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ON A CONJECTURE OF DE GIORGI CONCERNING NONLINEAR WAVE EQUATIONS

Abstract. We discuss a conjecture by De Giorgi, which states that global weak solutions to the Cauchy problem associated to certain nonlinear wave equations can be obtained as limits of minimizers of suitable convex functionals. There is no restriction on the growth of the nonlinearity, and the method is easily extended to more general equations.

Dedicated to Angelo Negro on the occasion of his 70th birthday.

1. The conjecture

In this talk I will report on a joint work with Paolo Tilli, discussing a conjecture of Ennio De Giorgi related to some classes of nonlinear wave equations.
We consider minimization/evolution problems in space time, $\mathbb{R}\times\mathbb{R}^n$, $n\geq 0$; the accent on *minimization* or *evolution* depends on the point of view, and as we will see this is at the core of the problem.

In a paper published in the Duke Mathematical Journal, [1], De Giorgi stated the following conjecture.

CONJECTURE 1. Let $p\in\mathbb{N}$ be an even number. For $\varepsilon>0$, let $v_\varepsilon(t,x)$ denote the minimizer of the convex functional

$$F_\varepsilon(v)=\int_0^\infty\int_{\mathbb{R}^n}e^{-t/\varepsilon}\big\{|v''(t,x)|^2+\frac{1}{\varepsilon^2}|\nabla v(t,x)|^2+\frac{1}{\varepsilon^2}|v(t,x)|^p\big\}\,dxdt$$

subject to the boundary conditions

$$v(0,x)=\alpha(x),\quad v'(0,x)=\beta(x),\quad x\in\mathbb{R}^n,$$

where $\alpha,\beta\in C_0^\infty(\mathbb{R}^n)$ are given functions. Then, for almost every $(t,x)\in\mathbb{R}^+\times\mathbb{R}^n$, the limit

$$w(t,x)=\lim_{\varepsilon\downarrow 0}v_\varepsilon(t,x)$$

exists and the function $w(t,x)$ solves in $\mathbb{R}^+\times\mathbb{R}^n$ the nonlinear wave equation

$$w''-\Delta w+\frac{p}{2}w^{p-1}=0 \tag{1}$$

with initial conditions

$$w(0,x)=\alpha(x),\quad w'(0,x)=\beta(x),\quad x\in\mathbb{R}^n. \tag{2}$$

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REMARK 1. Existence and uniqueness of a minimizer for the functional F_ε are straightforward. Basically one can consider the largest space of $L^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^n)$ functions where F_ε is finite and minimize among functions that satisfy the boundary condition (in a suitable sense). Coercivity and strict convexity easily provide existence and uniqueness of a minimizer v_ε.

The above conjecture casts a completely new bridge between *hard evolution problems* and more easily tractable *convex minimization problems*. Indeed, if proven true, it provides a method to approximate nonlinear (defocusing) wave equations by convex minimization problems. The variational approach is by genuine minimization, and not by Critical Point Theory, where one would have to use functionals that behave rather badly from the point of view of existence results. Notice also that the nonlinearity exponent p can be arbitrarily large.

We also point out that the approach is new (in spirit) even for the linear wave equation $w'' - \Delta w = 0$ or for the linear Klein–Gordon equation $w'' - \Delta w + w = 0$.

A further point of interest is the possibility to extend the method to other classes of evolution equations.

A proof of this conjecture has to face a series of difficulties. Among others, we list the following ones.

- The functionals involve first order spatial derivatives, but second order time derivatives.
- The weight $e^{-t/\varepsilon}$ in each single functional (ε fixed) decays very rapidly as $t \to \infty$.
- For fixed $t_2 > t_1$, the weight ratio $e^{-t_1/\varepsilon}/e^{-t_2/\varepsilon}$ diverges as $\varepsilon \to 0$.
- The time–scale depends on ε, making it difficult to compare two minimizers v_{ε_1} and v_{ε_2}.
- As $\varepsilon \to 0$, $e^{-t/\varepsilon}$ concentrates close to $t = 0$, and rescaled functionals Γ-converge to a constant functional, thereby exhibiting a strong loss of information.

The following is our main result.

THEOREM 1 ([2]). *For every real $p \geq 2$ and for initial data α, β in $H^1 \cap L^p$, the conjecture is true, up to subsequences.*

REMARK 2. Passing to subsequences is *not* necessary if the Cauchy problem (1)–(2) has uniqueness. However uniqueness for this problem is not known for large p.

REMARK 3. The solution of the Cauchy problem (1)–(2) obtained in the above theorem is of *energy class*, i.e. the function

$$\mathcal{E}(t) := \int_{\mathbb{R}^n} \left(|w'(t,x)|^2 + |\nabla w(t,x)|^2 + |w(t,x)|^p\right) dx$$

satisfies the energy inequality $\mathcal{E}(t) \leq \mathcal{E}(0)$.

We recall that *conservation* of energy for the Cauchy problem (1)–(2) is not known for large p.

2. The main ideas of the proof

We now sketch some of the main ideas involved in the proof. It is clear that, in order to pass to the limit in the Euler–Lagrange equation associated to the functionals F_ε, some estimates are needed. The type of estimates that we obtain, and that are sufficient to complete the limit procedure, can be summarized in the following list.

- A localized L^2 estimate for ∇v_ε, with values in $L^2(\mathbb{R}^n)$:

$$\int_t^{t+T} \int_{\mathbb{R}^n} |\nabla v_\varepsilon(s,x)|^2 \, dx\, ds \leq CT, \qquad t \geq 0, \quad T \geq \varepsilon.$$

- A localized L^p estimate for v_ε, with values in $L^p(\mathbb{R}^n)$:

$$\int_t^{t+T} \int_{\mathbb{R}^n} |v_\varepsilon(s,x)|^p \, dx\, ds \leq CT, \qquad t \geq 0, \quad T \geq \varepsilon.$$

- A global L^∞ estimate for v'_ε, with values in $L^2(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} |v'_\varepsilon(t,x)|^2 \, dx \leq C, \qquad t \geq 0.$$

These estimates provide convergence (up to subsequences) to some $w(t,x)$, with

$$w \in L^\infty(\mathbb{R}^+; L^p), \quad \nabla w \in L^\infty(\mathbb{R}^+; L^2), \quad w' \in L^\infty(\mathbb{R}^+; L^2),$$

for which the energy function

$$\mathcal{E}(t) := \int_{\mathbb{R}^n} \big(|w'|^2 + |\nabla w|^2 + |w|^p\big)\, dx$$

is finite for a.e. $t > 0$.

Moreover, w solves (in weak sense) the wave equation

$$w'' - \Delta w + \frac{p}{2}|w|^{p-2} w = 0,$$

as one sees by passing to the limit in the Euler–Lagrange equation of v_ε. In this context, it is interesting to note that the weight $e^{-t/\varepsilon}$ can be absorbed inside the test function during the limit process.

Indeed, let $\eta \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$ be a test function. Since v_ε is the global minimizer for F_ε, it satisfies the Euler–Lagrange equation that, written in weak form, is

$$\int_0^\infty \int_{\mathbb{R}^n} e^{-t/\varepsilon} \big(\varepsilon^2 v''_\varepsilon \eta'' + \nabla v_\varepsilon \nabla \eta + \frac{p}{2}|v_\varepsilon|^{p-2} v_\varepsilon \eta\big)\, dx\, dt = 0.$$

Integrating once by parts in time yields

$$\int_0^\infty \int_{\mathbb{R}^n} \big(-\varepsilon^2 v_\varepsilon'\big(e^{-t/\varepsilon}\eta''\big)' + e^{-t/\varepsilon}(\nabla v_\varepsilon \nabla \eta + \frac{p}{2}|v_\varepsilon|^{p-2}v_\varepsilon \eta)\big)\,dx\,dt = 0.$$

Now choosing $\eta = e^{t/\varepsilon}\varphi$, with $\varphi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$, the preceding identity reads

$$\int_0^\infty \int_{\mathbb{R}^n} \big(-v_\varepsilon'(\varepsilon^2\varphi'' + 2\varepsilon\varphi' + \varphi)' + \nabla v_\varepsilon \nabla \varphi + \frac{p}{2}|v_\varepsilon|^{p-2}v_\varepsilon \varphi\big)\,dx\,dt = 0$$

As $\varepsilon \to 0$, from $v_\varepsilon \to w$ (weakly in H^1, strongly in $L^{p-1}, \dots$) we obtain

$$\int_0^\infty \int_{\mathbb{R}^n} \big(-w'\varphi' + \nabla w \nabla \varphi + \frac{p}{2}|w|^{p-2}w\varphi\big)\,dx\,dt = 0 \qquad \forall \varphi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n),$$

namely the weak form of the wave equation

$$w'' - \Delta w + \frac{p}{2}|w|^{p-2}w = 0.$$

REMARK 4. Also the two initial conditions

$$v_\varepsilon(0,x) = \alpha(x) \qquad \text{and} \qquad v_\varepsilon'(0,x) = \beta(x)$$

pass to the limit as $\varepsilon \to 0$. For the former, the $L^\infty(\mathbb{R}^+; L^2)$ bound on v_ε' is enough. For the latter, we need estimates on v_ε'', uniform in ε. These are obtained in L^∞, with values in the dual of $H^1 \cap L^p$, by a careful choice of test functions in the Euler–Lagrange equation for v_ε.

We now sketch the main argument to obtain the *a priori* estimates that allowed us to carry out the preceding limit procedure. First of all it is convenient to get rid of the parameter ε in the weight: setting

$$u_\varepsilon(t,x) = v_\varepsilon(\varepsilon t, x),$$

we see that v_ε minimizes F_ε if and only if u_ε minimizes

$$J_\varepsilon(u) = \int_0^\infty \int_{\mathbb{R}^n} e^{-t}\big(|u''|^2 + \varepsilon^2|\nabla u|^2 + \varepsilon^2|u|^p\big)\,dx\,dt$$

with boundary contidions

$$\begin{cases} u(0,x) = \alpha \\ u'(0,x) = \varepsilon\beta \end{cases}$$

Precisely, $J_\varepsilon(u_\varepsilon) = \varepsilon F_\varepsilon(v_\varepsilon)$.

Now a crucial role is played by the function

$$E(t) = \int_{\mathbb{R}^n} |u_\varepsilon'|^2\,dx - 2\int_{\mathbb{R}^n} u_\varepsilon' u_\varepsilon''\,dx + e^t \int_t^\infty e^{-s}L(s)\,ds$$

where L is

$$L(s)=\int_{\mathbb{R}^n}|u''_\varepsilon(s,x)|^2+\varepsilon^2|\nabla u_\varepsilon(s,x)|^2+\varepsilon^2|u_\varepsilon(s,x)|^p\,dx.$$

The function $E:\mathbb{R}^+\to\mathbb{R}$ is a sort of *energy*, and indeed it is strongly related to the energy $\mathcal{E}$ of the wave equation. Its properties are summarized int he following result.

THEOREM 2 (Energy lemma). *Let u_ε be the minimizer for J_ε and let*

$$E(t)=\int_{\mathbb{R}^n}|u'_\varepsilon|^2\,dx-2\int_{\mathbb{R}^n}u'_\varepsilon u''_\varepsilon\,dx+e^t\int_t^\infty e^{-s}L(s)\,ds.$$

Then E is positive and decreasing; precisely

$$E'=-4\int_{\mathbb{R}^n}|u''_\varepsilon|^2\,dx \qquad \textit{in the sense of distributions}$$

and

$$0\le\frac{1}{\varepsilon^2}E(t)\le\mathcal{E}(0)+O(\varepsilon),$$

where

$$\mathcal{E}(0):=\int_{\mathbb{R}^n}\left(\beta^2+|\nabla\alpha|^2+|\alpha|^p\right)dx.$$

The proof of this result *could* be obtained, formally, by multiplying by u'_ε the Euler–Lagrange equation, but the integral

$$\int_{\mathbb{R}^n}|u_\varepsilon|^{p-2}u_\varepsilon u'_\varepsilon\,dx$$

is (a priori) meaningless for large p.

Instead, we make use of inner variations: we build competitors for u_ε of the form

$$U_\delta(t,x)=u_\varepsilon(t+\delta\eta(t),x),\quad \eta\in C_0^\infty(\mathbb{R}^+),$$

and compute $\dfrac{d}{d\delta}J_\varepsilon(U_\delta)$ at $\delta=0$. This is essentially the procedure that is used to derive the Du Bois–Reymond equation in the Calculus of Variations.

The other tools to complete the argument are the following.

- A level estimate:

$$J_\varepsilon(u_\varepsilon)\le J_\varepsilon(\alpha+\varepsilon t\beta)\le C\varepsilon^2.$$

- An energy estimate:

$$E(0)=\varepsilon^2\mathcal{E}(0)+O(\varepsilon^3)\le C\varepsilon^2.$$

- A consequence of the Energy lemma:

$$\int_{\mathbb{R}^n}|u'_\varepsilon(t)|^2dx+e^t\int_t^\infty\int_s^\infty e^{-\tau}L(\tau)\,d\tau\,ds\le E(t)\le E(0)\le C\varepsilon^2.$$

Setting
$$H(t)=\int_t^\infty e^{-\tau}L(\tau)\,d\tau,$$
the last inequality can be written more concisely

$$\int_{\mathbb{R}^n}|u'_\varepsilon(t)|^2dx+e^t\int_t^\infty H(s)\,ds\le C\varepsilon^2. \tag{3}$$

From this we first derive a pointwise estimate on H. Since H is decreasing by definition,
$$H(t+1)\le\int_t^{t+1}H(s)\,ds\le\int_t^\infty H(s)\,ds$$
Multiplying by e^{t+1} and using (3) yields
$$e^{t+1}H(t+1)\le e\,e^t\int_t^\infty H(s)\,ds\le C\varepsilon^2,$$
that is,
$$e^tH(t)\le C\varepsilon^2\qquad\forall t\ge 1.$$
But if $t\in[0,1]$,
$$e^tH(t)\le eH(t)\le eH(0)=eJ_\varepsilon(u_\varepsilon)\le C\varepsilon^2,$$
so that
$$e^tH(t)\le C\varepsilon^2\qquad\forall t\ge 0.$$
We are now in a position to conclude. Due to the preceding discussion we can proceed by estimating
$$\begin{aligned}C\varepsilon^2&\ge e^tH(t)=e^t\int_t^\infty e^{-s}L(s)\,ds\ge e^t\int_t^{t+1}e^{-s}L(s)\,ds\\&\ge e^te^{-t-1}\int_t^{t+1}L(s)\,ds=e^{-1}\int_t^{t+1}\int_{\mathbb{R}^n}|u''_\varepsilon|^2+\varepsilon^2|\nabla u_\varepsilon|^2+\varepsilon^2|u_\varepsilon|^p\,dx\,ds\\&\ge e^{-1}\varepsilon^2\int_t^{t+1}\int_{\mathbb{R}^n}|\nabla u_\varepsilon|^2+|u_\varepsilon|^p\,dx\,ds.\end{aligned}$$
Dividing by ε^2 we obtain
$$\int_t^{t+1}\int_{\mathbb{R}^n}|\nabla u_\varepsilon|^2dx\,ds\le C\qquad\forall t\ge 0,$$
$$\int_t^{t+1}\int_{\mathbb{R}^n}|u_\varepsilon|^pdx\,ds\le C\qquad\forall t\ge 0$$
and, directly from (3),
$$\int_{\mathbb{R}^n}|u'_\varepsilon(t)|^2\,dx\le C\varepsilon^2\qquad\forall t\ge 0.$$

When we scale back to v_ε by $u_\varepsilon(s,x) = v_\varepsilon(\varepsilon s, x)$ and we change variables these estimates take the form

$$\frac{1}{\varepsilon}\int_{\varepsilon t}^{\varepsilon t+\varepsilon}\int_{\mathbb{R}^n}|\nabla v_\varepsilon|^2 dx\,ds \le C \qquad \forall t \ge 0,$$

$$\frac{1}{\varepsilon}\int_{\varepsilon t}^{\varepsilon t+\varepsilon}\int_{\mathbb{R}^n}|v_\varepsilon|^p dx\,ds \le C \qquad \forall t \ge 0$$

and

$$\varepsilon^2\int_{\mathbb{R}^n}|v_\varepsilon'(t)|^2\,dx \le C\varepsilon^2 \qquad \forall t \ge 0.$$

Since t is arbitrary, we can rename εt by t and obtain

$$\int_{t}^{t+\varepsilon}\int_{\mathbb{R}^n}|\nabla v_\varepsilon|^2 dx\,ds \le C\varepsilon \qquad \forall t \ge 0,$$

$$\int_{t}^{t+\varepsilon}\int_{\mathbb{R}^n}|v_\varepsilon|^p dx\,ds \le C\varepsilon \qquad \forall t \ge 0,$$

$$\int_{\mathbb{R}^n}|v_\varepsilon'(t)|^2\,dx \le C \qquad \forall t \ge 0.$$

The last one is the global L^2 estimate on v_ε'. As for the remaining two, given $T \ge \varepsilon$, the interval $[t, t+T]$ can be covered by $O(T/\varepsilon)$ adjacent subintervals of length ε. On each of these intervals we use the above estimates and we add the results, arriving at

$$\int_{t}^{t+T}\int_{\mathbb{R}^n}|\nabla v_\varepsilon|^2\,dx\,ds \le C\varepsilon O(T/\varepsilon) \le CT, \qquad t \ge 0, \qquad T \ge \varepsilon$$

$$\int_{t}^{t+T}\int_{\mathbb{R}^n}|v_\varepsilon|^p\,dx\,ds \le C\varepsilon O(T/\varepsilon) \le CT, \qquad t \ge 0, \qquad T \ge \varepsilon,$$

which are the localized estimates we were looking for.

3. Some open problems

Here is a very short list of open problems that arise from the preceding discussion.

- Proving the conjecture without passing to subsequences. This is related, as we said, to the presence of uniqueness for the Cauchy problem (1)–(2), when p is large. If there is uniqueness, we know that there is no need for subsequences. If, on the contrary, there is no uniqueness, the situation could be even more interesting. Indeed, if one could prove the conjecture without passing to subsequences, then one would have a way to select a privileged solution to the Cauchy problem that could be referred to, for example, as the “Variational Solution”.

- Other equations. Just to make an example, what about

$$w'' - \frac{2}{q}\operatorname{div}\big(|\nabla w|^{q-2}\nabla w\big) + \frac{p}{2}|w|^{p-2}w = 0,$$

  the wave equation for the q-Laplacian with defocusing nonlinearity?

  This would correspond to the functional

$$\int_0^\infty \int_{\mathbb{R}^n} e^{-t/\varepsilon}\big(\varepsilon^2|v''|^2 + |\nabla v|^q + |v|^p\big)\,dx\,dt.$$

  As far as we know, even the *existence* of global weak solutions (to the Cauchy problem) for large q is unknown. Does the method of De Giorgi work to solve this problem?

- The abstract form of De Giorgi's Conjecture. Consider any convex functional of the Calculus of Variations,

$$F(u) = \int_\Omega f(x,u,\nabla u,\dots)\,dx$$

  Let $v_\varepsilon(t,x)$ be the minimizer of

$$\int_0^\infty e^{-t/\varepsilon}\left(\int_\Omega \varepsilon^2|v_\varepsilon''(t,x)|^2\,dx + F(v_\varepsilon(t,\cdot))\right)dt$$

  with given boundary conditions $v_\varepsilon(0,\cdot)$ and $v_\varepsilon'(0,\cdot)$

  As $\varepsilon \to 0$, does v_ε converge to some w, which solves the Cauchy Problem for the equation

$$w'' + \nabla F(w) = 0 \quad ?$$

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