Rend. Sem. Mat. Univ. Politec. Torino
Vol. 71, 2 (2013), 219 – 226

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ON THE NEWTON-NELSON TYPE EQUATIONS ON VECTOR BUNDLES WITH CONNECTIONS

Abstract. An equation of Newton-Nelson type on the total space of vector bundle with a connection, whose right-hand side is generated by the curvature form, is described and investigated. An existence of solution theorem is obtained.

Introduction

In [5] (see also [6]) a certain second order differential equation on the total space of vector bundle with a connection was constructed and investigated. In some cases it was interpreted as an equation of motion of a classical particle in the classical gauge field. The form of this equation allowed one to apply the quantization procedure in the language of Nelson's Stochastic Mechanics (see, e.g., [8, 9]). In [7] this procedure was realized for the vector bundles over Lorentz manifolds with complex fibers. The corresponding relativistic-type Newton-Nelson equation (the equation of motion in Stochastic Mechanics) was constructed and the existence of its solutions under some natural conditions was proved. The results of [7] were interpreted as the description of motion of a quantum particle in the gauge field.

In this paper we consider the analogous non-relativistic Newton-Nelson equation in the situation where the base of the bundle is a Riemannian manifold and the fiber is a real linear space. In this case some deeper results are obtained under some less restrictive conditions with respect to the case of [7].

We refer the reader to [2, 6] for the main facts of the geometry of manifolds and to [4, 6] for general facts of Stochastic Analysis on Manifolds.

1. Necessary facts from the Geometry of Manifolds

Recall that for every bundle E over a manifold M, in each tangent space $T_{(m,x)}E$ to the total space E there is a special sub-space $\mathsf{V}_{(m,x)}$, called *vertical*, that consists of the vectors tangent to the fiber E_m (called also vertical). In the case of principal or vector bundle, a connection H on E is a collection of sub-spaces in tangent spaces to E such that $T_{(m,x)}E = \mathsf{H}_{(m,x)} \oplus \mathsf{V}_{(m,x)}$ at each $(m,x) \in E$ and this collection possesses some properties of smoothness and invariance (see, e.g., [6]).

Denote by $\mathcal{M}$ a Riemannian manifold with metric tensor $g(\cdot,\cdot)$. Let $\Pi : \mathcal{E} \to \mathcal{M}$ be a principal bundle over $\mathcal{M}$ with a structure group G. By $\mathfrak{g}$ we denote the Lie algebra of G. Let a connection H with connection form θ and curvature form $\Phi = D\theta$ be given

*The research is supported in part by the RFBR Grants 10-01-00143 and 12-01-00183

on $\mathcal{E}$. Here D is the covariant differential (see, e.g., [2]). Recall that the 1-form θ and the 2-form Φ are equivariant and take values in the algebra $\mathfrak{g}$ of G and that Φ is horizontal (equals zero on vertical vectors).

We suppose G to be a subgroup of $GL(k,\mathbb{R})$ for a certain k. Let $\mathcal{F}$ be a k-dimensional real vector space, on which G acts from the left, and let on $\mathcal{F}$ an inner product $h(\cdot,\cdot)$, invariant with respect to the action of G, be given. We suppose that a mapping $e:\mathcal{F}\to\mathfrak{g}^*$ (where $\mathfrak{g}^*$ is the co-algebra) having constant values on the orbits of G, is given. This mapping is called *charge*.

Consider the vector bundle $\pi: Q\to\mathcal{M}$ with standard fiber $\mathcal{F}$, associated to $\mathcal{E}$. We denote by Q_m the fiber at $m\in\mathcal{M}$. Consider the factorization $\lambda:\mathcal{E}\times\mathcal{F}\to Q$ that yields the bundle Q (see [2]). The tangent mapping $T\lambda$ translates the connection H from the tangent spaces to $\mathcal{E}$ to tangent spaces to Q. This connection on Q is denoted by H^π. Recall that the spaces of connection are the kernels of operator $K^\pi: TQ\to Q$ called *connector*, that is constructed as follows. Consider the natural expansion of the tangent vector $X\in T_{(m,q)}Q$ at $(m,q)\in Q$ into horizontal and vertical components $X=\mathsf{H}X+\mathsf{V}X$, where $\mathsf{H}X\in\mathsf{H}^\pi_{(m,q)}$ and $\mathsf{V}X\in\mathsf{V}_{(m,q)}$. Introduce the operator $\mathbf{p}:\mathsf{V}_{(m,q)}\to Q_m$, the natural isomorphism of the linear tangent space $\mathsf{V}_{(m,q)}=T_qQ_m$ to the fiber Q_m of Q onto the fiber (linear space) Q_m. Then $K^\pi X=\mathbf{p}\mathsf{V}X$.

On the manifold Q (the total space of bundle) we construct the Riemannian metric g^Q as follows: in the horizontal subspaces H^π we introduce it as the pull-back $T\pi^*g$, in the vertical subspaces V – as h and define that H^π are V orthogonal to each other.

We denote the projection of tangent bundle $T\mathcal{M}$ to $\mathcal{M}$ by $\tau: T\mathcal{M}\to\mathcal{M}$ and by H^τ the Levi-Civita connection of metric g on $\mathcal{M}$. Its connector is denoted by $K^\tau: T^2\mathcal{M}\to T\mathcal{M}$. The construction of K^τ is quite analogous to that of K^π where Q is replaced by $T\mathcal{M}$ and TQ by $T^2\mathcal{M}=TT\mathcal{M}$.

Recall the standard construction of a connection on the total space of bundle Q, based on the connections H^π and H^τ (see, e.g., [3, 6]). The connector $K^Q: T^2Q\to TQ$ of this connection has the form: $K^Q=K^H+K^V$ where $K^H: T^2Q\to\mathsf{H}^\pi$ and $K^V: T^2Q\to\mathsf{V}$, and the latter connectors are introduced as: $K^H=T\pi^{-1}\circ K^\tau\circ T^2\pi$ where $T^2\pi=T(T\pi): T^2Q\to T^2\mathcal{M}$ and $T\pi^{-1}$ is the linear isomorphism of tangent spaces to $\mathcal{M}$ onto the spaces of connection H^π; $K^V=\mathbf{p}^{-1}\circ K^\pi\circ TK^\pi$.

Recall that λ is a one-to-one mapping of the standard fiber $\mathcal{F}$ onto the fibers of bundle Q, hence the charge e is well-defined on the entire Q. Since $T\lambda$ is also a one-to-one mapping of the connections and Φ is equivariant, we can introduce the differential form $\tilde{\Phi}$ on Q with values in $\mathfrak{g}$ as follows. Consider $(m,q)=\lambda((m,p),f)$ for $(m,p)\in\mathcal{E}$ and $f\in\mathcal{F}$. For $X,Y\in T_{(m,q)}Q$ we denote by $\mathsf{H}X$ and $\mathsf{H}Y$ their horizontal components. Then we define $\tilde{\Phi}_{(m,q)}(X,Y)=\Phi_{(m,p)}(T\lambda^{-1}\mathsf{H}X, T\lambda^{-1}\mathsf{H}Y)$.

Denote by $\bullet$ the coupling of elements of $\mathfrak{g}$ and $\mathfrak{g}^*$.Consider the vector $((m,q),X)$ tangent to Q at (m,q). It is clear that $e((m,q))\bullet\tilde{\Phi}_{(m,q)}(\cdot,X)$ is an ordinary 1-form (i.e., differential form with values in real line). Denote by $\overline{e((m,q))\bullet\tilde{\Phi}_{(m,q)}(\cdot,X)}$ the tangent vector to the total space of Q physically equivalent to the form $e((m,q))\bullet\tilde{\Phi}_{(m,q)}(\cdot,X)$ (i.e., obtained by lifting the indices with the use of Riemannian metric g^Q).

LEMMA 1 ([5]). *The vector field* $\overline{e((m,q))\bullet\tilde{\Phi}_{(m,q)}(\cdot,X)}$ *is horizontal, i.e., it belongs to the spaces of connection* H^{π}.

THEOREM 1 ([7]). *Let* $(m(t),q(t))$ *be a smooth curve in* Q. *Let* $X(t)$ *be the parallel translation of the vector* $X\in T_{(m(t_0),q(t_0))}Q$ *along* $(m(t),q(t))$ *with respect to* H^{Q}. *(i) Both the horizontal* $\mathsf{H}X(t)$ *and vertical* $\mathsf{V}X(t)$ *components of* $X(t)$ *are parallel along* $(m(t),q(t))$ *with respect to* H^{Q}. *(ii) The parallel translation of horizontal vectors preserves constant the norms and scalar products with respect to* g^{Q}. *(iii) The vector field* $T\pi X(t)$ *is parallel along* $m(t)$ *on* $\mathcal{M}$ *with respect to* H^{τ}.

2. Mean derivatives on manifolds and vector bundles

Consider a stochastic process $\xi(t)$ with values in $\mathcal{M}$, given on a certain probability space $(\Omega,\mathfrak{F},\mathsf{P})$. By $\mathfrak{N}_t^{\xi}$ we denote the minimal σ-sub-algebra of σ-algebra $\mathfrak{F}$ generated by the pre-images of Borel sets in $\mathcal{M}$ under the mapping $\xi(t):\Omega\to\mathcal{M}$ (the "present" or "now" of $\xi(t)$) and by $E(\cdot\mid\mathfrak{N}_t^{\xi})$ the conditional expectation with respect to $\mathfrak{N}_t^{\xi}$. Recall that the conditional expectation of a random element ϑ with respect to $\mathfrak{N}_t^{\xi}$ can be represented as $\Theta(\xi(t))$ where Θ is the so-called *regression* introduced by the formula $\Theta(m)=E(\theta\mid\xi(t)=m)$ (see, e.g., [10]).

Specify a point in $\mathcal{M}$ and consider the normal chart U_m at this point with respect to the exponential mapping of Levi-Civita connection on $\mathcal{M}$. In U_m construct the following regressions

(1) $$Y^{U_m}(t,m')=\lim_{\Delta t\downarrow 0}E\left(\frac{\xi(t+\Delta t)-\xi(t)}{\Delta t}\mid\xi(t)=m'\right);$$

(2) $${}_{*}^{U_m}(t,m')=\lim_{\Delta t\downarrow 0}E\left(\frac{\xi(t)-\xi(t-\Delta t)}{\Delta t}\mid\xi(t)=m'\right).$$

Introduce $X^0(t,m)=Y^{U_m}(t,m)$ and $X^0_*(t,m)=Y^{U_m}_*(t,m)$. Note that $X^0(t,m)$ and $X^0_*(t,m)$ are vector fields on $\mathcal{M}$, i.e., under the coordinate changes they transform like cross-sections of the tangent bundle $T\mathcal{M}$.

Forward and backward mean derivatives of $\xi(t)$ are defined by the formulae $D\xi(t)=X^0(t,\xi(t))$ and $D_*\xi(t)=X^0_*(t,\xi(t))$.

The vector $v^{\xi}(t)=\frac{1}{2}(D+D_*)\xi(t)$ is called the *current velocity* of $\xi(t)$. From the properties of conditional expectation it follows that there exists a Borel measurable vector field (regression) $v^{\xi}(t,m)$ on $\mathcal{M}$ such that $v^{\xi}(t)=v^{\xi}(t,\xi(t))$.

Introduce the increment $\Delta\xi(t)$ of process $\xi(t)$: $\Delta\xi(t)=\xi(t+\Delta t)-\xi(t)$ and the so called quadratic mean derivative D_2 (see [1,6]) $D_2\xi(t)=\lim_{\Delta\tau\downarrow 0}E(\frac{\Delta\xi(t)\otimes\Delta\xi(t)}{\Delta t}\mid\mathfrak{N}_t^{\xi})$. If $D_2\xi(t)$ exists, it takes values in $(2,0)$-tensors.

Everywhere below we are dealing with processes, along which the parallel translation with respect to an appropriate connection is well-posed. Here we use $\xi(\cdot)$ and parallel translation with respect to the connection H^{τ} and such an assumption is true, for example, if $\xi(t)$ is an Itô process on $\mathcal{M}$, i.e., an Itô development of an Itô process in a certain tangent space to $\mathcal{M}$ as it is defined in [6]. Denote by $\Gamma_{t,s}$ the operator of such parallel translation along $\xi(\cdot)$ of tangent vectors from the (random) point $\xi(s)$ of the process to the (random) point $\xi(t)$.

For a vector field $Z(t,m)$ on $\mathcal{M}$ the covariant forward and backward mean derivatives $\mathbf{D}Z(t,\xi(t))$ and $\mathbf{D}_*Z(t,\xi(t))$ are constructed by the formulae

$$\mathbf{D}Z(t,\xi(t)) = \lim_{\Delta t\downarrow 0} E\left(\frac{\Gamma_{t,t+\Delta t}Z(t+\Delta t,\xi(t+\Delta t)) - Z(t,\xi(t))}{\Delta t} \mid \mathfrak{N}_t^{\xi}\right); \tag{3}$$

$$\mathbf{D}_*Z(t,\xi(t)) = \lim_{\Delta t\downarrow 0} E_t^{\xi}\left(\frac{Z(t,\xi(t)) - \Gamma_{t,t-\Delta t}Z(t-\Delta t,\xi(t-\Delta t))}{\Delta t} \mid \mathfrak{N}_t^{\xi}\right). \tag{4}$$

From formulae (1), (2), (3) and (4) it evidently follows that $T\pi\mathbf{D}Z(t,\xi(t)) = D\xi(t)$ and $T\pi\mathbf{D}_*Z(t,\xi(t)) = D_*\xi(t)$.

Now consider a stochastic process $\eta(t)$ in the total space of bundle Q and introduce the process $\xi(t) = \pi\eta(t)$ on $\mathcal{M}$. Denote by $\Gamma^{\pi}_{t,s}$ the parallel translation of random vectors from the fiber $Q_{\xi(s)}$ to the fiber $Q_{\xi(t)}$ along $\xi(\cdot)$ with respect to connection H^{π}. For $\eta(t)$ we introduce the covariant mean derivatives by formulae

$$\mathbf{D}\eta(t) = \lim_{\Delta t\downarrow 0} E\left(\frac{\Gamma^{\pi}_{t,t+\Delta t}\eta(t+\Delta t) - \eta(t)}{\Delta t} \mid \mathfrak{N}_t^{\xi}\right); \tag{5}$$

$$\mathbf{D}_*\eta(t) = \lim_{\Delta t\downarrow 0} E\left(\frac{\eta(t) - \Gamma^{\pi}_{t,t-\Delta t}\eta(t-\Delta t)}{\Delta t} \mid \mathfrak{N}_t^{\xi}\right). \tag{6}$$

(analogs of (3) and (4)). As above, $v^{\eta}(t) = \frac{1}{2}(\mathbf{D}+\mathbf{D}_*)\eta(t)$ is called the *current velocity* of $\eta(t)$.

In order to define the mean derivatives of a vector field along $\eta(t)$ on Q we use the parallel translation $\Gamma^{Q}_{t,s}$ of vectors tangent to Q at $\eta(s)$, to vectors tangent to Q at $\eta(t)$ along $\eta(\cdot)$ with respect to connection H^{Q}. By analogy with formulae (3) and (4) for a vector field $Z(t,(m,q))$ on Q we introduce the covariant mean derivatives by formulae

$$\mathbf{D}^{Q}Z(t,\eta(t)) = \lim_{\Delta t\downarrow 0} E\left(\frac{\Gamma^{Q}_{t,t+\Delta t}Z(t+\Delta t,\eta(t+\Delta t)) - Z(t,\eta(t))}{\Delta t} \mid \mathfrak{N}_t^{\xi}\right); \tag{7}$$

$$\mathbf{D}^{Q}_*Z(t,\eta(t)) = \lim_{\Delta t\downarrow 0} E\left(\frac{Z(t,\eta(t)) - \Gamma^{Q}_{t,t-\Delta t}Z(t-\Delta t,\eta(t-\Delta t))}{\Delta t} \mid \mathfrak{N}_t^{\xi}\right). \tag{8}$$

LEMMA 2. *$\Gamma^{Q}_{t,s}$ translates $\mathsf{H}^{\pi}_{\eta(s)}$ onto $\mathsf{H}^{\pi}_{\eta(t)}$ and $\mathsf{V}_{\eta(s)}$ onto $\mathsf{V}_{\eta(t)}$; the parallel translation of horizontal components preserves the norms and inner products with respect to g^{Q}.*

The assertion of Lemma 2 follows from Theorem 1 and from the fact that (see [3, 6]) that the parallel translation along random processes can be described as the limit of parallel translations along the processes whose sample paths are piece-wise geodesic approximations of the sample paths of the process under consideration.

By symbols $\mathbf{D}^H$ and $\mathbf{D}^H_*$ we denote the derivatives introduced by formulae (7) and (8), respectively, for the horizontal components of vectors (i.e., taking values in H^π). By symbols $\mathbf{D}^V$ and $\mathbf{D}^V_*$ we denote the derivatives for vertical components (i.e., taking values in V). Thus, $\mathbf{D}^Q = \mathbf{D}^H + \mathbf{D}^V$ and $\mathbf{D}^Q_* = \mathbf{D}^H_* + \mathbf{D}^V_*$.

3. The Newton-Nelson equation on the total space of vector bundle

In the problem under consideration the Newton-Nelson equation takes the form

$$\begin{cases} \frac{1}{2}(\mathbf{D}^Q\mathbf{D}_* + \mathbf{D}^Q_*\mathbf{D})\eta(t) = \overline{e(\eta(t)) \bullet \tilde{\Phi}_{\eta(t)}(\cdot, v^\eta(t))} \\ D_2\xi(t) = \frac{\hbar}{\mathrm{m}} I \end{cases}, \tag{9}$$

where $\xi(t) = \pi\eta(t)$ (cf. [8, 9]).

Expand the current velocity v^η in the right-hand side of (9) into the sum of vertical and horizontal components: $v^\eta = v^H_\eta + v^V_\eta$, where $v^H_\eta \in \mathsf{H}^\pi$ and $v^V_\eta \in \mathsf{V}$. Since $\tilde{\Phi}_{\eta(t)}(\cdot,\cdot)$ is linear in both arguments, $\tilde{\Phi}_{\eta(t)}(\cdot, v^\eta) = \tilde{\Phi}_{\eta(t)}(\cdot, v^H_\eta) + \tilde{\Phi}_{\eta(t)}(\cdot, v^V_\eta)$. Then, since the form $\tilde{\Phi}$ is horizontal (see Lemma 1) we obtain that $\tilde{\Phi}_{\eta(t)}(\cdot, v^V_\eta) = 0$. Thus, the first equation of system (9) is equivalent to the following system:

$$\frac{1}{2}(\mathbf{D}^H\mathbf{D}_* + \mathbf{D}^H_*\mathbf{D})\eta(t) = \overline{e(\eta(t)) \bullet \tilde{\Phi}_{\eta(t)}(\cdot, v^H_\eta(t))}, \tag{10}$$

$$\frac{1}{2}(\mathbf{D}^V\mathbf{D}_* + \mathbf{D}^V_*\mathbf{D})\eta(t) = 0. \tag{11}$$

For simplicity of presentation we denote $\overline{e(\eta(t)) \bullet \tilde{\Phi}_{\eta(t)}(\cdot, v^H_\eta(t))}$ by $\alpha_{(t,\eta(t))}v^H_\eta$ where, by construction, $\alpha_{(t,(m',q'))}(\cdot)$ is a linear operator in $\mathsf{H}^\pi_{(m',q')}$ ((1,1)-tensor).

Introduce the horizontal (1,2)-tensor field $\nabla^H\alpha(\cdot,\cdot) = K^H T\alpha(\cdot)$ on Q. The vector $\mathrm{tr}\nabla^H\alpha(\alpha\cdot,\cdot)$ is horizontal by construction.

THEOREM 2. *Let for the tensor field $\alpha_{(t,(m,q))}(\cdot)$ there exist a constant $C > 0$ such that $\int_0^T(\|\alpha_{(t,x(t))}(\cdot)\|^2 + \|\mathrm{tr}\nabla^H\alpha_{(t,x(t))}(\alpha\cdot,\cdot)\|^2)dt < C$ for a certain $T > 0$ and every continuous curve $x(t)$ in Q given on $t \in [0,T]$. Here $\|\alpha_{(t,x)}(\cdot)\|$ is the operator norm (all the norms are generated by g^Q). Let also the connections H^τ and H^π be stochastically complete (see [6]). Then for every point $(m,q) \in Q$, every vector $\beta_0 \in \mathsf{H}^\pi_{(m,q)}$ and every time instant $t_0 \in (0,T)$ there exists a stochastic process $\eta(t)$ in Q such that: (i) it is well-defined on $[0,T]$; (ii) $\eta(0) = (m,q)$ and $D\eta(0) = \beta_0$; (iii) for all $t \in (t_0,T)$ the processes $\eta(t)$ and $\xi(t) = \pi\eta(t)$ satisfy* (9)*; (iv) along $\eta(t)$ the charge $e(\eta(t))$ is constant.*

Proof. For simplicity and without loss of generality we suppose that $\frac{\hbar}{\mathrm{m}} = 1$.

Consider on the space of continuous curves $C^0([0,T], T_mM)$ the filtration $\mathcal{P}_t$ where for every $t \in [0,T]$ the σ-algebra $\mathcal{P}_t$ is generated by cylinder sets with bases over $[0,t]$. Consider the Wiener measure ν on the measure space $(C^0([0,T],T_mM),\mathcal{P}_T)$ and so the standard Wiener process $W_m(t)$ in T_mM as the coordinate process on the probability space $(C^0([0,T],T_mM),\mathcal{P}_T,\nu)$. Since H^τ is stochastically complete, the Itô development $W^M(t)$ of $W_m(t)$ with respect to H^τ on M is well-posed. Since H^π is also stochastically complete, the horizontal lift $W^Q(t)$ of $W^M(t)$ onto Q with respect to H^π with initial condition (m,q) is also well-posed. A detailed description of the construction of processes $W^M(t)$ and $W^Q(t)$ can be found in [6].

Since $T\pi : \mathsf{H}^\pi_{(m,q)} \to T_mM$ is a linear isomorphism that defines the metric tensor g^Q in $\mathsf{H}^\pi_{(m,q)}$ by the pull back of g from T_mM, we can translate the Wiener measure and the Wiener process from T_mM to $\mathsf{H}^\pi_{(m,q)}$. Denote by $W(t)$ the Wiener process obtained by this construction. This is a coordinate process on the space of continuous curves in $\mathsf{H}^\pi_{(m,q)}$ with σ-algebra $\mathcal{P}_T$ and Wiener measure.

For $t_0 \geq 0$ we introduce the real-valued function $t_0(t)$ that equals $\frac{1}{t_0}$ for $t < t_0$ and $\frac{1}{t}$ for $t \geq t_0$. Its derivative $t_0'(t)$ is equal to 0 for $t < t_0$ and to $-\frac{1}{t^2}$ for $t \geq t_0$.

Now consider the following Itô equation in $\mathsf{H}^\pi_{(m,q)}$:

$$\beta(t) = \beta_0 + \frac{1}{2}\int_0^t \Gamma^Q_{0,s} tr\nabla^H \alpha_{(s,W^Q(s))}(\alpha\cdot,\cdot)ds + \int_0^t \Gamma^Q_{0,s}\alpha_{(s,W^Q(s))}dW(s)$$
$$-\frac{1}{2}\int_0^t t_0(s)\beta(s)ds - \frac{1}{2}\int_0^t t_0'(s)W(s)ds. \tag{12}$$

Since equation (12) is linear in β, it has a strong and strongly unique solution $\beta(t)$. Since this solution is strong, it can be given on the space of continuous curves in $\mathsf{H}^\pi_{(m,q)}$ equipped with Wiener measure. Consider the following density on the latter space of curves $\theta(l) = \exp\left(-\frac{1}{2}\int_0^T \beta(s)^2 ds + \int_0^T (\beta(s)\cdot dW(s))\right)$. From the hypothesis and from Lemma 2 it follows that it is well-posed. Introduce the measure that has this density with respect to the Wiener measure. It is well-known that with the new measure the coordinate process takes the form $\zeta(t) = \int_0^t \beta(s)ds + w(t)$ where $w(t)$ is a certain Wiener process adapted to $\mathcal{P}_t$. Denote $W^Q(t)$, considered with respect to the new measure, by the symbol $\eta(t)$ and introduce the process $\xi(t) = \pi\eta(t)$; $\xi(t)$ is obtained from $W^M(t)$ by the change of measure. Equation (12) turns into

$$\beta(t) = \beta_0 + \frac{1}{2}\int_0^t \Gamma^Q_{0,s} tr\nabla^H \alpha_{(s,\eta(t))}(\alpha\cdot,\cdot)ds + \int_0^t \Gamma^Q_{0,s}\alpha_{(s,\eta(s))}\beta(s)ds$$
$$+\int_0^t \left(\Gamma^Q_{0,s}\alpha_{(s,\eta(s))}(\cdot) + \frac{1}{2}t_0(s)\right)dw(s) - \frac{1}{2}\int_0^t t_0(s)\beta(s)ds - \frac{1}{2}\int_0^t t_0'(t)\zeta(t)ds.$$

By construction, $\eta(0) = (m,q)$ and $D\eta(t) = \beta_0$. The process $\eta(t)$ satisfies (11) also by construction. The fact that for $t \in (t_0,T)$ the processes $\eta(t)$ and $\xi(t) = \pi\eta(t)$ satisfy (10) and that $D_2\xi(t) = I$ follows from the formulae for mean derivatives obtained in [6, Chapters 12 and 18].

Evidently $\eta(t)$ is the horizontal lift of the process $\xi(t)$ with respect to connection H^π with the initial condition (m,q). Recall that the horizontal lift $\eta(t)$ of $\xi(t)$ is a parallel translation of (m,q) along $\xi(\cdot)$ with respect to H^{π}. Hence, it can be presented in the form $(\xi(t), b_t(f))$ where b_t is the horizontal lift of $\xi(t)$ to $\mathcal{E}$ with respect to connection H and f is a certain vector in the standard fiber $\mathcal{F}$. Thus, the sample paths of $\eta(t)$ belong to an orbit of G and so the charge e is constant along $\eta(t)$. $\square$

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AMS Subject Classification: 58J65; 60H30; 60H10

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Lavoro pervenuto in redazione il 22.05.2013