/\varepsilon, \nabla u^\varepsilon) - \chi(x/\varepsilon, \nabla w_\varepsilon^\lambda)) \cdot (\nabla u^\varepsilon - \nabla w_\varepsilon^\lambda) \right) \varphi(x) \, dx \longrightarrow \\ & \longrightarrow \int_{\Omega} \left(\left(\chi^0 - \frac{1}{|Y|} \int_Y \chi(y, \lambda + \nabla w^\lambda(y)) \, dy \right) \cdot (\nabla u^0 - \lambda) \right) \varphi(x) \, dx. \end{aligned}$$

Proof. Taking into account equations (1) and (3), we have

$$\begin{cases} -\operatorname{div}(\chi(x/\varepsilon, \nabla u^\varepsilon) - \chi(x/\varepsilon, \nabla w_\varepsilon^\lambda)) = f_\varepsilon \\ \operatorname{curl}(\nabla u^\varepsilon - \nabla w_\varepsilon^\lambda) = 0 \end{cases}$$

and, in order to apply the Lemma 1, we put

$$\begin{cases} V^\varepsilon = (\chi(x/\varepsilon, \nabla u^\varepsilon) - \chi(x/\varepsilon, \nabla w_\varepsilon^\lambda)) \\ U^\varepsilon = (\nabla u^\varepsilon - \nabla w_\varepsilon^\lambda) \end{cases}$$

and the result is achieved. We can add another proof of this Proposition. We observe that the limit (8) has a direct development by integration by parts

$$(9) \quad \begin{aligned} & \int_{\Omega} \left((\chi(x/\varepsilon, \nabla u^\varepsilon) - \chi(x/\varepsilon, \nabla w_\varepsilon^\lambda)) \cdot (\nabla u^\varepsilon - \nabla w_\varepsilon^\lambda) \right) \varphi(x) \, dx = \\ & =_{H^{-1}} \langle -\operatorname{div}(\chi(x/\varepsilon, \nabla u^\varepsilon) - \chi(x/\varepsilon, \nabla w_\varepsilon^\lambda)), (u^\varepsilon - w_\varepsilon^\lambda) \varphi \rangle_{H_0^1} + \\ & - \int_{\Omega} \left((\chi(x/\varepsilon, \nabla u^\varepsilon) - \chi(x/\varepsilon, \nabla w_\varepsilon^\lambda)) \cdot \nabla \varphi(x) \right) (u^\varepsilon - w_\varepsilon^\lambda) \, dx \end{aligned}$$

for every $\varphi \in C_0^\infty$. At the left-hand side of equation (9) there is product of two sequences which converge only in the weak topology, but at the right-hand side the convergence of $u^\varepsilon - w_\varepsilon^\lambda$ is weakly in H^1 and strongly in L^2 :

$$(10) \quad u^\varepsilon - w_\varepsilon^\lambda \quad \longrightarrow \quad u^0 - (\lambda \cdot x).$$

The strong convergence (10) permits to pass to the limit and the statement is achieved. \square

COROLLARY 1. *Let u^ε , u^0 , w_ε^λ , w^λ , χ and χ^0 be defined as in section 2, 3 and let $B(\lambda)$ be defined by*

$$B(\lambda) = \frac{1}{|Y|} \int_Y \chi(y, \lambda + \nabla w^\lambda(y))$$

for every $\lambda \in \mathbb{R}^3$. Then

$$(11) \quad \chi^0 = B(\nabla u^0)$$

and

$$(12) \quad \begin{cases} -\operatorname{div}(B(\nabla u^0)) = f & \text{on } \Omega \\ u^0 \in H_0^1(\Omega). \end{cases}$$

Proof. From (5) and (8) we get

$$(13) \quad \int_{\Omega} \left((\chi^0 - B(\lambda)) \cdot (\nabla u^0 - \lambda) \right) \varphi(x) \, dx \geq 0.$$

By taking into account that B is maximal monotone the last inequality (13) ensures identity (11) and then, taking the limit, in the problem (1) we get the homogenized problem (12). \square

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Marco CODEGONE

Department of Mathematical Sciences, Politecnico di Torino
 Corso Duca degli Abruzzi, 24, 10129 Torino, ITALY
 e-mail: marco.codegone@polito.it

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