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NONLINEAR DIFFERENTIAL POLYNOMIALS SHARING TWO VALUES IM

The authors dedicate the paper to their teacher respected Prof. Indrajit Lahiri who is their main inspiration for research work.

Abstract. In the Paper, we study the uniqueness of meromorphic functions concerning some general nonlinear differential polynomials sharing fixed point and infinity in which multiplicity is not taken into account. The results of the paper improve and generalize some recent results due to the present first author [Bull. Math. Anal. Appl., 2(2010), 106-118] and as well as of J. Wang, W. LU and Y. Chen [Applied Math. E-Notes, 11(2011), 91-100].

1. Introduction, Definitions and Results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [5], [16] and [19]. For a nonconstant meromorphic function f, we denote by $T(r, f)$ the Nevanlinna characteristic of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ possibly outside a set of finite linear measure.

If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with same multiplicities then we say that f and g share the value a CM (counting multiplicities). If we do not take the multiplicities into account, f and g are said to share the value a IM (ignoring multiplicities). A finite value z_0 is said to be a fixed point of $f(z)$ if $f(z_0) = z_0$. For a positive integer m and a number μ, let $m^* = \chi_\mu m$, where $\chi_\mu = 0$ if $\mu = 0$ and $\chi_\mu = 1$ if $\mu \neq 0$. Throughout this paper, we need the following definition.

$$\Theta(a, f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r, a; f)}{T(r, f)},$$

where a is a value in the extended complex plane.

In 1959, W.K. Hayman (see [4], Corollary of Theorem 9) proved the following theorem.

Theorem A. Let f be a transcendental meromorphic function and $n(\geq 3)$ is an integer. Then $f^n f' = 1$ has infinitely many solutions.

Corresponding to which, C.C. Yang and X.H. Hua obtained the following result in 1997.

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Theorem B. [15] Let f and g be two nonconstant meromorphic functions, $n \geq 11$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.

Using the idea of sharing fixed points, M.L. Fang and H.L. Qiu proved the following result in 2002.

Theorem C. [3] Let f and g be two nonconstant meromorphic functions, and let $n \geq 11$ be a positive integer. If $f^n f' - z$ and $g^n g' - z$ share 0 CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $4(c_1 c_2)^{n+1} c^2 = -1$ or $f = tg$ for a complex number t such that $t^{n+1} = 1$.

For the last couple of years a handful number of astonishing results have been obtained regarding the value sharing of nonlinear differential polynomials which are mainly the k-th derivative of some linear expression of f and g (see. [2], [11]). In 2010, J.F. Xu, F. Lu and H.X. Yi proved the following results.

Theorem D. [13] Let f and g be two nonconstant meromorphic functions, and let n, k be two positive integers with $n > 3k + 10$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, f and g share ∞ IM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4n^2 (c_1 c_2)^n c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^n = 1$.

Theorem E. [13] Let f and g be two nonconstant meromorphic functions satisfying $\Theta(\infty, f) > \frac{2}{n}$, and let n, k be two positive integers with $n \geq 3k + 12$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share z CM, f and g share ∞ IM, then $f \equiv g$.

Naturally one may ask the following question.
Question 1. What can be said if we do not consider the multiplicity into account in Theorem D and Theorem E ?

Recently, J. Wang, W. Lu and Y. Chen proved the following theorems which dealt with above question.

Theorem F. [12] Let f and g be two nonconstant meromorphic functions, and n, k, m be three positive integers with $n > 9k + 6m^* + 13$. Suppose $(f^n(\mu f^m + \lambda))^{(k)}$, $(g^n(\mu g^m + \lambda))^{(k)}$ share 1 IM, where λ, μ are constants such that $|\lambda| + |\mu| \neq 0$, and f, g share ∞ IM.
(i) If $\lambda\mu \neq 0$, $m > 1$ and $(n, n+m) = 1$, or while $m = 1$ and $\Theta(\infty, f) > 2/n$, then $f \equiv g$;
(ii) if $\lambda\mu = 0$, then either $f = tg$, where t is a constant satisfying $t^{n+m^*} = 1$ or $f = c_1 e^{cz^2}$, $g = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants such that $(-1)^k \lambda^2 (c_1 c_2)^{n+m^*} [(n + m^*)c]^{2k} = 1$ or $(-1)^k \mu^2 (c_1 c_2)^{n+m^*} [(n+m^*)c]^{2k} = 1$.

Theorem G. [12] Let f, g be two transcendental meromorphic functions, and n, k, m be three positive integers with $n > 9k + 4m + 15$. If $(f^n(f-1)^m)^{(k)}$, $(g^n(g-1)^m)^{(k)}$ share 1 IM and f, g share ∞ IM, then either $f \equiv g$ or $f^n(f-1)^m \equiv g^n(g-1)^m$.

Recently, the present first author haved also worked on the above direction and obtained the following result which improves as well as generalizes Theorems F and G.

Theorem H. [10] Let f and g be two transcendental meromorphic functions, and let n, k and m be three positive integers such that $n > 9k + 4m + 13$. Let $P(z) = a_m z^m + ... + a_1 z + a_0$, where $a_0(\neq 0)$, a_1, ... , $a_m(\neq 0)$ are complex constants. Suppose that $[f^n P(f)]^{(k)}$, $[g^n P(g)]^{(k)}$ share z IM and f, g share ∞ IM. Then either $f(z) = tg(z)$ for a constant t such that $t^d = 1$, where $d = (n+m, ..., n+m-i, ..., n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, 2, ..., m$ or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where

$$R(f,g) = f^n P(f) - g^n P(g). \tag{1.1}$$

Now observing the above results the following question is inevitable.

Question 2. Is it possible in any way to further reduce the lower bound of n in Theorem H?

In the paper, taking the possible answer of the above question into background we will discuss all forms of the polynomial as mentioned in Theorem H and thus provide a compact result in this perspective. Indeed the following theorem which is the main result of the paper justify our claim.

Theorem 1. *Let f and g be two transcendental meromorphic functions, and let n, k and m be three positive integers such that $n > 9k + 4m + 11$. Let $P(z) = a_m z^m + ... + a_1 z + a_0$, where $a_0(\neq 0)$, a_1, ... , a_m are complex constants. Suppose that $[f^n P(f)]^{(k)}$, $[g^n P(g)]^{(k)}$ share z IM and f, g share ∞ IM. Then the following statements are valid.*
(i) When $m = 0$, $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4a_0^2 (c_1 c_2)^n (nc)^2 = -1$ or $f \equiv tg$ for a constant t such that $t^n = 1$.
(ii) When $m = 1$, $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$ then $f \equiv g$.
(iii) When $m \geq 2$, then either $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = (n+m, ..., n+m-i, ..., n+1, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, 2, ..., m$ or f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(f, g)$ is given by (1.1).

Corollary 1. *Under the same condition of Theorem 1, we set $P(z) = \mu z^m + \lambda$, where λ and μ are two constants such that $|\lambda| + |\mu| \neq 0$ and m be a positive integer. If $n > 9k + 4m^* + 11$, then the following statements are valid.*
(i) Suppose $\lambda\mu \neq 0$. If $m = 1$ and $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$, then $f \equiv g$. If $m > 1$ then $f \equiv tg$, where t is a constant satisfying $t^d = 1$, $d = (n+m, n)$.
(ii) When $\lambda\mu = 0$, then either $f \equiv tg$, where t is a constant satisfying $t^{n+m^*} = 1$, or $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(\lambda + \mu)^2 (c_1 c_2)^{n+m^*} [(n+m^*)c]^2 = -1$.

Corollary 2. *Under the same condition of Theorem 1, if $P(z) = (z-1)^m$, then the conclusion of Theorem 1 holds where $R(f, g)$ is given by $R(w_1, w_2) = w_1^n (w_1 - 1)^m - w_2^n (w_2 - 1)^m$.*

Remark 1. Theorem 1 improves Theorem H by reducing the lower bound of n.

Remark 2. Corollary 1 and Corollary 2 improves Theorems F and G respectively.

We now explain following definitions and notations which are used in the paper.

DEFINITION 1. *[6] Let* $a \in \mathbb{C} \cup \{\infty\}$. *For a positive integer* p *we denote by* $N(r,a;f \mid\leq p)$ *the counting function of those a-points of f (counted with multiplicities) whose multiplicities are not greater than* p. *By* $\overline{N}(r,a;f \mid\leq p)$ *we denote the corresponding reduced counting function.*

In an analogous manner we can define $N(r,a;f \mid\geq p)$ *and* $\overline{N}(r,a;f \mid\geq p)$.

DEFINITION 2. *[8] Let k be a positive integer or infinity. We denote by* $N_k(r,a;f)$ *the counting function of a-points of f, where an a-point of multiplicity m is counted m times if* $m \leq k$ *and k times if* $m > k$. *Then*

$$N_k(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geq 2) + \ldots + \overline{N}(r,a;f \mid\geq k).$$

Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.

DEFINITION 3. *[7, 8] Let k be a nonnegative integer or infinity. For* $a \in \mathbb{C} \cup \{\infty\}$ *we denote by* $E_k(a;f)$ *the set of all a-points of f where an a-point of multiplicity m is counted m times if* $m \leq k$ *and k+1 times if* $m > k$. *If* $E_k(a;f) = E_k(a;g)$, *we say that f, g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k, then z_0 *is an a-point of f with multiplicity* $m(\leq k)$ *if and only if it is an a-point of g with multiplicity* $m(\leq k)$ *and* z_0 *is an a-point of f with multiplicity* $m(> k)$ *if and only if it is an a-point of g with multiplicity* $n(> k)$, *where m is not necessarily equal to n.*

We write f, g share (a,k) *to mean that f, g share the value a with weight k. Clearly if f, g share* (a,k) *then f, g share* (a,p) *for any integer p,* $0 \leq p < k$. *Also we note that f, g share a value a IM or CM if and only if f, g share* $(a,0)$ *or* (a,∞) *respectively.*

DEFINITION 4. *[1] Let f and g be two nonconstant meromorphic functions such that f and g share the value a IM for* $a \in \mathbb{C} \cup \{\infty\}$. *Let* z_0 *be an a-point of f with multiplicity p and also an a-point of g with multiplicity q. We denote by* $\overline{N}_L(r,a;f)$ $(\overline{N}_L(r,a;g))$ *the reduced counting function of those a-points of f and g, where* $p > q \geq 1$ $(q > p \geq 1)$. *Also we denote by* $\overline{N}_E^{(1}(r,a;f)$ *the reduced counting function of those a-points of f and g, where* $p = q \geq 1$.

DEFINITION 5. *[7, 8] Let f and g be two nonconstant meromorphic functions such that f and g share the value a IM. We denote by* $\overline{N}_*(r,a;f,g)$ *the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g. Clearly* $\overline{N}_*(r,a;f,g) = \overline{N}_*(r,a;g,f)$ *and* $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$.

2. Lemmas

Let F and G be two nonconstant meromorphic functions defined in $\mathbb{C}$. We denote by H the function as follows:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

LEMMA 1. *[14] Let f be a nonconstant meromorphic function and let $a_n(z)(\not\equiv 0)$, $a_{n-1}(z)$, ... , $a_0(z)$ be meromorphic functions such that $T(r,a_i(z)) = S(r,f)$ for $i = 0,1,2,...,n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0) = nT(r,f) + S(r,f).$$

LEMMA 2. *[20] Let f be a nonconstant meromorphic function, and p, k be positive integers. Then*

$$N_p\left(r,0;f^{(k)}\right) \le T\left(r,f^{(k)}\right) - T(r,f) + N_{p+k}(r,0;f) + S(r,f), \tag{2.1}$$

$$N_p\left(r,0;f^{(k)}\right) \le k\overline{N}(r,\infty;f) + N_{p+k}(r,0;f) + S(r,f). \tag{2.2}$$

LEMMA 3. *[5, 16] Let f be a transcendental meromorphic function, and let $a_1(z)$, $a_2(z)$ be two distinct meromorphic functions such that $T(r,a_i(z)) = S(r,f)$, i=1,2. Then*

$$T(r,f) \le \overline{N}(r,\infty;f) + \overline{N}(r,a_1;f) + \overline{N}(r,a_2;f) + S(r,f).$$

LEMMA 4. *[18] Let f and g be two nonconstant meromorphic functions that share 1 IM. Then*

$$\overline{N}_L(r,1;f) \le \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + S(r,f).$$

The similar result holds for g also.

LEMMA 5. *[10] Let F, G be two nonconstant meromorphic functions that share $1, \infty$ IM and $H \not\equiv 0$. Then*
(i) $T(r,F) \le N_2(r,0;F) + N_2(r,0;G) + 3\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G)$
$+\overline{N}_*(r,\infty;F,G) + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,F) + S(r,G)$.
(ii) $T(r,G) \le N_2(r,0;F) + N_2(r,0;G) + 2\overline{N}(r,\infty;F) + 3\overline{N}(r,\infty;G)$
$+\overline{N}_*(r,\infty;F,G) + \overline{N}(r,0;F) + 2\overline{N}(r,0;G) + S(r,F) + S(r,G)$;

LEMMA 6. *[17] Suppose that f and g be two nonconstant meromorphic functions. Let*

$$V = \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right), \tag{2.3}$$

where $F = \frac{(f^nP(f))^{(k)}}{z}$, $G = \frac{(g^nP(g))^{(k)}}{z}$, $n(\ge 1)$, $k(\ge 1)$, $m(\ge 0)$ are positive integers and $P(z)$ be defined as in Theorem H. If F, G share ∞ IM and $V \equiv 0$, then $F \equiv G$.

LEMMA 7. *Suppose that f and g be two nonconstant meromorphic functions. Let V be given by (2.3), F, G are defined as in Lemma 6 and $V \not\equiv 0$. If f, g share ∞ IM and F, G share 1 IM, then the poles of F and G are zeros of V and*

$$\begin{aligned}(n+m-3k-3)\overline{N}(r,\infty;f\mid\geq 1) &= (n+m-3k-3)\overline{N}(r,\infty;g\mid\geq 1)\\ &\leq 2(k+m+1)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).\end{aligned}$$

Proof. Since f, g share ∞ IM, it follows that F, G share (∞,k) and so a pole of F with multiplicity $p(\geq k+1)$ is a pole of G with multiplicity $q(\geq k+1)$ and vice versa. It is clear that F and G have no pole of multiplicity r where $k<r<n+m+k$. Now using the Milloux theorem [5], p. 55, and Lemma 1, we obtain from the definition of V that

$$m(r,V)=S(r,f)+S(r,g).$$

Thus using Lemma 4 and (2.2) we get

$$\begin{aligned}(n+m+k-1)\overline{N}(r,\infty;f\mid\geq 1) &= (n+m+k-1)\overline{N}(r,\infty;g\mid\geq 1)\\ &= (n+m+k-1)\overline{N}(r,\infty;F\mid\geq n+m+k)\\ &\leq N(r,0;V)\\ &\leq T(r,V)+O(1)\\ &\leq N(r,\infty;V)+m(r,V)+O(1)\\ &\leq \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}_*(r,1;F,G)\\ &\quad +S(r,f)+S(r,g)\\ &\leq 2\overline{N}(r,0;F)+2\overline{N}(r,0;G)+\overline{N}(r,\infty;F)\\ &\quad +\overline{N}(r,\infty;G)+S(r,f)+S(r,g)\\ &\leq 2N_{k+1}(r,0;f^nP(f))+(2k+1)\overline{N}(r,\infty;f\mid\geq 1)\\ &\quad +2N_{k+1}(r,0;g^nP(g))+(2k+1)\overline{N}(r,\infty;g\mid\geq 1)\\ &\quad +S(r,f)+S(r,g).\end{aligned}$$

This gives

$$\begin{aligned}(n+m-3k-3)\overline{N}(r,\infty;f\mid\geq 1) &= (n+m-3k-3)\overline{N}(r,\infty;g\mid\geq 1)\\ &\leq 2(k+m+1)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).\end{aligned}$$

This completes the proof of the lemma. □

LEMMA 8. *[10] Let f and g be two transcendental meromorphic functions and let n, k be two positive integers. Suppose that F, G are defined as in Lemma 6. If there exist two nonzero constants c_1 and c_2 such that $\overline{N}(r,c_1;F_1)=\overline{N}(r,0;G_1)$ and $\overline{N}(r,c_2;G_1)=\overline{N}(r,0;F_1)$, then $n\leq 3k+m+3$.*

LEMMA 9. *[10] Let f and g be two transcendental meromorphic functions and let n, k be two positive integers. Suppose that $F_1=(f^nP(f))^{(k)}$ and $G_1=(g^nP(g))^{(k)}$ where $P(z)$ be defined as in Theorem H. If there exist two nonzero constants d_1 and d_2 such that $\overline{N}(r,d_1;F_1)=\overline{N}(r,0;G_1)$ and $\overline{N}(r,d_2;G_1)=\overline{N}(r,0;F_1)$, then $n\leq 3k+m+3$.*

LEMMA 10. *Let f and g be two nonconstant meromorphic functions such that*

$$\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n},$$

where $n(\geq 3)$ is an integer. Then

$$f^n(af + b) \equiv g^n(ag + b)$$

implies $f \equiv g$, where a, b are two nonzero constants.

Proof. We omit the proof since it can be carried out in the line of Lemma 6 [9]. □

LEMMA 11. *Let f and g be two nonconstant meromorphic functions and $n(\geq 2)$, $m(\geq 2)$ be two distinct integers satisfying $n + m \geq d + 7$. Then for two nonzero constants a, b,*

$$f^n(af^m + b) \equiv g^n(ag^m + b)$$

implies $f \equiv tg$, for some constant t, satisfying $t^d \equiv 1$, where $d = (n + m, n)$.

Proof. Suppose $F = f^n(af^m + b)$ and $G = g^n(ag^m + b)$. Let $f \not\equiv tg$ for a constant t satisfying $t^d = 1$. We put $h = \frac{f}{g}$. Then $h^d \not\equiv 1$. First suppose that h is constant. Also $F \equiv G$ implies

$$g^m = -\frac{b}{a}\,\frac{h^n - 1}{h^{n+m} - 1}.$$

We note that the numerator and the denominator has d common factors namely $h - v_k$, $k = 0, 1, 2, \ldots, d - 1$, where $v_k = exp\left(\frac{2k\pi i}{d}\right)$. Since $(h - v_1)(h - v_2)\ldots(h - v_k) \neq 0$, it follows that g is a constant, which is impossible. So h is nonconstant. We observe that since a nonconstant meromorphic function can not have more than two Picard exceptional values h can take at least $n + m - d - 2$ values among $u_j = exp\left(\frac{2j\pi i}{n+m}\right)$, where $j = 0, 1, 2, \ldots, n + m - 1$. Since f^m has no simple pole $h - u_j$ has no simple zero for at least $n + m - d - 2$ values of u_j, for $j = 0, 1, 2, \ldots, n + m - 1$ and for these values of j we have $\Theta(u_j; h) \geq \frac{1}{2}$, which leads to a contradiction. This proves the lemma. □

3. Proof of the Theorem

Proof of Theorem 1. Let F and G be given as in Lemma 6. Then F, G are transcendental meromorphic functions that share 1 and ∞ IM. So

$$\overline{N}_*(r, \infty; F, G) \leq \overline{N}(r, \infty; F \mid\geq n + m + k) = \overline{N}(r, \infty; f \mid\geq 1).$$

If possible, we suppose that $H \not\equiv 0$. Then $F \not\equiv G$. So from Lemma 6 we have $V \not\equiv 0$. From Lemma 1 and (2.1) we obtain

$$\begin{aligned} N_2(r, 0; F) &\leq N_2\left(r, 0; (f^n P(f))^{(k)}\right) + S(r, f) \\ &\leq T\left(r, (f^n P(f))^{(k)}\right) - (n + m)T(r, f) + N_{k+2}(r, 0; f^n P(f)) + S(r, f) \\ &\leq T(r, F) - (n + m)T(r, f) + N_{k+2}(r, 0; f^n P(f)) \\ &\quad + O\{\log r\} + S(r, f). \end{aligned} \tag{3.1}$$

In a similar way we obtain

$$N_2(r,0;G) \leq T(r,G) - (n+m)T(r,g) + N_{k+2}(r,0;g^nP(g)) + O\{\log r\} + S(r,g). \tag{3.2}$$

Again by (2.2) we have

$$N_2(r,0;F) \leq k\overline{N}(r,\infty;f) + N_{k+2}(r,0;f^nP(f)) + S(r,f). \tag{3.3}$$

$$N_2(r,0;G) \leq k\overline{N}(r,\infty;g) + N_{k+2}(r,0;g^nP(g)) + S(r,g). \tag{3.4}$$

From (3.1) and (3.2) we get

$$\begin{aligned}(n+m)\{T(r,f)+T(r,g)\} \leq\ & T(r,F)+T(r,G)+N_{k+2}(r,0;f^nP(f))\\ &+N_{k+2}(r,0;g^nP(g)) - N_2(r,0;F) - N_2(r,0;G)\\ &+O\{\log r\} + S(r,f) + S(r,g).\end{aligned} \tag{3.5}$$

Then using Lemma 1, Lemma 5, (3.3) and (3.4) we obtain from (3.5)

$$\begin{aligned}(n+m)\{T(r,f)+T(r,g)\} \leq\ & N_2(r,0;F)+N_2(r,0;G)+5\overline{N}(r,\infty;F)+5\overline{N}(r,\infty;G)\\ &+2\overline{N}_*(r,\infty;F,G)+3\overline{N}(r,0;F)+3\overline{N}(r,0;G)\\ &+N_{k+2}(r,0;f^nP(f))+N_{k+2}(r,0;g^nP(g))\\ &+O\{\log r\}+S(r,f)+S(r,g)\\ \leq\ & 2N_{k+2}(r,0;f^nP(f))+2N_{k+2}(r,0;g^nP(g))\\ &+3N_{k+1}(r,0;f^nP(f))+3N_{k+1}(r,0;g^nP(g))\\ &+(4k+5)\overline{N}(r,\infty;f)+(4k+5)\overline{N}(r,\infty;g)\\ &+2\overline{N}_*(r,\infty;F,G)+O\{\log r\}+S(r,f)+S(r,g)\\ \leq\ & (5k+5m+7)\{T(r,f)+T(r,g)\}+(4k+6)(\overline{N}(r,\infty;f)\\ &+\overline{N}(r,\infty;g))+O\{\log r\}+S(r,f)+S(r,g).\end{aligned} \tag{3.6}$$

Since f and g are transcendental meromorphic functions, we have

$$\log r = o\{T(r,f)\}. \tag{3.7}$$

Using Lemma 2, Lemma 7 and (3.7) we obtain from (3.6)

$$\begin{aligned}(n-9k-4m-7)\{T(r,f)+T(r,g)\} \leq\ & 12\overline{N}(r,\infty;f\mid\leq 1)\\ \leq\ & \frac{24(k+m+1)}{n+m-3k-3}\{T(r,f)+T(r,g)\}\\ &+S(r,f)+S(r,g).\end{aligned}$$

This gives

$$\begin{aligned}[(n-9k-4m-7)(n+m-3k-3)-24(k+m+1)]\{T(r,f)+T(r,g)\}\\ \leq S(r,f)+S(r,g),\end{aligned}$$

which leads to a contradiction as $n > 9k+4m+11$.
We now assume that $H \equiv 0$. That is

$$\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) = 0.$$

Integrating both sides of the above equality twice we get

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{3.8}$$

where $A(\neq 0)$ and B are constants. Suppose $m \geq 1$. We now discuss following three cases separately.

Case 1. Let $B \neq 0$ and $A = B$. Then from (3.8) we get

$$\frac{1}{F-1} = \frac{BG}{G-1}. \tag{3.9}$$

If $B = -1$, then from (3.9) we obtain

$$FG = 1,$$

i.e.,

$$(f^n P(f))^{(k)} (g^n P(g))^{(k)} = z^2. \tag{3.10}$$

From our assumption it is clear that $f \neq 0$ and $f \neq \infty$. Let $f(z) = e^{\alpha}$, where α is a nonconstant entire function. Then by induction we get

$$(a_m f^{n+m})^{(k)} = t_m(\alpha', \alpha'', ..., \alpha^{(k)}) e^{(n+m)\alpha}, \tag{3.11}$$

.

$$(a_0 f^n)^{(k)} = t_0(\alpha', \alpha'', ..., \alpha^{(k)}) e^{n\alpha}, \tag{3.12}$$

where $t_i(\alpha', \alpha'', ..., \alpha^{(k)})$ $(i = 0, 1, ..., m)$ are differential polynomials in $\alpha', \alpha'', \ldots, \alpha^{(k)}$.
Obviously

$$t_i(\alpha', \alpha'', ..., \alpha^{(k)}) \not\equiv 0$$

for $i = 0, 1, 2, ..., m$, and

$$(f^n P(f))^{(k)} \neq 0.$$

From (3.11) and (3.12) we obtain

$$\overline{N}(r, 0; t_m(\alpha', \alpha'', ..., \alpha^{(k)}) e^{m\alpha(z)} + ... + t_0(\alpha', \alpha'', ..., \alpha^{(k)})) \leq N(r, 0; z^2) = S(r, f). \tag{3.13}$$

Since α is an entire function, we obtain $T(r,\alpha^{(j)}) = S(r,f)$ for $j = 1,2,...,k$. Hence $T(r,t_i) = S(r,f)$ for $i = 0,1,2,...,m$.
So from (3.13), Lemmas 1 and 3 we obtain

$$\begin{aligned} mT(r,f) &= T(r,t_m e^{m\alpha} + ... + t_1 e^{\alpha}) + S(r,f) \\ &\leq \overline{N}(r,0;t_m e^{m\alpha} + ... + t_1 e^{\alpha}) + \overline{N}(r,0;t_m e^{m\alpha} + ... + t_1 e^{\alpha} + t_0) + S(r,f) \\ &\leq \overline{N}(r,0;t_m e^{(m-1)\alpha} + ... + t_1) + S(r,f) \\ &\leq (m-1)T(r,f) + S(r,f), \end{aligned}$$

which is a contradiction.
If $B \neq -1$, from (3.9), we have $\frac{1}{F} = \frac{BG}{(1+B)G-1}$ and so $\overline{N}(r,\frac{1}{1+B};G) = \overline{N}(r,0;F)$. Now from Nevanlinna's second fundamental theorem, we get

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,0;G) + \overline{N}\left(r,\frac{1}{1+B};G\right) + \overline{N}(r,\infty;G) + S(r,G) \\ &\leq \overline{N}(r,0;G) + \overline{N}(r,0;F) + \overline{N}(r,\infty;G) + S(r,G). \end{aligned}$$

Then using (2.1 and (2.2) we obtain

$$\begin{aligned} T(r,G) \leq\ & N_{k+1}(r,0;f^nP(f)) + N_{k+1}(r,0;g^nP(g)) + k\overline{N}(r,\infty;f) + T(r,G) \\ & -(n+m)T(r,g) + \overline{N}(r,\infty;g) + O\{\log r\} + S(r,g). \end{aligned}$$

Using (3.7) we obtain

$$(n+m)T(r,g) \leq (2k+m+1)T(r,f) + (k+m+2)T(r,g) + S(r,g).$$

Thus we obtain

$$(n-3k-m-3)\{T(r,f) + T(r,g)\} \leq S(r,f) + S(r,g),$$

a contradiction as $n > 9k + 4m + 11$.

Case 2. Let $B \neq 0$ and $A \neq B$. Then from (3.8) we get $F = \frac{(B+1)G-(B-A+1)}{BG+(A-B)}$ and so $\overline{N}(r,\frac{B-A+1}{B+1};G) = \overline{N}(r,0;F)$. Proceeding as in Case 1 we obtain a contradiction.

Case 3. Let $B = 0$ and $A \neq 0$. Then from (3.8) we get $F = \frac{G+A-1}{A}$ and $G = AF - (A-1)$. If $A \neq 1$, we have $\overline{N}(r,\frac{A-1}{A};F) = \overline{N}(r,0;G)$ and $\overline{N}(r,1-A;G) = \overline{N}(r,0;F)$. So by Lemma 8 we have $n \leq 3k+m+3$, a contradiction. Thus $A = 1$ and hence $F = G$. That is

$$[f^nP(f)]^{(k)} = [g^nP(g)]^{(k)}.$$

Integrating we get

$$[f^nP(f)]^{(k-1)} = [g^nP(g)]^{(k-1)} + c_{k-1},$$

where c_{k-1} is a constant. If $c_{k-1} \neq 0$, from Lemma 9 we obtain $n \leq 3k+m$, a contradiction. Hence $c_{k-1} = 0$. Repeating k-times, we obtain

$$f^nP(f) = g^nP(g). \tag{3.14}$$

When $m=1$ the theorem follows from Lemma 10. When $m \geq 2$ let $h=\frac{f}{g}$. If h is a constant, by putting $f=gh$ in (3.14) we get

$$a_m g^{n+m}(h^{n+m}-1)+a_{m-1}g^{n+m-1}(h^{n+m-1}-1)+...+a_0 g^n(h^n-1)=0,$$

which implies $h^d=1$, where $d=(n+m,...,n+m-i,...,n+1,n)$, $a_{m-i}\neq 0$ for some $i=0,1,...,m$. Thus $f=tg$ for a constant t such that $t^d=1, d=(n+m,...,n+m-i,...,n+1,n)$, $a_{m-i}\neq 0$ for some $i=0,1,...,m$.

If h is not a constant, then from (3.14) we can say that f and g satisfy the algebraic equation $R(f,g)=0$, where $R(f,g)$ is given by (1.1). So the theorem is proved for $m\geq 1$.

For $m=0$ we proceed as follows. Since f, g share ∞ IM, from (3.10) it is clear that $f\neq\infty$ and $g\neq\infty$. Suppose that z_0 is a zero of f of multiplicity p. Then z_0 must be a zero of $(a_0 f^n)^{(k)}$ of multiplicity $np-k$. Since $n>9k+4m+11>k+2$, it follows from (3.10) that z_0 is a zero of z^2 of order at least 3, which is impossible. So f and g are two nonconstant entire functions having no zero or poles. Then using the same argument as in the proof of Theorem 1 (See. [13], p.15) we obtain $f(z)=c_1e^{cz^2}$, $g(z)=c_2e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4a_0^2(c_1c_2)^{n+m}(nc)^2=-1$. On the other hand following the same procedure as above we can see that only (3.14) holds good and hence $f=tg$, where $t^n=1$. This completes the proof of the theorem. □

Proof of Corollary 1. **Case i.** Let $\lambda\mu\neq 0$. For $m=1$ we get the conclusion from Theorem 1(ii). When $m>1$, in view of Lemma 11 we can obtain the conclusion of the corollary.

Case ii. Let $\lambda\mu=0$.

Subcase(i) We first assume that $\lambda=0$ and $\mu\neq 0$. Then by Theorem 1 (ii) we obtain either $f=tg$, where $t^{n+m}=1$ or $f(z)=c_1e^{cz^2}$, $g(z)=c_2e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4\mu^2(c_1c_2)^{n+m}[(n+m)c]^2=-1$.

Subcase(ii) If $\lambda\neq 0$ and $\mu=0$, then by similar as above subcase, either $f=tg$, where $t^n=1$ or f and g must satisfy $f(z)=c_1e^{cz^2}$, $g(z)=c_2e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4\lambda^2(c_1c_2)^n(nc)^2=-1$.

Combining the two subcases Corollary 1(ii) follows. □

Proof of Corollary 2. Since in view of Theorem 1 the proof of the Corollary 2 is obvious, we omit the proof. □

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