Rend. Sem. Mat. Univ. Politec. Torino
Vol. 70, 2 (2012), 151 – 164

L. Radzhabova

CAUCHY TYPE PROBLEMS FOR A CLASS OF INTEGRAL EQUATIONS OF VOLTERRA TYPE

Abstract. We examine 2-dimensional integral equations of Volterra type with two singular interior lines corresponding to $x = a$ and $y = b$. The non-homogeneous integral equation that we can consider involves functions $A(x)$, $B(y)$, $C(x,y)$. Given certain inequalities for $A(a)$ and $B(b)$, it always has solutions on suitable domains that contain arbitrary functions of one variable. With other hypotheses, the equation has a unique solution in some domain.

1. Introduction and preliminaries

Consider the rectangle

$$D_0 = \{a_0 < x < a_1,\, b_0 < y < b_1\},$$

and the straight lines

$$\Gamma_1 = \{a_0 < x < a,\, y = b\},$$
$$\Gamma_2 = \{a < x < a_1,\, y = b\},$$
$$\Gamma_3 = \{x = a,\, b_0 < y < b\},$$
$$\Gamma_4 = \{x = a,\, b < y < b_1\},$$

where $a_0 < a < a_1$, $b_0 < b < b_1$.

In the domain $D = D_0 \setminus \{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4\}$, we consider the 2-dimensional integral equation

$$(1)\quad u(x,y) + \int_a^x \frac{A(t)u(t,y)}{|t-a|^\alpha} dt - \int_y^b \frac{B(s)u(x,s)}{|b-s|^\beta} ds + \int_a^x \frac{dt}{|t-a|^\alpha} \int_y^b \frac{C(t,s)u(t,s)}{|b-s|^\beta} ds \;=\; f(x,y),$$

where $A(x), B(y), C(x,y)$ are given functions in D_0, $f(x,y) \in C(\overline{D})$, and both α and β are positive constants.

The solution of many problems having a significance in applications can be figured out by the help of integral equations in explicit form. For that reason, this article is dedicated to this area.

For the equation (1), we find the solution of a second-order hyperbolic equation with two super-singular lines in the domain $D_2 = \{a < x < a_0,\, b_0 < y < b\}$ and for types of functions, approaching infinity on singular lines.

Problems concerning 2-dimensional Volterra-type integral equations

$$u(x,y) + \lambda \int_a^x \frac{u(t,y)}{(t-a)^\alpha} dt - \mu \int_y^b \frac{u(x,s)}{(b-s)^\beta} ds + \delta \int_a^x \frac{dt}{(t-a)^\alpha} \int_y^b \frac{u(t,s)}{(b-s)^\beta} ds = f(x,y)$$

with two boundary singular and super-singular lines in the domain D_2, are investigated in [1, 2, 4].

Integral equations of type (1) with boundary singular and super-singular lines, and cases

$$\alpha = 1,\ \beta > 1; \qquad \alpha > 1,\ \beta = 1; \qquad \alpha < 1,\ \beta > 1$$

are investigated in [3, 5].

References [6, 7] are dedicated to the problem of finding continuous solutions of second-order hyperbolic equation with two boundary singular or super-singular lines Γ_1 and Γ_2 corresponding to the study of integral equations (1) in the domain D_2 with $\alpha \geq 1$ and $\beta \geq 1$.

Finally, [8, 9] deal with the integral equation

$$u(x,y) + \int_a^x \frac{K_1(x,y;t)u(y,y)}{(t-a)^\alpha}dt - \int_y^b \frac{K_2(x,y;s)u(x,s)}{(b-s)^\beta}\,ds + \int_a^x \frac{dt}{(t-a)^\alpha}\int_y^b \frac{K_3(x,y;t,s)u(t,s)}{(b-s)^\beta}ds = f(x,y),$$

in the domain D_2 in the cases $\alpha = 1, \beta = 1$ where

$$\lambda := K_1(a,b;\ a), \quad \mu := K_2(a,b;\ b), \quad \delta := K_3(a,b;\ a,b) = -\lambda\mu.$$

For $\alpha > 1, \beta > 1$ we set

$$A(t) = K_1(a,b;t), \quad B(s) = K_2(a,b;s), \quad C(t,s) = K_3(a,b;t,s)$$

and require $C_1(t,s) := C(t,s) + A(t)B(s)$ not to be identically zero.

In this paper, we find the solution of the 2-dimensional Volterra type linear integral equation with interior singularities for exponents $\alpha = 1$ and $\beta = 1$ in the kernels in (1), when $C(x,y) \neq A(x)B(y)$.

In this case we shall prove that, when the coefficients of the integral equation are related in a determined way, the homogeneous integral equation (1) has infinitely many linear independent solutions given conditions on $A(a), B(b)$. For other values of these quantities, the homogeneous integral equation (1) has non-zero solution in some of the domains D_j.

These solutions are found by resolving known integral equations of Volterra type with weak singularity lines.

In the domain D_0, if we fix lines $x = a$ and $y = b$, the domain D_0 is divided into four domains as

$$\begin{aligned} D_1 &= \{a_0 < x < a,\ b_0 < y < b\}, \\ D_2 &= \{a < x < a_1,\ b_0 < y < b\}, \\ D_3 &= \{a_0 < x < a,\ b < y < b_1\}, \\ D_4 &= \{a < x < a_1,\ b < y < b_1\}. \end{aligned}$$

If $(x,y) \in D_1$, then we integrate over s and t that satisfy $a_0 < x < t < a$ and $b_0 < y < s < b$. The left-hand side of (1) takes the form:

$$u(x,y) - \int_x^a \frac{A(t)u(t,y)}{a-t}dt - \int_y^b \frac{B(s)u(x,s)}{b-s}ds - \int_x^a \frac{dt}{a-t}\int_y^b \frac{C(t,s)u(t,s)}{b-s}ds.$$

Similarly, if $(x,y) \in D_2, D_3, D_4$ then the left-hand side of (1) becomes respectively

$$u(x,y) - \int_a^x \frac{A(t)u(t,y)}{a-t}dt - \int_y^b \frac{B(s)u(x,s)}{b-s}ds + \int_a^x \frac{dt}{a-t}\int_y^b \frac{C(t,s)u(t,s)}{b-s}ds,$$
$$u(x,y) - \int_x^a \frac{A(t)u(t,y)}{a-t}dt + \int_b^y \frac{B(s)u(x,s)}{s-b}ds - \int_x^a \frac{dt}{a-t}\int_b^y \frac{C(t,s)u(t,s)}{s-b}ds,$$
$$u(x,y) + \int_a^x \frac{A(t)u(t,y)}{t-a}dt + \int_b^y \frac{B(s)u(x,s)}{s-b}ds - \int_a^x \frac{dt}{t-a}\int_b^y \frac{C(t,s)u(t,s)}{s-b}ds.$$

In this way, the study of these four integral equations in the respective domains D_1, D_2, D_3, D_4 is investigated in [1, 2].

2. A first theorem

Our first result is the following statement that deals with the case in which $A(a) < 0$ and $B(b) > 0$.

THEOREM 1. *Given (1), suppose that $A(x)$ and $B(y)$ define continuous functions on $\Gamma_1 \cup \Gamma_2$ and $\Gamma_3 \cup \Gamma_4$ respectively, and that they satisfy the Hölder condition, $A(a) < 0$ and $B(b) > 0$. Suppose that $C(x,y)$ defines a continuous function on D_0 and that*

$$C_1(x,y) = C(x,y) + A(x)B(y)$$

is continuous on $\overline{D_0}$ and not identically zero. Suppose that $C_1(x,y)$ vanishes on Γ_1, Γ_2, Γ_3, Γ_4, with

$$C_1(x,y) = O\big[|a-x|^{\varepsilon}|b-y|^{\varepsilon}\big] \quad \text{as } x \to a \pm 0,\ y \to b \pm 0, \varepsilon > 0$$

Moreover, suppose that $f(x,y)$ is a continuous function on $\overline{D_0}$ that satisfies $f(a,b) = 0$ and has the following asymptotic behavior:

$$f(x,y) = \begin{cases} o\big[(x-a)^{\delta_1}\big] & \text{as } x \to a+o, \quad \text{where } \delta_1 > |A(a)|, \\ o[(a-x)^{\varepsilon}] & \text{as } x \to a-o, \\ o[(b-y)^{\gamma_1}] & \text{as } y \to b-o, \quad \text{where } \gamma_1 > B(b), \\ o[(y-b)^{\varepsilon}] & \text{as } y \to b+o. \end{cases}$$

Then (1) always admits a solution with $u \in C(\overline{D_0})$ and $u(x,y) \to 0$ as $(x,y) \to \Gamma_j$ for $j = 1,2,3,4$, and its general solution contains four arbitrary functions each of one variable.

We claim that the solution is given by means of following formulas:

$$
(2)\quad \begin{cases}
u(x,y) = \exp[-w_b^{-1}(y)](b-y)^{B(b)}\Bigg[\varphi_1(x) + \int\limits_x^a \left(\frac{a-x}{a-t}\right)^{A(a)} \exp[w_b^{-1}(t) - \\
-w_a^{-1}(x)]\frac{A(t)}{a-t}\varphi_1(t)\,dt\Bigg] + K_{a,b}^{-;-}(f(x,y)) + \exp[-w_a^{-1}(x) - w_a^{-1}(y)] \\
(a-x)^{A(a)}(b-y)^{B(b)}\int\limits_x^a dt\int\limits_y^b \Gamma_{11}(x,y;t,s)E_1[\varphi_1(t),f(t,s)]\,ds \\
\text{when } (x,y)\in D_1, \\
u(x,y) = \exp[-w_a^{+1}(x)](x-a)^{-A(a)}\psi_1(y) + \exp[-w_b^{+1}](b-y)^{B(b)}[\varphi_2(x) - \\
-\int_a^x \exp[-w_a^{+1}(x)w_a^{+1}(t)]\left(\frac{x-a}{t-a}\right)^{A(a)}\frac{A(t)}{t-a}\varphi_2(t)\,dt\Bigg] \\
+K_{a,b}^{+;-}(f(x,y)) + \exp[-w_a^{+1}(x) - w_b^{-1}(y)](x-a)^{-A(a)}(b-y)^{B(b)} \\
\qquad\qquad \times\int_a^x dt\int_y^b \Gamma_{12}(x,y;t,s)E_2[\varphi_2(t),\psi_1(s),f(t,s)]\,ds. \\
\text{when } (x,y)\in D_2, \\
u(x,y) = \exp[-w_a^{-1}(x) - w_b^{+1}(y)](a-x)^{A(a)}(y-b)^{-B(b)} \\
\qquad \times\left[K_{a,b}^{-;+}(f(x,y)) - \int_x^a dt\int_b^y \Gamma_{13}(x,y;t,s)E_3[f(t,s)]\,ds\right]. \\
\text{when } (x,y)\in D_3, \\
u(x,y) = \exp[-w_a^{+1}(x)](x-a)^{-A(a)}\psi_2(y) + K_{a,b}^{+;+}(f(x,y)) \\
\qquad -\exp[-w_a^{+1}(x) - w_b^{+1}(y)](x-a)^{-A(a)}(y-b)^{-B(b)} \\
\qquad\qquad \times\int_a^x dt\int_b^x \Gamma_{14}(x,y;t,s)E_4[\psi_2(s),f(t,s)]\,ds. \\
(x,y)\in D_4.
\end{cases}
$$

In these formulas,

$$K_{a,b}^{-;-}(f(x,y)),\quad K_{a,b}^{+;-}(f(x,y)),\quad K_{a,b}^{-;+}(f(x,y)),\quad K_{a,b}^{+;+}(f(x,y)),$$

$$E_1[\varphi_1(x),f(x,y)],\quad E_2[\varphi_2(x),\psi_1(y),f(x,y)],\quad E_3[f(x,y)],\quad E_4[\psi_2(y),f(x,y)]$$

are all known integral operators, and

$$\Gamma_{11}(x,y;t,s),\quad \Gamma_{12}(x,y;t,s),\quad \Gamma_{13}(x,y;t,s),\quad \Gamma_{14}(x,y;t,s)$$

are all resolvents of known integral equation of Volterra type with weak singularity lines,

$$w_a^{+1}(x) = \int_a^x \frac{A(t)-A(a)}{t-a}dt,\qquad w_a^{-1}(x) = \int_x^a \frac{A(a)-A(t)}{a-t}dt,$$

$$w_b^{+1}(y)=\int_b^y \frac{B(s)-B(b)}{s-b}ds,\quad w_b^{-1}(y)=\int_y^b \frac{B(b)-B(s)}{b-s}ds,$$

$\varphi_j(x),\ \psi_j(y),\ j=1,2$ *are continuous functions on* Γ_1,Γ_2 *and on* Γ_3,Γ_4 *satisfying* $\varphi_j(x)\to 0$ *as* $x\to a$ *and* $\psi_j(y)\to 0$ *as* $y\to b$*, and such that their behavior is governed by the following asymptotic formulas:*

$$\begin{aligned}
&\varphi_1(x)=o[(a-x)^{\varepsilon}] && \text{as } x\to a-o,\\
&\varphi_2(x)=o[(x-a)^{\delta_2}] && \text{as } x\to a+o, \quad \text{where } \delta_2>|A(a)|,\\
&\psi_1(y)=o[(b-y)^{\gamma_3}] && \text{as } y\to b-o, \quad \text{where } \gamma_3>B(b),\\
&\psi_2(y)=o[(y-b)^{\varepsilon}] && \text{as } y\to b+o,
\end{aligned}$$

3. Three more theorems

We now state three theorems analogous to the first, corresponding to the other possible signs of $A(a)$ and $B(b)$.

THEOREM 2. *In equation (1), let* $A(x)\in C(\Gamma_1\cup\Gamma_2)$*,* $B(y)\in C(\Gamma_3\cup\Gamma_4)$ *for the points* $x=a$*,* $y=b$ *they satisfy the Helder's condition,* $A(a)>0, B(b)<0$*,* $C(x,y)\in C(\overline{D_0})$*,* $C_1(x,y)=C(x,y)+A(x)B(y)\neq 0$*,* $C_1(x,y)\in C(\overline{D_0})$ *and* $C_1(x,y)=0$ *with following asymptotic behavior on the boundaries* $\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4$ *:*

$$C_1(x,y)=o[|a-x|^{\varepsilon}|b-y|^{\varepsilon}] \quad \textit{as } x\to a\pm o,\ y\to b\pm o$$

Moreover, let $f(x,y)\in C(\overline{D_0})$*,* $f(a,b)=0$ *with following asymptotic behavior:*

$$f(x,y)=\begin{cases}
o[(a-x)^{\delta_3}] & \textit{as } x\to a-o, \quad \delta_3>A(a)\\
o[(x-a)^{\varepsilon}] & \textit{as } x\to a+o,\\
o[(b-y)^{\varepsilon}] & \textit{as } y\to b-o,\\
o[(y-b)^{\gamma_3}] & \textit{as } y\to b+o. \quad \gamma_3>|B(b)|.
\end{cases}$$

The non-homogeneous integral equation (1) in the class $C(\overline{D_0})$*, approaching zero in* $\Gamma_j\ j=1,2,3,4$*, is always solvable and its general solution contains four arbitrary functions with one variable.*

The solutions are in fact given by means of following formulas:

$$\begin{aligned}
&u(x,y)=\exp[-w_a^{-1}(x)](a-x)^{A(a)}\psi_1(y)+K_{a,b}^{-,-}(f(x,y))\\
&-\exp[-w_a^{-1}-w_b^{-1}(y)](a-x)^{A(a)}(b-y)^{B(b)}\\
&\qquad\times\int_x^a dt\int_y^b \Gamma_{21}(x,y;t,s)E_5[\psi_1(s),f(t,s)]\,ds.\\
&(x,y)\in D_1\\
&u(x,y)=K_{a,b}^{+,-}(f(x,y))-\exp[-w_a^{+1}(x)-w_b^{-1}(y)](x-a)^{-A(a)}(b-y)^{B(b)}\\
&\qquad\times\int_a^x dt\int_y^b \Gamma_{22}(x,y;t,s)E_6[f(t,s)]\,ds.\\
&(x,y)\in D_2
\end{aligned} \tag{3}$$

$$u(x,y)=\exp[-w_a^{-1}(y)](a-x)^{A(a)}\psi_2(y)$$
$$+\exp[-w_b^{+1}(y)](y-b)^{-B(b)}\left[\varphi_1(x)+\int_x^a \exp[w_a^{-1}(t)-w_a^{-1}(x)]\times\right.$$
$$\left.\times\left(\frac{a-x}{a-t}\right)^{A(a)}\frac{A(t)}{a-t}\varphi_1(t)dt\right]+K_{a,b}^{-;+}(f(x,y))-\exp[w_a^{-1}(x)-w_b^{-1}(y)]\times$$
$$\times(a-x)^{A(a)}(y-b)^{-B(b)}\int_x^a dt\int_b^y \Gamma_{23}(x,y;t,s)E_7[\varphi_1(t),\psi_2(s),f(t,s)]\,ds.$$

$(x,y)\in D_3$ (3)

$$u(x,y)=\exp[-w_b^{+1}(y)](y-b)^{-B(b)}\left[\varphi_2(x)-\int_a^x\left(\frac{t-a}{x-a}\right)^{A(a)}\times\right.$$
$$\left.\times\exp[w_a^{+1}(t)-w_a^{+1}(x)]\frac{A(t)}{t-a}\varphi_2(t)\,dt\right]$$
$$+K_{a,b}^{+;+}(f(x,y))+\exp[-w_a^{+1}(x)-w_b^{+1}(y)](x-a)^{-A(a)}(b-y)^{-B(b)}$$
$$\times\int_a^x dt\int_b^y \Gamma_{24}(x,y;t,s)E_8[\varphi_2(t),f(t,s)]\,ds.$$

$(x,y)\in D_4$

Here,

$$K_{a,b}^{-;-}(f(x,y)),\quad K_{a,b}^{+;-}(f(x,y)),\quad K_{a,b}^{-;+}(f(x,y)),\quad K_{a,b}^{+;+}(f(x,y))$$

and

$$E_5[\psi_1(y),f(x,y)],\quad E_6[f(x,y)],\quad E_7[\varphi_2(x),\psi_2(y),f(x,y)],\quad E_8[\varphi_2(x),f(x,y)]$$

are known integral operators, and

$$\Gamma_{21}(x,y;t,s),\quad \Gamma_{22}(x,y;t,s),\quad \Gamma_{23}(x,y;t,s),\quad \Gamma_{24}(x,y;t,s)$$

are resolvents of known integral equation of Volterra type with weak singularity lines, $\varphi_j(x),\psi_j(y)$, $j=1,2$ are arbitrary continuous functions for the boundaries Γ_1, Γ_2, Γ_3, Γ_4. Moreover, at $x\to a$, $y\to b$, $\varphi_j(x)$ and $\psi_j(y)$ approach zero and their behavior is determined by the following asymptotic formulas

$$\begin{aligned}
&\varphi_1(x)=o[(a-x)^{\delta_4}] && \text{as } x\to a-o, \quad \delta_4>A(a)\\
&\varphi_2(x)=o[(x-a)^{\varepsilon}] && \text{as } x\to a+o,\\
&\psi_1(y)=o[(b-y)^{\varepsilon}] && \text{as } y\to b-o,\\
&\psi_2(y)=o[(y-b)^{\gamma_4}] && \text{as } y\to b+o. \quad \gamma_4>|B(b)|.
\end{aligned}$$

THEOREM 3. *Let in equation (1) $A(x)\in C(\Gamma_1\cup\Gamma_2)$, $B(y)\in C(\Gamma_3\cup\Gamma_4)$ for the points $x=a$, $y=b$ they satisfy the Helder's condition, $A(a)>0, B(b)>0$, $C(x,y)\in$*

$C(\overline{D_0})$, $C_1(x,y) = C(x,y) + A(x)B(y) \neq 0$, $C_1(x,y) \in C(\overline{D_0})$ and $C_1(x,y) = o$ with following asymptotic behavior on boundaries $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$:

$$C_1(x,y) = o[|a-x|^{\varepsilon}|b-y|^{\varepsilon}] \text{ at } x \to a \pm 0,\ y \to b \pm 0.$$

Moreover, let $f(x,y) \in C(\overline{D_0})$, $f(a,b) = 0$ with

$$\begin{aligned} &f(x,y) = o[(a-x)^{\delta_5}],\ \delta_5 > A(a) \text{ at } x \to a - o,\\ &f(x,y) = o[(x-a)^{\varepsilon}],\ \text{at } x \to a + o,\\ &f(x,y) = o[(b-y)^{\gamma_5}],\ \gamma_5 > B(b) \text{ at } y \to b - o,\\ &f(x,y) = o[(y-b)^{\varepsilon}],\ \text{at } y \to b + o.\end{aligned}$$

Then the non homogeneous integral equation (1) in the class $C(\overline{D_0})$, approaching zero in Γ_j $j = 1,2,3,4$, is always solvable and its general solution contains four arbitrary functions of one variable.

The solution is given by means of the following formulas:

$$\begin{cases} u(x,y) = (a-x)^{A(a)} \exp[-w_a^{-1}(x)]\psi_1(y) \\ + \exp[-w_b^{-1}(y)](b-y)^{B(b)} \left[\varphi_1(x) + \int_x^a \exp[w_a^{-1}(t) - w_a^{-1}] \right. \\ \left. \times \left(\dfrac{a-x}{a-t}\right)^{A(a)} \dfrac{A(t)}{a-t}\varphi_1(t)dt\right] \\ + K_{a,b}^{-,-}(f(x,y)) - \exp[-w_a^{-1}(x) - w_b^{-1}(y)](a-x)^{A(a)}(b-y)^{B(b)} \\ \quad \times \int_x^a dt \int_y^b \Gamma_{31}(x,y;t,s)E_9[\varphi_1(t), \psi_1(s), f(t,s)]\,ds. \\ \text{when } (x,y) \in D_1, \\ u(x,y) = \exp[-w_b^{-1}(y)](b-y)^{B(b)} \left[\varphi_2(x) - \int_a^x \left(\dfrac{t-a}{x-a}\right)^{A(a)} \right. \\ \left. \times \exp[-w_a^{+1}(t)] \dfrac{A(t)}{t-a}\varphi_2(t)dt\right] \\ + K_{a,b}^{+,-}(f(x,y)) - \exp[-w_a^{+1}(x) - w_b^{+1}(y)](x-a)^{-A(a)}(b-y)^{B(b)} \\ \quad \times \int_a^x dt \int_y^b \Gamma_{32}(x,y;\ t,s)E_{10}[\varphi_2(t), f(t,s)]\,ds. \\ \text{when } (x,y) \in D_2, \\ u(x,y) = \exp[-w_a^{-1}(x)](a-x)^{A(a)}\psi_2(y) \\ + K_{a,b}^{-,+}(f(x,y)) - \exp[-w_a^{-1}(x) - w_b^{-1}(y)](a-x)^{A(a)}(y-b)^{-B(b)} \\ \quad \times \int_x^a dt \int_b^y \Gamma_{33}(x,y;t,s)E_{11}[\psi_2(s), f(t,s)]\,ds \\ \text{when } (x,y) \in D_3, \\ u(x,y) = K_{a,b}^{+,+}(f(x,y)) - \exp[-w_a^{+1} - w_b^{+1}(y)](x-a)^{-A(a)}(y-b)^{-B(b)} \\ \qquad \times \int_a^x dt \int_b^y \Gamma_{34}(x,y;t,s)E_{34}(x,y;t,s)E_{12}[f(t,s)]\,ds. \\ (x,y) \in D_4. \end{cases} \tag{4}$$

Here

$$K^{-,-}_{a,b}(f(x,y)),\quad K^{+,-}_{a,b}(f(x,y)),\quad K^{-,+}_{a,b}(f(x,y)),\quad K^{+,+}_{a,b}(f(x,y))$$

and

$$E_9[\varphi_1(x),\psi_1(y),f(x,y)],\quad E_{10}[\varphi_1(x),f(x,y)],\quad E_{11}[\psi_2(y),(f(x,y))],\quad E_{12}[f(x,y)]$$

are known integral operators, and

$$\Gamma_{31}(x,y;t,s),\quad \Gamma_{32}(x,y;t,s),\quad \Gamma_{33}(x,y;t,s),\quad \Gamma_{34}(x,y;t,s)$$

are resolvents of known integral equation of Volterra type with weak singularity lines, $\varphi_j(x),\psi_j(y)$, $j=1,2$, are arbitrary contiuous function for the boundaries Γ_1, Γ_2, Γ_3, Γ_4. Moreover, at $x\to a$, $y\to b$, $\varphi_j(x)$ and $\psi_j(y)$ approach zero and their behavior is determined by the following asymptotic formula

$$\begin{aligned}
&\varphi_1(x)=o[(a-x)^{\delta_6}] && \text{as } x\to a-o, \quad \delta_6>A(a),\\
&\varphi_2(x)=o[(x-a)^{\varepsilon}] && \text{as } x\to a+o,\\
&\psi_1(y)=o[(b-y)^{\gamma_6}] && \text{as } y\to b-o, \quad \gamma_6>B(b),\\
&\psi_2(y)=o[(y-b)^{\varepsilon}] && \text{as } y\to b+o.
\end{aligned}$$

THEOREM 4. *Let in equation (1) $A(x)\in C(\Gamma_1\cup\Gamma_2)$, $B(y)\in C(\Gamma_3\cup\Gamma_4)$ for the points $x=a$, $y=b$ they satisfy the Helder's condition, $A(a)<0$, $B(b)<0$, $C(x,y)\in C(\overline{D_0})$, $C_1(x,y)=C(x,y)+A(x)B(y)\neq 0$, $C_1(x,y)\in C(\overline{D_0})$ and $C_1(x,y)=0$ with following asymptotic behavior on boundaries Γ_1, Γ_2, Γ_3, Γ_4 :*

$$C_1(x,y)=o[|a-x|^{\varepsilon}\,|b-y|^{\varepsilon}] \quad \text{as } x\to a\pm o\ y\to b\pm o$$

Moreover, let $f(x,y)\in C(\overline{D_0}), f(a,b)=0$ with following asymptotic behavior:

$$\begin{aligned}
&f(x,y)=o[(a-x)^{\varepsilon}] && \text{as } x\to a-o,\\
&f(x,y)=o[(x-a)^{\delta_7}] && \text{as } x\to a+o, \quad \delta_7>|A(a)|,\\
&f(x,y)=o[(b-y)^{\varepsilon}] && \text{as } y\to b-o,\\
&f(x,y)=o[(y-b)^{\gamma_7}] && \text{as } y\to b+o. \quad \gamma_7>|B(b)|.
\end{aligned}$$

Then the non-homogeneous integral equation (1) in the class $C(\overline{D_0})$, approaching zero in Γ_j for $j=1,2,3,4$, is always solvable and its general solution contain four arbitrary functions of one variable.

The solution is given by means of following formulas:

$$\begin{cases} u(x,y) = K_{a,b}^{-;-}(f(x,y)) - \exp[-w_A^{-1}(x) - w_B^{-1}(y)](a-x)^{A(a)}(b-y)^{B(b)} \\ \qquad \times \int_x^a dt \int_y^b \Gamma_{41}(x,y;t,s) E_{13}[f(t,s)]\, ds, \quad (x,y) \in D_1 \\ u(x,y) = \exp[-w_A^{+1}(x)](x-a)^{-A(a)}\psi_1(y) \\ +K_{a,b}^{+;-}(f(x,y)) - \exp[-w_A^{+1}(x) - w_B^{-1}(y)](x-a)^{-A(a)}(b-y)^{B(b)} \\ \qquad \times \int_a^x dt \int_y^b \Gamma_{42}(x,y;t,s) E_{14}[\psi_1(s), f(t,s)]\, ds, \quad (x,y) \in D_2 \\ u(x,y) = \exp[-w_B^{+1}(y)](y-b)^{-B(b)} \left[\varphi_1(x) - \int_x^a \left(\frac{a-x}{a-t}\right)^{A(a)} \right. \\ \left. \times \exp[w_A^{-1}(t) - w_A^{-1}(x)] \frac{A(t)}{a-t}\varphi_1(t)dt \right] \\ +K_{a,b}^{-;+}(f(x,y)) - \exp[-w_A^{-1}(x) - w_B^{+1}(y)](a-x)^{A(a)}(y-b)^{-B(b)} \\ \qquad \times \int_x^a dt \int_b^y \Gamma_{43}(x,y;t,s) E_{15}[\varphi_1(t), f(t,s)]\, ds, \quad (x,y) \in D_3 \\ u(x,y) = \exp[-w_A^{+1}(y)](x-a)^{-A(a)}\psi_2(y) \\ +\exp[-w_B^{+1}(y)](y-b)^{-B(b)} \left[\varphi_2(x) - \int_a^x \exp[w_A^{+1}(t) - w_A^{+1}(x)] \right. \\ \left. \times \left(\frac{t-a}{x-a}\right)^{A(a)} \frac{A(t)}{t-a}\varphi_2(t)dt \right] \\ +K_{a,b}^{+;+}(f(x,y)) - \exp[-w_A^{+1}(x) - w_B^{+1}(y)](x-a)^{-A(a)}(y-b)^{-B(b)} \\ \qquad \times \int_a^x dt \int_b^y \Gamma_{44}(x,y;t,s) E_{16}[\varphi_2(t), \psi_2(s), f(t,s)]\, ds, \quad (x,y) \in D_4. \end{cases} \tag{5}$$

Here,

$$K_{a,b}^{-;-}(f(x,y)), \quad K_{a,b}^{+;-}(f(x,y)), \quad K_{a,b}^{-;+}(f(x,y)), \quad K_{a,b}^{+;+}(f(x,y))$$

and

$$E_{13}[f(x,y)], \quad E_{14}[\psi_1(y), f(x,y)], \quad E_{15}[\varphi_1(x), f(x,y)], \quad E_{16}[\varphi_2(x), \psi_2(y), f(x,y)]$$

are known integral operators, and

$$\Gamma_{41}(x,y;t,s), \quad \Gamma_{42}(x,y;t,s), \quad \Gamma_{43}(x,y;t,s), \quad \Gamma_{44}(x,y;t,s)$$

are resolvents of known integral equations of Volterra type with weak singularity lines, $\varphi_j(x), \psi_j(y),\ j=1,2$ *are arbitrary contiuous functions for the boundaries* $\Gamma_1, \Gamma_2, \Gamma_3$

and Γ_4. *Moreover, at* $x \to a,\ y \to b,\ \varphi_j(x)$ *and* $\psi_j(y)$ *approach zero and their behavior is determined by the following asymptotic formulas*

$$\begin{aligned}
&\varphi_1(x) = o[(a-x)^{\varepsilon}] \quad \text{as } x \to a-o, \\
&\varphi_2(x) = o[(x-a)^{\delta_8}] \quad \text{as } x \to a+o, \quad \delta_8 > |A(a)|, \\
&\psi_1(y) = o[(b-y)^{\varepsilon}] \quad \text{as } y \to b-o, \\
&\psi_2(y) = o[(y-b)^{\gamma_8}] \quad \text{as } y \to b+o. \quad \gamma_8 > |B(b)|.
\end{aligned}$$

4. Four problems

PROBLEMS Find a solution of the integral equation (1), belonging to the class $C(\overline{D_0})$ and approaching zero on Γ_j for $j = 1,2,3,4$, such that it satisfies one of the following conditions. These respect in order the inequalities $(A(a), B(b))$ of Theorems 1,2,3,4.

$$\mathbf{P_1}: \ A(a) < 0,\ B(b) > 0 \quad \begin{cases}
(b-y)^{-B(b)}u(x,y)\Big|_{y\to b-o} = f_1(x) \ \text{ for } x \in \Gamma_1, \\
(b-y)^{-B(b)}u(x,y)\Big|_{y\to b-o} = f_2(x) \ \text{ for } x \in \Gamma_2, \\
(x-a)^{A(a)}u(x,y)\Big|_{x\to a+o} = g_1(y) \ \text{ for } y \in \Gamma_3, \\
(x-a)^{A(a)}u(x,y)\Big|_{x\to a+o} = g_2(y) \ \text{ for } y \in \Gamma_4, \\
(b-y)^{-B(b)}u(x,y)\Big|_{\substack{y\to b-o \\ x\to a-o}} = 0.
\end{cases}$$

$$\mathbf{P_2}: \ A(a) > 0,\ B(b) < 0 \quad \begin{cases}
(y-b)^{B(b)}u(x,y)\Big|_{y\to b+o} = f_1(x) \ \text{ for } x \in \Gamma_1, \\
(y-b)^{B(b)}u(x,y)\Big|_{y\to b+o} = f_2(x) \ \text{ for } x \in \Gamma_2, \\
(a-x)^{-A(a)}u(x,y)\Big|_{x\to a-o} = g_1(y) \ \text{ for } y \in \Gamma_3, \\
(a-x)^{-A(a)}u(x,y)\Big|_{x\to a-o} = g_2(y) \ \text{ for } y \in \Gamma_4, \\
(y-b)^{B(b)}u(x,y)\Big|_{\substack{y\to b+o \\ x\to a+o}} = 0.
\end{cases}$$

$$\mathbf{P_3}: \ A(a) > 0,\ B(b) > 0 \quad \begin{cases}
(b-y)^{-B(b)}u(x,y)\Big|_{y\to b-o} = f_1(x) \ \text{ for } x \in \Gamma_1, \\
(b-y)^{-B(b)}u(x,y)\Big|_{y\to b-o} = f_2(x) \ \text{ for } x \in \Gamma_2, \\
(a-x)^{-A(a)}u(x,y)\Big|_{x\to a-o} = g_1(y) \ \text{ for } y \in \Gamma_3, \\
(a-x)^{-A(a)}u(x,y)\Big|_{x\to a-o} = g_2(y) \ \text{ for } y \in \Gamma_4, \\
(b-y)^{-B(b)}u(x,y)\Big|_{\substack{y\to b-o \\ x\to a+o}} = 0.
\end{cases}$$

$$\mathbf{P_4}: \ A(a)<0,\ B(b)<0 \quad \begin{cases} (y-b)^{B(b)}u(x,y)\Big|_{y\to b+o} = f_1(x) \ \text{ for } \ x\in\Gamma_1, \\ (y-b)^{B(b)}u(x,y)\Big|_{y\to b+o} = f_2(x) \ \text{ for } \ x\in\Gamma_2, \\ (x-a)^{A(a)}u(x,y)\Big|_{x\to a+o} = g_1(y) \ \text{ for } \ y\in\Gamma_3, \\ (x-a)^{A(a)}u(x,y)\Big|_{x\to a+o} = g_2(y) \ \text{ for } \ y\in\Gamma_4, \\ (y-b)^{B(b)}u(x,y)\Big|_{\substack{y\to b+o\\ x\to a-o}} = 0. \end{cases}$$

5. Unique solutions

THEOREM 5. *Consier the integral equation (1) and suppose that $A(x)$, $B(y)$, $C(x,y)$, $f(x,y)$ satisfy the hypotheses of Theorem 1. If*
1) $f_1(x)\in C(\Gamma_1),\ f_2(x)\in C(\Gamma_2),\ g_1(y)\in C(\Gamma_3),\ g_2(y)\in C(\Gamma_4)$.
2) $f_1(a)=0,\ f_2(a)=0,\ g_1(b)=0,\ g_2(b)=0$*, with asymptotic behavior*

$$\begin{aligned} &f_1(x)=o[(a-x)^{\varepsilon}] && \text{as } x\to a-o, && \\ &f_2(x)=o[(x-a)^{\delta_2}] && \text{as } x\to a-o, && \delta_2>|A(a)|, \\ &g_1(x)=o[(b-y)^{\gamma_2}] && \text{as } y\to b-o, && \gamma_2>B(b), \\ &g_2(x)=o[(y-b)^{\varepsilon}] && \text{as } y\to b+o. && \end{aligned}$$

then problem P_1 has a unique solution and it can be written in the form (2), in the case in which

$$\begin{aligned} &\psi_1(y)=g_1(y), && \text{when } y\in\Gamma_3, \\ &\psi_2(y)=g_2(y), && \text{when } y\in\Gamma_4, \\ &\varphi_1(x)=c_1+f_1(x)-\int_x^a \frac{A(t)}{a-t} f_1(t)\,dt, && \text{when } x\in\Gamma_1, \\ &\varphi_2(x)=f_2(x)-\int_a^x \frac{A(t)}{t-a} f_2(t)\,dt, && \text{when } x\in\Gamma_2,\ c_1=0. \end{aligned}$$

THEOREM 6. *Suppose that $A(x), B(y), C(x,y), f(x,y)$ satisfy the hypotheses of Theorem 2. If*
1) $f_1(x)\in C(\Gamma_1),\ f_2(x)\in C(\Gamma_2),\ g_1(y)\in C(\Gamma_3),\ g_2(y)\in C(\Gamma_4)$,
2) $f_1(a)=0,\ f_2(a)=0,\ g_1(b)=0,\ g_2(b)=0$*, with asymptotic behavior*

$$\begin{aligned} &f_1(x)=o[(a-x)^{\delta_4}] && \text{as } x\to a-o, && \delta_4>A(a), \\ &f_2(x)=o[(x-a)^{\varepsilon}] && \text{as } x\to a-o, && \\ &g_1(x)=o[(b-y)^{\varepsilon}] && \text{as } y\to b-o, && \\ &g_2(x)=o[(y-b)^{\gamma_4}] && \text{as } y\to b+o. && \gamma_4>|B(b)|. \end{aligned}$$

then problem P_2 has a unique solution and it can be written in the form (3), in the case when

$$\begin{aligned}
&\psi_1(y) = g_1(y) && \text{when } y \in \Gamma_3,\\
&\psi_2(y) = g_2(y) && \text{when } y \in \Gamma_4,\\
&\varphi_1(x) = f_1(x) - \int_x^a \frac{A(t)}{a-t} f_1(t)\,dt, && \text{when } x \in \Gamma_1,\\
&\varphi_2(x) = c_2 + f_2(x) + \int_a^x \frac{A(t)}{t-a} f_2(t)\,dt, && \text{when } x \in \Gamma_2,\ c_2 = 0.
\end{aligned}$$

THEOREM 7. *Suppose that $A(x), B(y), C(x,y), f(x,y)$ satisfy the hypotheses of Theorem 3. If*

1) $f_1(x) \in C(\Gamma_1),\ f_2(x) \in C(\Gamma_2),\ g_1(y) \in C(\Gamma_3),\ g_2(y) \in C(\Gamma_4)$,

2) $f_1(a) = 0,\ f_2(a) = 0,\ g_1(b) = 0,\ g_2(b) = 0$ *with asymptotic behavior*

$$\begin{aligned}
&f_1(x) = o[(a-x)^{\delta_6}] && \text{as } x \to a - o, \quad \delta_6 > A(a),\\
&f_2(x) = o[(x-a)^{\varepsilon}] && \text{as } x \to a - o,\\
&g_1(x) = o[(b-y)^{\gamma_6}] && \text{as } y \to b - o, \quad \gamma_6 > B(b),\\
&g_2(x) = o[(y-b)^{\varepsilon}] && \text{as } y \to b + o.
\end{aligned}$$

then problem P_3 has a unique solution and it can be written in the form (4) in the case when

$$\begin{aligned}
&\psi_1(y) = g_1(y) && \text{when } y \in \Gamma_3,\\
&\psi_2(y) = g_2(y) && \text{when } y \in \Gamma_4,\\
&\varphi_1(x) = f_1(x) - \int_x^a \frac{A(t)}{a-t} f_1(t)\,dt, && \text{when } x \in \Gamma_1,\\
&\varphi_2(x) = c_3 + f_2(x) + \int_a^x \frac{A(t)}{t-a} f_2(t)\,dt, && \text{when } x \in \Gamma_2,\ c_3 = 0.
\end{aligned}$$

THEOREM 8. *Let the integral equation (1) $A(x), B(y), C(x,y), f(x,y)$ satisfy the hypotheses of Theorem 4. If*

1) $f_1(x) \in C(\Gamma_1),\ f_2(x) \in C(\Gamma_2),\ g_1(y) \in C(\Gamma_3),\ g_2(y) \in C(\Gamma_4)$,

2) $f_1(a) = 0,\ f_2(a) = 0,\ g_1(b) = 0,\ g_2(b) = 0$*, with asymptotic behavior*

$$\begin{aligned}
&f_1(x) = o[(a-x)^{\varepsilon}] && \text{as } x \to a - o,\\
&f_2(x) = o[(x-a)^{\delta_8}] && \text{as } x \to a - o, \quad \delta_8 > |A(a)|,\\
&g_1(x) = o[(b-y)^{\varepsilon}] && \text{as } y \to b - o,\\
&g_2(x) = o[(y-b)^{\gamma_8}] && \text{as } y \to b + o. \quad \gamma_8 > |B(b)|.
\end{aligned}$$

then problem P_4 has a unique solution and it can be written in the form (5) in the case *when*

$$\begin{aligned}
&\psi_1(y) = g_1(y) && \textit{when } \ y \in \Gamma_3,\\
&\psi_2(y) = g_2(y) && \textit{when } \ y \in \Gamma_4,\\
&\varphi_1(x) = c_4 + f_1(x) - \int_x^a \frac{A(t)}{a-t} f_1(t)\,dt, && \textit{when } \ x \in \Gamma_1,\\
&\varphi_2(x) = f_2(x) + \int_a^x \frac{A(t)}{t-a} f_2(t)\,dt && \textit{when } \ x \in \Gamma_2, \ c_4 = 0.
\end{aligned}$$

References

[1] RAJABOV, N., AND RAJABOVA, L. Explicit solution one class two dimensional linear volterra type integral equation with two boundary singular lines. *Differential Equations and its applications* (2002), 286–288.

[2] RAJABOV, N., AND RAJABOVA, L. Investigation one class two dimensional integral equation with fixed singular kernels, connection with hyperbolic equation. *Academy of Science 391*, 1 (2003), 20–22.

[3] RAJABOV, N., AND RAJABOVA, L. To theory one class two dimensional, non model volterra type integral equation with boundary super - singular lines on the kernels. *Academy of Science 400*, 5 (2005), 602–605.

[4] RAJABOV, N., RONTO, M., AND RAJABOVA, L. On some two dimensional volterra type linear integral equations with super - singularity. *Mathematical Notes 4*, 1 (2003), 65–76.

[5] RAJABOVA, L. Explicit solution one class non model volterra type integral equation with one singular and one weak - singular. In *Differential Equations with partial derivatives and related problems analysis and information*, vol. 4. A work of Inter. Sci.Conf., Tashkent, 2004, pp. 78–80.

[6] RAJABOVA, L. About one class hyperbolic equation with singular lines. *Vestnik National University (science magazine)*, 5 (2006), 44–51.

[7] RAJABOVA, L. To theory one class hyperbolic equation with singular lines. *Academy of Sciences 49*, 8 (2006), 710–717.

[8] RAJABOVA, L. About general two dimensional volterra type integral equation with singular and super - singular in boundary domain. In *Differential Equations, theory function and applications*. Proceeding Intern. Sci. Conf., Novosibirsk, 2007, pp. 458–459.

[9] RAJABOVA, L. About one two dimensional volterra type integral equation with singulars in boundary domain. *Vestnik Tajik State National University (science magazine)*, 3 (2007), 30–37.

[10] RAJABOVA, L. A study about one kind of two dimensional integral equation of volterra type with two interior or singular lines. In *Progress in Analysis and its applications*. *Proceedings of the 7-th International ISAAC Congress*. World Scientific, 2010, pp. 96–104.

AMS Subject Classification:

Lutfiya RADZHABOVA
Tajik Technical University
Dushanbe, TAJIKISTAN
e-mail: lutfya62@mail.ru

Lavoro pervenuto in redazione il 23.04.2013.