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**SOME COINCIDENCE AND COMMON FIXED POINT
THEOREMS FOR PREŠIĆ-REICH TYPE MAPPINGS IN CONE
METRIC SPACES**

Abstract. The purpose of this paper is to prove coincidence and common fixed point theorems for mappings satisfying Prešić-Reich type contraction condition in cone metric spaces, without assuming the normality of cone. Our results generalize several known results in metric and cone metric spaces. An example illustrate the case when new result can be applied while old one can not.

1. Introduction

The well known Banach contraction mapping principle states that if (X, d) is a complete metric space and $T : X \rightarrow X$ is a self mapping such that

$$(1.1) \quad d(Tx, Ty) \leq \lambda d(x, y),$$

for all $x, y \in X$, where $0 \leq \lambda < 1$, then there exists a unique $x \in X$ such that $T(x) = x$. This point x is called the fixed point of mapping T .

On the other hand, for mappings $T : X \rightarrow X$ Kannan [10] introduced the contractive condition:

$$(1.2) \quad d(Tx, Ty) \leq \lambda[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$, where $\lambda \in (0, 1/2)$ is a constant, and proved a fixed point theorem using (1.2) instead of (1.1). The conditions (1.1) and (1.2) are independent, as it was shown by two examples in [11].

Reich [20], for mappings $T : X \rightarrow X$ generalized Banach and Kannan fixed point theorems, using contractive condition: for all $x, y \in X$,

$$(1.3) \quad d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty)$$

where α, β, γ are nonnegative reals with $\alpha + \beta + \gamma < 1$. An example in [20] shows that the condition (1.3) is a proper generalization of (1.1) and (1.2).

In recent years many generalizations of Banach contraction mapping principle have appeared. In 1965 S.B. Prešić [16, 17] extended Banach contraction mapping principle to mappings defined on product spaces and proved following theorem.

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THEOREM 1. (*Prešić-Banach type*) Let (X, d) be a complete metric space, k a positive integer and $T : X^k \rightarrow X$ a mapping satisfying the following contractive type condition:

$$(1.4) \quad d(T(x_1, x_2, \dots, x_k), T(x_2, x_3, \dots, x_{k+1})) \leq \sum_{i=1}^k q_i d(x_i, x_{i+1}),$$

for every $x_1, x_2, \dots, x_{k+1} \in X$, where q_1, q_2, \dots, q_k are nonnegative constants such that $q_1 + q_2 + \dots + q_k < 1$. Then there exists a unique point $x \in X$ such that $T(x, x, \dots, x) = x$. Moreover if x_1, x_2, \dots, x_k are arbitrary points in X and for $n \in \mathbb{N}$, $x_{n+k} = T(x_n, x_{n+1}, \dots, x_{n+k-1})$, then the sequence $\{x_n\}$ is convergent and $\lim x_n = T(\lim x_n, \lim x_n, \dots, \lim x_n)$.

Remark that condition (1.4) in the case $k = 1$ reduces to the well-known Banach contraction mapping principle. So, Theorem 1 is a generalization of the Banach fixed point theorem. Some generalization of theorem 1 can be seen in [3, 5].

Let k be a positive integer and $f : X^k \rightarrow X$ be a mapping, then f is called Prešić-Kannan type contraction if,

$$(1.5) \quad d(f(x_0, \dots, x_{k-1}), f(x_1, \dots, x_k)) \leq a \sum_{i=0}^k d(x_i, f(x_i, \dots, x_i)),$$

for all $x_0, x_1, \dots, x_k \in X$, where real constant a is such that $0 \leq ak(k+1) < 1$. In a similar manner to that used by S.B. Prešić when extending Banach contractions to product spaces, Mădălina Păcurar [18] generalized the Kannan's theorem in product spaces and proved fixed point theorem for Prešić-Kannan type contractions.

f is called Prešić-Reich type contraction if,

$$(1.6) \quad d(f(x_0, \dots, x_{k-1}), f(x_1, \dots, x_k)) \leq \sum_{i=1}^k \alpha_i d(x_{i-1}, x_i) + \sum_{i=0}^k \beta_i d(x_i, f(x_i, \dots, x_i)),$$

for all $x_0, x_1, \dots, x_k \in X$, where α_i, β_i are nonnegative constants such that

$$\sum_{i=1}^k \alpha_i + k \sum_{i=0}^k \beta_i < 1.$$

Note that for $k = 1$ above definition reduced to the definition due to Reich. Also Prešić-Banach type contraction and Prešić-Kannan type contraction are particular cases of Prešić-Reich type contractions.

K-metric and K-normed spaces were introduced in the mid-20th century (see [2, 22, 12, 15]) by using an ordered Banach space instead of the set of real numbers, as the

codomain for a metric. Indeed this idea of replacement of real numbers by an ordered "set" can be seen in [13], [14] (see also their references). Huang and Zhang [6] re-introduced such spaces under the name of cone metric spaces, defining convergent and Cauchy sequence in terms of interior points of underlying cone. They proved the basic version of the fixed point theorem with the assumption that the cone is normal, which were generalized by several authors (see, e.g. [7, 8, 9, 1, 23, 5, 21, 19]). Rezapour and Hambarani [21] removed the assumption of normality of cone and generalized the results of Huang and Zhang in non-normal cone metric spaces.

Cone metric version of Prešić's theorem can be seen in [5]. The purpose of this paper is to generalize and extend Prešić's theorem in cone metric spaces, by proving a Prešić-Reich type common fixed point theorem for two maps, without assuming the normality of cone. Theorems of this paper generalize several known results in metric and cone metric spaces. An example is included which illustrate that the generalization is proper.

2. Preliminaries

We need the following definitions and results, consistent with [4] and [6].

DEFINITION 1. [6] *Let E be a real Banach space and P be a subset of E . The set P is called a cone if:*

- (i) P is closed, nonempty and $P \neq \{\theta\}$, here θ is the zero vector of E ;
- (ii) $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta \geq 0$, $x, y \in P \Rightarrow \alpha x + \beta y \in P$;
- (iii) $x \in P$ and $-x \in P \Rightarrow x = \theta$.

Given a cone $P \subset E$, we define a partial ordering " \leq " with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$. While $x \ll y$ if and only if $y - x \in P^0$, where P^0 denotes the interior of P .

Let P be a cone in a real Banach space E , then P is called normal, if there exist a constant $K > 0$ such that for all $x, y \in E$,

$$\theta \leq x \leq y \Rightarrow \|x\| \leq K\|y\|.$$

The least positive number K satisfying the above inequality is called the normal constant of P .

DEFINITION 2. [6] *Let X be a nonempty set, E be a real Banach space. Suppose that the mapping $d : X \times X \rightarrow E$ satisfies:*

- (i) $\theta \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
- (ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (iii) $d(x, y) \leq d(x, z) + d(y, z)$, for all $x, y, z \in X$.

Then d is called a cone metric on X , and (X, d) is called a cone metric space. If the underlying cone is normal then (X, d) is called a normal cone metric space.

In the following we always suppose that E is a real Banach space, P is a solid cone in E i.e. $P^0 \neq \emptyset$ and " \leq " is partial ordering with respect to P .

The concept of cone metric space is more general than that of a metric space, because each metric space is a cone metric space with $E = \mathbb{R}$ and $P = [0, +\infty)$. For examples and basic properties of normal and non-normal cones and cone metric spaces we refer [6] and [21].

The following remark will be useful in sequel.

REMARK 1. [9] Let P be a cone in a real Banach space E , and $a, b, c \in P$.

- (a) If $a \leq b$ and $b \ll c$ then $a \ll c$.
- (b) If $a \ll b$ and $b \ll c$ then $a \ll c$.
- (c) If $\theta \leq u \ll c$ for every $c \in P^0$ then $u = \theta$.
- (d) If $c \in P^0$, $\theta \leq a_n$ and $a_n \rightarrow \theta$ then there exist $n_0 \in \mathbb{N}$ such that, for all $n > n_0$ we have $a_n \ll c$.
- (e) If $\theta \leq a_n \leq b_n$ for each n and $a_n \rightarrow a$, $b_n \rightarrow b$ then $a \leq b$.
- (f) If $a \leq \lambda a$ where $0 \leq \lambda < 1$ then $a = \theta$.

DEFINITION 3. Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$.

- (a) If for every $c \in E$ with $\theta \ll c$ (or equivalently $c \in P^0$) there is $n_0 \in \mathbb{N}$ such that, $d(x_n, x) \ll c$ for all $n > n_0$. Then the sequence $\{x_n\}$ is said to be convergent and converges to x . We denote this by $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$ as $n \rightarrow \infty$.
- (b) If for every $c \in E$ with $\theta \ll c$ there is $n_0 \in \mathbb{N}$ such that, $d(x_n, x_m) \ll c$, for all $n, m > n_0$. Then the sequence $\{x_n\}$ is called a Cauchy sequence in X .
- (c) (X, d) is said to be a complete cone metric space, if every Cauchy sequence in X is convergent in X .

DEFINITION 4. Let (X, d) be a cone metric space, k a positive integer and $f : X^k \rightarrow X$ be a mapping. If $f(x, x, \dots, x) = x$, then $x \in X$ is called a fixed point of f .

DEFINITION 5. Let (X, d) be a cone metric space, k a positive integer, $f : X^k \rightarrow X$ and $g : X \rightarrow X$ be mappings.

- (i) An element $x \in X$ said to be a coincidence point of f and g if $gx = f(x, \dots, x)$.
- (ii) If $w = gx = f(x, \dots, x)$, then w is called a point of coincidence of f and g .

- (iii) If $x = gx = f(x, \dots, x)$, then x is called a common fixed point of f and g .
- (iv) f and g are said to be commuting if $g(f(x, \dots, x)) = f(gx, \dots, gx)$ for all $x \in X$.
- (v) f and g are said to be weakly compatible if they commute at their coincidence points.

Remark that above definition in the case $k = 1$ reduces to the usual definitions of commuting and weakly compatible mappings in cone metric space.

PROPOSITION 1. Let (X, d) be a cone metric space with solid cone P . Suppose α, β, γ are positive reals and $\{x_n\}, \{y_n\}$ be two sequences in X . Let $a \in P$, $x_n \rightarrow x$, $y_n \rightarrow y$ as $n \rightarrow \infty$ and

$$\alpha a \leq \beta d(x_n, x) + \gamma d(y_n, y),$$

for all $n \in \mathbb{N}$, then $a = \theta$.

Now we can state our main results.

3. Main Results

THEOREM 2. Let (X, d) be a cone metric space, k a positive integer. Let $f : X^k \rightarrow X$, $g : X \rightarrow X$ be two mappings such that $f(X^k) \subset g(X)$, $g(X)$ is complete subspace of X and

$$(3.1) \quad d(f(x_0, x_1, \dots, x_{k-1}), f(x_1, x_2, \dots, x_k)) \leq \sum_{i=1}^k \alpha_i d(gx_{i-1}, gx_i) + \sum_{i=0}^k \beta_i d(gx_i, f(x_i, x_i, \dots, x_i))$$

for all $x_0, x_1, \dots, x_k \in X$, where α_i, β_i are nonnegative constants such that

$$(3.2) \quad \sum_{i=1}^k \alpha_i + k \sum_{i=0}^k \beta_i < 1.$$

Then

1. f and g have a unique point of coincidence say $v \in X$;
2. the sequence $\{y_n\}$ defined by $x_0 \in X$ and

$$y_n = gx_n = f(x_{n-1}, \dots, x_{n-1}), \quad n \geq 1,$$

converges to v ;

3. if in addition f and g are weakly compatible, then v is their unique common fixed point.

Proof. Let x_0 be any arbitrary element of X . Then $f(x_0, \dots, x_0) \in f(X^k) \subset g(X)$, so there exists $x_1 \in X$ such that $y_1 = gx_1 = f(x_0, \dots, x_0)$. In similar manner we can define a sequence $\{y_n\}$ such that

$$(3.3) \quad y_n = gx_n = f(x_{n-1}, \dots, x_{n-1}), \quad n \geq 1.$$

We shall show that $\{y_n\}$ is a Cauchy sequence.

If $y_n = y_{n+1}$ for any n , then

$$\begin{aligned} d(y_{n+1}, y_{n+2}) &= d(f(x_n, \dots, x_n), f(x_{n+1}, \dots, x_{n+1})) \\ &\leq d(f(x_n, \dots, x_n), f(x_n, \dots, x_n, x_{n+1})) \\ &\quad + d(f(x_n, \dots, x_n, x_{n+1}), f(x_n, \dots, x_n, x_{n+1}, x_{n+1})) \\ &\quad + \dots + d(f(x_n, x_{n+1}, \dots, x_{n+1}), f(x_{n+1}, \dots, x_{n+1})). \end{aligned}$$

Using (3.1), above inequality implies that

$$\begin{aligned} d(y_{n+1}, y_{n+2}) &\leq \alpha_k d(gx_n, gx_{n+1}) + \beta_0 d(gx_n, f(x_n, \dots, x_n)) + \dots \\ &\quad + \beta_{k-1} d(gx_n, f(x_n, \dots, x_n)) + \beta_k d(gx_{n+1}, f(x_{n+1}, \dots, x_{n+1})) \\ &\quad + \alpha_{k-1} d(gx_n, gx_{n+1}) + \beta_0 d(gx_n, f(x_n, \dots, x_n)) + \dots \\ &\quad + \beta_{k-1} d(gx_{n+1}, f(x_{n+1}, \dots, x_{n+1})) + \beta_k d(gx_{n+1}, f(x_{n+1}, \dots, x_{n+1})) \\ &\quad + \dots + \alpha_1 d(gx_n, gx_{n+1}) + \beta_0 d(gx_n, f(x_n, \dots, x_n)) \\ &\quad + \beta_1 d(gx_{n+1}, f(x_{n+1}, \dots, x_{n+1})) + \dots + \beta_k d(gx_{n+1}, f(x_{n+1}, \dots, x_{n+1})), \end{aligned}$$

i.e.

$$\begin{aligned} d(y_{n+1}, y_{n+2}) &\leq \left[\sum_{i=1}^k \alpha_i \right] d(y_n, y_{n+1}) + \beta_0 d(y_n, y_{n+1}) + \dots + \beta_{k-1} d(y_n, y_{n+1}) \\ &\quad + \beta_k d(y_{n+1}, y_{n+2}) + \beta_0 d(y_n, y_{n+1}) + \dots + \beta_{k-1} d(y_{n+1}, y_{n+2}) \\ &\quad + \beta_k d(y_{n+1}, y_{n+2}) + \dots + \beta_0 d(y_n, y_{n+1}) + \beta_1 d(y_{n+1}, y_{n+2}) \\ &\quad + \dots + \beta_k d(y_{n+1}, y_{n+2}) \\ &\leq [k\beta_k + (k-1)\beta_{k-1} + \dots + \beta_1] d(y_{n+1}, y_{n+2}) \quad (\text{since } y_n = y_{n+1}). \end{aligned}$$

In view of (3.2), we have $k\beta_k + (k-1)\beta_{k-1} + \dots + \beta_1 = \sum_{i=0}^k i\beta_i \leq k \sum_{i=0}^k \beta_i < 1$, so using (f) of remark 1 in above inequality we obtain $d(y_{n+1}, y_{n+2}) = \theta$ i.e. $y_{n+1} = y_{n+2}$. Similarly it can be shown that

$$y_n = y_{n+1} = y_{n+2} = \dots = v \quad (\text{say}).$$

Therefore $\{y_n\}$ is a Cauchy sequence.

If $y_n \neq y_{n+1}$ for all n , then

$$\begin{aligned} d(y_n, y_{n+1}) &= d(f(x_{n-1}, \dots, x_{n-1}), f(x_n, \dots, x_n)) \\ &\leq d(f(x_{n-1}, \dots, x_{n-1}), f(x_{n-1}, \dots, x_{n-1}, x_n)) \\ &\quad + d(f(x_{n-1}, \dots, x_{n-1}, x_n), f(x_{n-1}, \dots, x_{n-1}, x_n, x_n)) \\ &\quad + \dots + d(f(x_{n-1}, x_n, \dots, x_n), f(x_n, \dots, x_n)). \end{aligned}$$

Using (3.1), above inequality implies that

$$\begin{aligned} d(y_n, y_{n+1}) &\leq \alpha_k d(gx_{n-1}, gx_n) + \beta_0 d(gx_{n-1}, f(x_{n-1}, \dots, x_{n-1})) + \dots \\ &\quad + \beta_{k-1} d(gx_{n-1}, f(x_{n-1}, \dots, x_{n-1})) + \beta_k d(gx_n, f(x_n, \dots, x_n)) \\ &\quad + \alpha_{k-1} d(gx_{n-1}, gx_n) + \beta_0 d(gx_{n-1}, f(x_{n-1}, \dots, x_{n-1})) + \dots \\ &\quad + \beta_{k-1} d(gx_n, f(x_n, \dots, x_n)) + \beta_k d(gx_n, f(x_n, \dots, x_n)) \\ &\quad + \dots + \alpha_1 d(gx_{n-1}, gx_n) + \beta_0 d(gx_{n-1}, f(x_{n-1}, \dots, x_{n-1})) \\ &\quad + \beta_1 d(gx_n, f(x_n, \dots, x_n)) + \dots + \beta_k d(gx_n, f(x_n, \dots, x_n)), \end{aligned}$$

i.e.

$$\begin{aligned} d(y_n, y_{n+1}) &\leq \left[\sum_{i=1}^k \alpha_i \right] d(y_{n-1}, y_n) + \beta_0 d(y_{n-1}, y_n) + \dots + \beta_{k-1} d(y_{n-1}, y_n) \\ &\quad + \beta_k d(y_n, y_{n+1}) + \beta_0 d(y_{n-1}, y_n) + \dots + \beta_{k-1} d(y_n, y_{n+1}) \\ &\quad + \beta_k d(y_n, y_{n+1}) + \dots + \beta_0 d(y_{n-1}, y_n) + \beta_1 d(y_n, y_{n+1}) \\ &\quad + \dots + \beta_k d(y_n, y_{n+1}). \end{aligned}$$

Writing $d_n = d(y_n, y_{n+1})$ we obtain

$$\begin{aligned} d_n &\leq \left[\sum_{i=1}^k \alpha_i + \sum_{i=0}^{k-1} (k-i)\beta_i \right] d_{n-1} + \left[\sum_{i=1}^k i\beta_i \right] d_n \\ d_n &\leq \frac{\sum_{i=1}^k \alpha_i + \sum_{i=0}^k (k-i)\beta_i}{1 - \sum_{i=0}^k i\beta_i} d_{n-1} \\ d_n &\leq \lambda d_{n-1} \text{ (say)}. \end{aligned}$$

Let $A = \sum_{i=1}^k \alpha_i, B = k \sum_{i=0}^k \beta_i, C = \sum_{i=0}^k i\beta_i$, then in view of (3.2) we have,

$$\lambda = \frac{\sum_{i=1}^k \alpha_i + \sum_{i=0}^k (k-i)\beta_i}{1 - \sum_{i=0}^k i\beta_i} = \frac{A+B-C}{1-C} < 1.$$

By repeating this process we obtain

$$(3.4) \quad d_n \leq \lambda^n d_0.$$

Let $m, n \in \mathbb{N}$ and $m > n$, then it follows from (3.4) that

$$\begin{aligned} d(y_n, y_m) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + \cdots + d(y_{m-1}, y_m) \\ &\leq d_n + d_{n+1} + d_{n+2} + \cdots \\ &\leq \lambda^n d_0 + \lambda^{n+1} d_0 + \lambda^{n+2} d_0 + \cdots \\ &= [1 + \lambda + \lambda^2 + \cdots] \lambda^n d_0 \\ d(y_n, y_m) &\leq \frac{\lambda^n}{1 - \lambda} d_0. \end{aligned}$$

As $\lambda < 1$, we have $\frac{\lambda^n}{1 - \lambda} d_0 \rightarrow \theta$ as $n \rightarrow \infty$. Therefore by (a) and (d) of remark 1, for every $c \in P^0$ there exists $n_0 \in \mathbb{N}$ such that, $d(y_n, y_m) \ll c$, for all $n > n_0$. Therefore $\{y_n\} = \{gx_n\}$ is a Cauchy sequence. As $g(X)$ is complete, there exist $v \in g(X), u \in X$ such that

$$\lim_{n \rightarrow \infty} y_n = gu = v.$$

We shall show that v is point of coincidence of f and g . By (3.1) we obtain

$$\begin{aligned} d(gu, f(u, \dots, u)) &\leq d(gu, y_{n+1}) + d(y_{n+1}, f(u, \dots, u)) \\ &\leq d(gu, y_{n+1}) + d(f(x_n, \dots, x_n), f(u, \dots, u)) \\ &\leq d(gu, y_{n+1}) + d(f(x_n, \dots, x_n), f(x_n, \dots, x_n, u)) \\ &\quad + d(f(x_n, \dots, x_n, u), f(x_n, \dots, x_n, u, u)) + \cdots \\ &\quad + d(f(x_n, u, \dots, u), f(u, \dots, u)), \end{aligned}$$

using (3.1) and (3.3), we obtain

$$\begin{aligned} d(gu, f(u, \dots, u)) &\leq \sum_{i=1}^k \alpha_i d(y_n, gu) + \sum_{i=0}^k (k-i) \beta_i d(y_n, y_{n+1}) \\ &\quad + \sum_{i=0}^k i \beta_i d(gu, f(u, \dots, u)) + d(gu, y_{n+1}), \end{aligned}$$

i.e.

$$\begin{aligned} (1 - C)d(gu, f(u, \dots, u)) &\leq Ad(y_n, gu) + (B - C)[d(y_n, gu) + d(gu, y_{n+1})] \\ &\quad + d(gu, y_{n+1}) \\ (1 - C)d(gu, f(u, \dots, u)) &\leq (A + B - C)d(y_n, gu) + (1 + B - C)d(y_{n+1}, gu). \end{aligned}$$

As $1 - C = 1 - \sum_{i=1}^k i \beta_i > 0$ and $\lim_{n \rightarrow \infty} y_n = gu$, by proposition 1 it follows that $d(gu, f(u, \dots, u)) = \theta$, i.e.

$$(3.5) \quad f(u, \dots, u) = gu = v.$$

Thus v is point of coincidence of f and g . We shall show that it is unique.

Let q is another point of coincidence of f and g i.e., there exists $p \in X$ such that

$$f(p, \dots, p) = gp = q.$$

From (3.1) it follows that

$$\begin{aligned} d(v, q) &= d(f(u, \dots, u), f(p, \dots, p)) \\ &\leq d(f(u, \dots, u), f(u, \dots, u, p)) + d(f(u, \dots, u, p), f(u, \dots, u, p, p)) + \dots \\ &\quad + d(f(u, p, \dots, p), f(p, \dots, p)) \\ &\leq \sum_{i=1}^k \alpha_i d(gu, gp) = Ad(v, q), \end{aligned}$$

as $A = \sum_{i=1}^k \alpha_i < 1$, from (f) of remark 1 it follows that

$$d(v, q) = \theta \text{ i.e. } v = q.$$

Thus f and g have a unique point of coincidence v .

If f and g are weakly compatible then by (3.5) we have $f(v, \dots, v) = f(gu, \dots, gu) = gf(u, \dots, u) = gv$, i.e. $f(v, \dots, v) = gv = v'$ (say). So v' is another point of coincidence of f and g and by uniqueness it follows that $v' = v$ i.e. $f(v, \dots, v) = gv = v$. Thus v is unique common fixed point of f and g . □

REMARK 2. For $k = 1$ mapping f in condition (3.1) reduces to g -weak contraction (see [23]), therefore above theorem is a generalization of result of Vetro [23] in product spaces.

Following is an example of Prešić-Reich type contraction which is neither a Prešić-Banach type contraction nor a Prešić-Kannan type contraction in product spaces (see [17, 18]).

EXAMPLE 1. Let $X = [0, 1]$, $E = C_{\mathbb{R}}^1[0, 1]$ with norm $\|f\| = \|f\|_{\infty} + \|f'\|_{\infty}$, $P = \{f : f(t) \geq 0 \text{ for all } t \in [0, 1]\}$. Define $d : X \times X \rightarrow E$ by $d(x, y) = |x - y|\varphi(t)$, where $\varphi(t) = e^t \in E$. Then (X, d) is a complete cone metric space with non-normal cone P . For $k = 2$ define $f : X^2 \rightarrow X$ and $g : X \rightarrow X$ by

$$f(x, y) = \begin{cases} \frac{1}{12} & \text{if } x = y = 1; \\ \frac{x+y}{6} & \text{otherwise,} \end{cases}$$

and

$$gx = x \text{ for all } x \in [0, 1].$$

Then

- (i) f and g satisfy (3.1) and all other conditions of theorem 2, with $\alpha_1 = \alpha_2 = \frac{1}{6}$ and $\beta_1 = \beta_2 = \beta_3 = \frac{1}{10}$;
- (ii) f is not a contraction in Prešić's sense i.e. f does not satisfy (1.4);
- (iii) f is not a contraction in Păcurar's sense i.e. f does not satisfy (1.5).

Proof. (i): For $k = 2$ and $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta_3 = \beta$ and $gx = x$, condition (3.1) becomes

$$(3.6) \quad d(f(x_0, x_1), f(x_1, x_2)) \leq \alpha[d(x_0, x_1) + d(x_1, x_2)] + \beta[d(x_0, f(x_0, x_0)) + d(x_1, f(x_1, x_1)) + d(x_2, f(x_2, x_2))],$$

for all $x_0, x_1, x_2 \in X$, where α, β are nonnegative constants such that $2\alpha + 6\beta < 1$.

Note that, if $x_0 = x_1 = x_2 = 1$, then (3.6) is satisfied trivially.

If $x_0, x_1, x_2 \in [0, 1)$, then for the validity of (3.6) it is sufficient that

$$\begin{aligned} d(f(x_0, x_1), f(x_1, x_2)) &\leq \alpha[d(x_0, x_1) + d(x_1, x_2)] \\ \Leftrightarrow d\left(\frac{x_0 + x_1}{6}, \frac{x_1 + x_2}{6}\right) &\leq \alpha[d(x_0, x_1) + d(x_1, x_2)] \\ \Leftrightarrow \frac{1}{6}|x_0 + x_1 - x_1 - x_2|e^t &\leq \alpha[|x_0 - x_1| + |x_1 - x_2|]e^t \\ \Leftrightarrow |x_0 - x_2|e^t &\leq 6\alpha[|x_0 - x_1| + |x_1 - x_2|]e^t, \end{aligned}$$

which is valid for $\alpha = \frac{1}{6}$ and $\beta \in [0, 1)$.

If any two of x_0, x_1, x_2 are equal to 1, e.g. if $x_0 = x_1 = 1$ and $x_2 \in [0, 1)$, then for the validity of (3.6) it is sufficient that

$$\begin{aligned} d(f(1, 1), f(1, x_2)) &\leq \beta[d(1, f(1, 1)) + d(1, f(1, 1)) + d(x_2, f(x_2, x_2))] \\ \Leftrightarrow d\left(\frac{1}{12}, \frac{1+x_2}{6}\right) &\leq \beta\left[d\left(1, \frac{1}{12}\right) + d\left(1, \frac{1}{12}\right) + d\left(x_2, \frac{2x_2}{6}\right)\right] \\ \Leftrightarrow \left|\frac{1}{12} - \frac{1+x_2}{6}\right|e^t &\leq \beta\left[\frac{11}{6} + \frac{2x_2}{3}\right]e^t \\ \Leftrightarrow \frac{1}{12}(1+2x_2)e^t &\leq \frac{1}{6}\beta(11+4x_2)e^t \\ \Leftrightarrow (1+2x_2)e^t &\leq 2\beta(11+4x_2)e^t \end{aligned}$$

which is valid for $\alpha \in [0, 1)$ and $\beta = \frac{1}{10}$. If we take $\alpha = \frac{1}{6}, \beta = \frac{1}{10}$ then

$$2\alpha + 6\beta = \frac{1}{3} + \frac{3}{5} = \frac{14}{15} < 1.$$

Similarly in all possible cases (3.6) is satisfied with $\alpha = \frac{1}{6}, \beta = \frac{1}{10}$.

Clearly $f(X^k) \subset g(X)$ and all the conditions of theorem 2 are satisfied and $v = 0$ is the unique common fixed point of f and g , i.e.

$$g0 = f(0, 0) = 0.$$

(ii): For $k = 2$, condition (1.4) becomes

$$(3.7) \quad d(f(x_0, x_1), f(x_1, x_2)) \leq \alpha_1 d(x_0, x_1) + \alpha_2 d(x_1, x_2),$$

for all $x_0, x_1, x_2 \in X$, where α_1, α_2 are nonnegative constants such that $\alpha_1 + \alpha_2 < 1$.

We shall show that condition (3.7) is not satisfied for certain points in $[0, 1]$.

Let $x_0 = x_1 = 1, x_2 = \frac{9}{10}$, then (3.7) becomes

$$\begin{aligned} d(f(1, 1), f(1, \frac{9}{10})) &\leq \alpha_1 d(1, 1) + \alpha_2 d(1, \frac{9}{10}) \\ \Leftrightarrow d(\frac{1}{12}, \frac{19}{60}) &\leq \alpha_2 \frac{1}{10} e^t \\ \Leftrightarrow \frac{7}{30} e^t &\leq \alpha_2 \frac{1}{10} e^t \\ \Leftrightarrow \frac{7}{3} &\leq \alpha_2. \end{aligned}$$

But $\alpha_1 + \alpha_2 < 1$, therefore above inequality will never hold. Thus f is not a contraction in Prešić's sense.

(iii): For $k = 2$, condition (1.5) becomes

$$(3.8) \quad d(f(x_0, x_1), f(x_1, x_2)) \leq a[d(x_0, f(x_0, x_0)) + d(x_1, f(x_1, x_1)) + d(x_2, f(x_2, x_2))]$$

for all $x_0, x_1, x_2 \in X$, where real constant a is such that $0 \leq a < \frac{1}{6}$.

Let $x_0 = x_1 = 0, x_2 \in (0, 1)$, then (3.8) becomes

$$\begin{aligned} d(f(0, 0), f(0, x_2)) &\leq a[d(0, f(0, 0)) + d(0, f(0, 0)) + d(x_2, f(x_2, x_2))] \\ \Leftrightarrow d(0, \frac{x_2}{6}) &\leq a[d(0, 0) + d(0, 0) + d(x_2, \frac{x_2}{3})] \\ \Leftrightarrow \frac{x_2}{6} e^t &\leq a \frac{2x_2}{3} e^t \\ \Leftrightarrow \frac{1}{4} &\leq a. \end{aligned}$$

But $0 \leq a < \frac{1}{6}$, therefore above inequality will never hold. Thus f is not a contraction in Păcurar's sense. □

Remark that for $g = I_X$ i.e. identity mapping of X , condition (3.1) reduces to the definition of Prešić-Reich type mapping. Therefore we obtain the following corollary.

COROLLARY 1. *Let (X, d) be a complete cone metric space, k a positive integer. Let $f : X^k \rightarrow X$ be a Prešić-Reich type mapping. Then f has a unique fixed point $v \in X$. Moreover the sequence $\{x_n\}$ defined by $x_0 \in X$ and $x_n = f(x_{n-1}, \dots, x_{n-1}), n \geq 1$, converges to v .*

Taking $\beta_i = 0$ for $i = 0, 1, \dots, k$ in theorem 2 we get following generalization of Prešić's result [17] in cone metric spaces.

COROLLARY 2. *Let (X, d) be a cone metric space, k a positive integer. Let $f : X^k \rightarrow X, g : X \rightarrow X$ be two mappings such that $f(X^k) \subset g(X), g(X)$ is complete subspace*

of X and

$$d(f(x_0, \dots, x_{k-1}), f(x_1, \dots, x_k)) \leq \sum_{i=1}^k \alpha_i d(gx_{i-1}, gx_i)$$

for all $x_0, \dots, x_k \in X$, where α_i are nonnegative constants such that $\sum_{i=1}^k \alpha_i < 1$. Then

1. f and g have a unique point of coincidence say $v \in X$;
2. the sequence $\{y_n\}$ defined by $x_0 \in X$ and

$$y_n = gx_n = f(x_{n-1}, \dots, x_{n-1}), n \geq 1,$$

converges to v ;

3. if in addition f and g are weakly compatible, then v is their unique common fixed point.

Again, taking $\alpha_i = 0$ for $i = 1, \dots, k$, we get following generalization of Prešić-Kannan type mapping in cone metric spaces (see [18], Theorem 3.1).

COROLLARY 3. Let (X, d) be a cone metric space, k a positive integer. Let $f : X^k \rightarrow X$, $g : X \rightarrow X$ be two mappings such that $f(X^k) \subset g(X)$, $g(X)$ is complete subspace of X and

$$d(f(x_0, \dots, x_{k-1}), f(x_1, \dots, x_k)) \leq \sum_{i=0}^k \beta_i d(gx_i, f(x_i, \dots, x_i))$$

for all $x_0, \dots, x_k \in X$, where β_i are nonnegative constants such that, $k \sum_{i=0}^k \beta_i < 1$. Then

1. f and g have a unique point of coincidence say $v \in X$;
2. the sequence $\{y_n\}$ defined by $x_0 \in X$ and

$$y_n = gx_n = f(x_{n-1}, \dots, x_{n-1}), n \geq 1,$$

converges to v ;

3. if in addition f and g are weakly compatible, then v is their unique common fixed point.

REMARK 3. Note that if $g = I_X$ and $\beta_i = a$ for $i = 1, \dots, k$, then f in above corollary reduced to Prešić-Kannan type contraction, while with $\beta_i = a$ for $i = 1, \dots, k$, above corollary gives an extension of Theorem 3.1 of Păcurar [18] in cone metric spaces.

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