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GENERALIZED SOLUTIONS TO EQUATIONS WITH MULTIPLICATIVE NOISE IN HILBERT SPACES

Abstract. We suggest a framework that allows to introduce multiplicative stochastic perturbation of the Gaussian white noise type into a linear differential equation in a Hilbert space and prove existence of the unique solution for the obtained stochastic problem in a certain space of generalized functions.

1. Introduction

Our model problem is

$$\frac{\partial u(t,s)}{\partial t} = -\frac{\partial u(t,s)}{\partial s} + \eta(s)u(t,s),\ 0 < s < 1,\ t > 0,\ u(t,0) = 0,\ u(0,s) = \varphi(s),$$

where $\eta \in L_\infty[0;1]$. It can be written as the Cauchy problem for an operator-differential equation in Hilbert space $H = L^2[0,1]$ in the following way:

$$\frac{du(t)}{dt} = Au(t)\,,\quad t > 0\,,\ u(0) = \varphi\,, \tag{1}$$

where

$$A = A_0 + B_0 = -\frac{d}{ds} + \eta(s),\ \ \mathrm{dom}A = \{x \in L^2[0,1]\,, \frac{dx}{ds} \in L^2[0,1]\,, x(0) = 0\}. \tag{2}$$

Operator A_0 is the generator of the right shift semigroup, which is a C_0-semigroup in H. Its perturbation by B_0, which is bounded in H, gives A which is also the generator of a C_0-semigroup. Such problems arise for example in population dynamics. In this case u represents population density with respect to a certain numerical characteristic, say age, or size of an individual, A_0 is usually the generator of a shift-type semigroup, B_0 is a multiplication operator (or a sum of multiplication operators) that reflects the influence of such phenomena as death and birth. We will be concerned with the situation when B_0 is subject to random fluctuations, so that instead of $\eta(s)$ we have $\eta(s) + \nu(t,s)$, where ν is a random process taking values in a certain space of functions on $[0,1]$. If we want multiplication by $\eta + \nu$ to be a bounded operator in H, ν must be a sufficiently smooth function of s. We use multiplication by smoothed values of an H-valued white noise. Namely, consider $B(\cdot) \in \mathcal{L}\big(H; \mathcal{L}(H)\big)$ defined by

$$[B(x)y](s) := \varepsilon \cdot x(s) \int_0^1 \psi(s - \tau) y(\tau)\, d\tau\,, \tag{3}$$

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where $\varepsilon > 0, \psi \in C_0^\infty(\mathbb{R})$. Thus we come to the following stochastic problem:

$$dX(t) = AX(t)dt + B(X(t))dW(t)\,,\ t \geq 0\,, \quad X(0) = \Phi\,. \tag{4}$$

where $W(t)$ is a cylindrical H-valued Wiener process on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ with normal filtration $\{\mathcal{F}_t\}$, Φ is an $\mathcal{F}_0$-measurable random variable.

In our work we introduce spaces of H-valued generalized random variables $(\mathcal{S})_{-\rho}(H), 0 \leq \rho \leq 1$, so that (4) can be written as

$$\frac{dX(t)}{dt} = AX(t) + B\big(X(t)\big) \diamond \mathbb{W}(t)\,,\ t \geq 0\,, \quad X(0) = \Phi\,, \tag{5}$$

where $\mathbb{W}(t)$ is H-valued cylindrical singular white noise and "$\diamond$" is the Wick product. Using S-transform we reduce the problem (5) to a deterministic one and thus prove the existence and uniqueness of its solution in $(\mathcal{S})_{-0}(H)$.

2. Framework

Let $(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \mu)$ be the white noise probability space, where $\mathcal{S}'$ is the space of tempered distributions over the space of rapidly decreasing functions $\mathcal{S}$, $\mathcal{B}(\mathcal{S}')$ is the σ-algebra of Borel subsets of $\mathcal{S}'$ and μ is the white noise probability measure on $\mathcal{B}(\mathcal{S}')$ (Minlos – Sasonov measure) with

$$\int_{\mathcal{S}'} e^{i\langle \omega, \theta \rangle} d\mu(\omega) = e^{-\frac{1}{2}|\theta|_0^2}\,, \quad \theta \in \mathcal{S}\,. \tag{6}$$

We denote by $|\cdot|_0 = \sqrt{(\cdot,\cdot)_0}$ the norm of $L^2(\mathbb{R})$. Let (L^2) be the space of μ-square integrable $\mathbb{R}$-valued functions (random variables) on $\mathcal{S}'$ with norm $\|\cdot\|_0$. It follows from (6) that for any $\theta, \eta \in \mathcal{S}$ we have $\big(\langle\cdot,\theta\rangle, \langle\cdot,\eta\rangle\big)_{(L^2)} = (\theta,\eta)_{L^2(\mathbb{R})}$, $\|\langle\cdot,\theta\rangle\|_0^2 = E\langle\cdot,\theta\rangle^2 = |\theta|_0^2$. It follows from here that the mapping $\theta \mapsto \langle\cdot,\theta\rangle$ can be extended by continuity from $\mathcal{S}$ to the whole $L^2(\mathbb{R})$, so that $\langle\cdot,\phi\rangle \in (L^2)$ is well defined for all $\phi \in L^2(\mathbb{R})$ and (6) is still valid for $\theta \in L^2(\mathbb{R})$.

Let $\{\xi_k\}_{k=1}^\infty$ be the orthonormal basis of $L^2(\mathbb{R})$, consisting of Hermite functions $\xi_k(x) = \dfrac{e^{-\frac{x^2}{2}} h_{k-1}(x)}{\pi^{\frac{1}{4}}((k-1)!)^{\frac{1}{2}}}$, where $h_k(x) = (-1)^k e^{\frac{x^2}{2}} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}}$ are Hermite polynomials.

Let $\mathcal{T} \subset \big(\mathbb{N} \cup \{0\}\big)^{\mathbb{N}}$ be the set of all finite multi-indices. Stochastic Hermite polynomials, defined by $\mathbf{h}_\alpha(\omega) := \prod_k h_{\alpha_k}(\langle\omega, \xi_k\rangle)\,, \omega \in \mathcal{S}'\,, \alpha \in \mathcal{T}$, form an orthogonal basis of (L^2) with $(\mathbf{h}_\alpha, \mathbf{h}_\beta)_{(L^2)} = \delta_{\alpha,\beta}\alpha!$, where $\alpha! := \prod_k \alpha_k!$.

The Gelfand triple

$$(\mathcal{S})_\rho \subset (L^2) \subset (\mathcal{S})_{-\rho}\,, \quad (0 \leq \rho \leq 1) \tag{7}$$

is widely used in white noise analysis (see [1, 3]). Here $(\mathcal{S})_\rho = \cap_{p\in\mathbb{N}}(\mathcal{S}_p)_\rho$ with projective limit topology, where

$$(\mathcal{S}_p)_\rho = \Big\{\varphi = \sum_{\alpha\in\mathcal{T}} \varphi_\alpha \mathbf{h}_\alpha \in (L^2) : \sum_{\alpha\in\mathcal{T}} (\alpha!)^{1+\rho} |\varphi_\alpha|^2 (2\mathbb{N})^{2p\alpha} < \infty\Big\}$$

and the norm $|\cdot|_{p,\rho}$, generated by the scalar product

$$(\varphi,\psi)_{p,\rho}=\sum_{\alpha\in\mathcal{T}}(\alpha!)^{1+\rho}\varphi_\alpha\psi_\alpha(2\mathbb{N})^{2p\alpha}\,,\quad \left(2\mathbb{N}\right)^{p\alpha}:=\prod_{i\in\mathbb{N}}(2i)^{p\alpha_i}\,;$$

$(\mathcal{S})_{-\rho}=\cup_{p\in\mathbb{N}}(\mathcal{S}_{-p})_{-\rho}$ with inductive limit topology, where $(\mathcal{S}_{-p})_{-\rho}$ is the adjoint to $(\mathcal{S}_p)_\rho$. The space $(\mathcal{S}_{-p})_{-\rho}$ can be identified with the Hilbert space of all formal expansions $\Phi=\sum_{\alpha\in\mathcal{T}}\Phi_\alpha\mathbf{h}_\alpha$ such that $\sum_{\alpha\in\mathcal{T}}(\alpha!)^{1-\rho}\frac{|\Phi_\alpha|^2}{(2\mathbb{N})^{2p\alpha}}<\infty$ with scalar product $(\Phi,\Psi)_{-p,-\rho}=\sum_{\alpha\in\mathcal{T}}(\alpha!)^{1-\rho}\frac{\Phi_\alpha\Psi_\alpha}{(2\mathbb{N})^{2p\alpha}}$. We will denote $|\cdot|^2_{-p,-\rho}=(\cdot,\cdot)_{-p,-\rho}$. For $\Phi=\sum_{\alpha\in\mathcal{T}}\Phi_\alpha\mathbf{h}_\alpha\in(\mathcal{S})_{-\rho},\varphi=\sum_{\alpha\in\mathcal{T}}\varphi_\alpha\mathbf{h}_\alpha\in(\mathcal{S})_\rho$ we have $\langle\Phi,\varphi\rangle=\sum_{\alpha\in\mathcal{T}}\alpha!\Phi_\alpha\varphi_\alpha$.

A set $M\subseteq(\mathcal{S})_\rho$ is called bounded if for any $\{\varphi_n\}\subseteq M$ and for any $\{\varepsilon_n\}\subset\mathbb{R}$ converging to 0, the sequence $\{\varepsilon_n\varphi_n\}$ converges to zero in $(\mathcal{S})_\rho$. It is easy to see that boundedness of a set in $(\mathcal{S})_\rho$ is equivalent to its boundedness in any $(\mathcal{S}_p)_\rho$.

Let H be a separable Hilbert space over $\mathbb{C}$ with scalar product $(\cdot,\cdot)$ and corresponding norm $\|\cdot\|$. Denote by $(L^2)(H)$ the space of H-valued functions on $\mathcal{S}'$, square Bochner integrable with respect to μ. Let $\{e_j\}_{j=1}^\infty$ be an orthonormal basis in H. The family $\{\mathbf{h}_\alpha e_j\}_{\alpha\in\mathcal{T},j\in\mathbb{N}}$ is an orthogonal basis in $(L^2)(H)$. Any $f\in(L^2)(H)$ can be expanded into Fourier series $f=\sum_{\alpha\in\mathcal{T},j\in\mathbb{N}}f_{\alpha,j}\mathbf{h}_\alpha e_j=\sum_{\alpha\in\mathcal{T}}f_\alpha\mathbf{h}_\alpha=\sum_{j=1}^\infty f_je_j$, where $f_{\alpha,j}\in\mathbb{R},\ f_\alpha=\sum_j f_{\alpha,j}e_j\in H,\ f_j=\sum_{\alpha\in\mathcal{T}}f_{\alpha,j}\mathbf{h}_\alpha\in(L^2)$, and we have $\|f\|^2_{(L^2)(H)}=\sum_{\alpha\in\mathcal{T},j\in\mathbb{N}}\alpha!|f_{\alpha,j}|^2=\sum_{\alpha\in\mathcal{T}}\alpha!\|f_\alpha\|^2_H=\sum_{j=1}^\infty\|f_j\|^2_{(L^2)}$.

Define the space $(\mathcal{S})_{-\rho}(H)$ of H-valued generalized functions over the space $(\mathcal{S})_\rho$ of test functions as the space of all linear continuous operators $\Phi:(\mathcal{S})_\rho\to H$ with the topology of uniform convergence on bounded subsets of $(\mathcal{S})_\rho$. We will denote by $\Phi[\varphi]$ the action of $\Phi\in(\mathcal{S})_{-\rho}(H)$ on a test function $\varphi\in(\mathcal{S})_\rho$.

Now we describe the structure of $(\mathcal{S})_{-\rho}(H)$. It is easy to prove the following proposition:

Proposition 1 *Any $\Phi\in(\mathcal{S})_{-\rho}(H)$ is bounded as an operator from $(\mathcal{S}_p)_\rho$ to H for some $p\in\mathbb{N}$.*

Since $(\mathcal{S})_\rho$ is a countably Hilbert nuclear space, it follows from Proposition 1:

Corollary 1 *Any $\Phi\in(\mathcal{S})_{-\rho}(H)$ is a Hilbert–Schmidt operator from $(\mathcal{S}_p)_\rho$ to H for some $p\in\mathbb{N}$.*

For any $\Phi\in(\mathcal{S})_{-\rho}(H)$ denote by Φ_j the linear functional, defined on $(\mathcal{S})_\rho$ by $\langle\Phi_j,\varphi\rangle:=(\Phi[\varphi],e_j)$. Let Φ be Hilbert–Schmidt from $(\mathcal{S}_p)_\rho$ to H, then all Φ_j, $j\in\mathbb{N}$, belong to the corresponding $(\mathcal{S}_{-p})_{-\rho}$ and thus we have

$$\Phi_j=\sum_{\alpha\in\mathcal{T}}\Phi_{\alpha,j}\mathbf{h}_\alpha\,,\quad \sum_{\alpha\in\mathcal{T}}(\alpha!)^{1-\rho}\frac{|\Phi_{\alpha,j}|^2}{(2\mathbb{N})^{2p\alpha}}<\infty\,.$$

For the Hilbert–Schmidt norm of Φ as an operator from $(\mathcal{S}_p)_\rho$ to H we have:

$$\|\Phi\|^2_{\mathrm{HS},p,\rho}=\sum_{\alpha\in\mathcal{T}}\left\|\Phi\left[\frac{\mathbf{h}_\alpha}{(\alpha!)^{\frac{1+\rho}{2}}(2\mathbb{N})^{p\alpha}}\right]\right\|^2=\sum_{\alpha\in\mathcal{T},j\in\mathbb{N}}(\alpha!)^{1-\rho}\frac{|\Phi_{\alpha,j}|^2}{(2\mathbb{N})^{2p\alpha}}\,.$$

Denote by $\mathrm{HS}\big((\mathcal{S}_p)_\rho;H\big)$ the space of Hilbert–Schmidt operators from $(\mathcal{S}_p)_\rho$ to H.It is a separable Hilbert space. The family of operators $\{\mathbf{h}_\alpha\otimes e_j\}_{\alpha\in\mathcal{T},j\in\mathbb{N}}$, defined by $(\mathbf{h}_\alpha\otimes e_j)\varphi := \big(\mathbf{h}_\alpha,\varphi\big)_{(L^2)}e_j\,,\ \varphi\in(\mathcal{S}_p)_\rho$ is an orthogonal basis of $\mathrm{HS}\big((\mathcal{S}_p)_\rho;H\big)$. It follows from Proposition 1 that $(\mathcal{S})_{-\rho}(H)=\bigcup\limits_{p\in\mathbb{N}}\mathrm{HS}\big((\mathcal{S}_p)_\rho;H\big)$. Any $\Phi\in(\mathcal{S})_{-\rho}(H)$ has the following decomposition:

$$\Phi[\cdot]=\sum_{j\in\mathbb{N}}\langle\Phi_j,\cdot\rangle e_j=\sum_{\alpha\in\mathcal{T},j\in\mathbb{N}}\Phi_{\alpha,j}(\mathbf{h}_\alpha\otimes e_j)=\sum_{\alpha\in\mathcal{T}}\Phi_\alpha(\mathbf{h}_\alpha,\cdot)_{(L^2)}\,,$$

where $\Phi_j=(\Phi[\cdot],e_j)\in(\mathcal{S}_{-p})_{-\rho}$ for some $p\in\mathbb{N}$, $\Phi_\alpha=\sum_{j\in\mathbb{N}}\Phi_{\alpha,j}e_j\in H$. We have

$$\|\Phi\|^2_{\mathrm{HS},p,\rho}=\sum_{j\in\mathbb{N}}|\Phi_j|^2_{-p,-\rho}=\sum_{\alpha\in\mathcal{T},j\in\mathbb{N}}(\alpha!)^{1-\rho}\frac{|\Phi_{\alpha,j}|^2}{(2\mathbb{N})^{2p\alpha}}=\sum_{\alpha\in\mathcal{T}}(\alpha!)^{1-\rho}\frac{\|\Phi_\alpha\|^2}{(2\mathbb{N})^{2p\alpha}}<\infty.$$

For all $p_1<p_2$ and $\Phi\in\mathrm{HS}\big((\mathcal{S}_{p_1})_\rho;H\big)$ we evidently have

$$\mathrm{HS}\big((\mathcal{S}_{p_1})_\rho;H\big)\subseteq\mathrm{HS}\big((\mathcal{S}_{p_2})_\rho;H\big)\,,\quad \|\Phi\|_{HS,p_1,\rho}\geq\|\Phi\|_{HS,p_2,\rho}.$$

A set $\mathcal{M}\subseteq(\mathcal{S})_{-\rho}(H)$ is called bounded if for any sequence $\{\Phi_n\}\subseteq\mathcal{M}$ and any $\{\varepsilon_n\}\subset\mathbb{R}$ convergent to zero, $\{\varepsilon_n\Phi_n\}$ converges to zero in $(\mathcal{S})_{-\rho}(H)$. It is easy to prove the following propositions:

Proposition 2 *A set $\mathcal{M}$ is bounded in $(\mathcal{S})_{-\rho}(H)$ if and only if for any bounded $M\subset(\mathcal{S})_\rho$ there exists $K>0$ such that $\|\Phi[\varphi]\|\leq K$ for any $\varphi\in M$, $\Phi\in\mathcal{M}$.*

Proposition 3 *If $\mathcal{M}$ is bounded in $(\mathcal{S})_{-\rho}(H)$, then there exist $p\in\mathbb{N}$ and $K>0$ such that $\|\Phi[\varphi]\|\leq K|\varphi|_{p,\rho}$ for all $\Phi\in\mathcal{M}$, $\varphi\in(\mathcal{S})_\rho$.*

Thus, if a set $\mathcal{M}$ is bounded in $(\mathcal{S})_{-\rho}(H)$, then all elements of $\mathcal{M}$ are bounded operators from $(\mathcal{S}_p)_\rho$ to H for some $p\in\mathbb{N}$ and $\mathcal{M}$ is bounded in $\mathcal{L}\big((\mathcal{S}_p)_\rho,H\big)$. Consequently we have

Proposition 4 *If $\mathcal{M}$ is bounded in $(\mathcal{S})_{-\rho}(H)$, then $\mathcal{M}\subset\mathrm{HS}\big((\mathcal{S}_p)_\rho;H\big)$ for some $p\in\mathbb{N}$, and $\mathcal{M}$ is bounded in $\mathrm{HS}\big((\mathcal{S}_p)_\rho;H\big)$.*

The next proposition, which we state omitting the proof, gives characterization of convergence in $(\mathcal{S})_{-\rho}(H)$.

Proposition 5 *Let $\Phi_n=\sum_\alpha\Phi_\alpha^{(n)}\mathbf{h}_\alpha\,,\Phi=\sum_\alpha\Phi_\alpha\mathbf{h}_\alpha\in(\mathcal{S})_{-\rho}(H)$. The following assertions are equivalent:*

(i) $\{\Phi_n\}$ converges to Φ in $(\mathcal{S})_{-\rho}(H)$;

(ii) All elements of the sequence $\{\Phi_n\}$ and Φ belong to $\mathrm{HS}\big((\mathcal{S}_p)_\rho;H\big)$ for some $p\in\mathbb{N}$ and $\lim\limits_{n\to\infty}\|\Phi_n-\Phi\|_{\mathrm{HS},p,\rho}=0$.

Let $\Phi(\cdot):\mathbb{R}\to(\mathcal{S})_{-\rho}(H)$. We will write $\Psi=\lim\limits_{t\to t_0}\Phi(t)$ if $\Phi(t_n)\to\Psi$ uniformly on any bounded subset of $(\mathcal{S})_\rho$ for any sequence $t_n\to t_0$. The derivative $\Phi'(t_0)$ will be understood in the same way. It is easy to derive from Proposition 5 the following

Corollary 2 *Let* $\Phi(t) = \sum_\alpha \Phi_\alpha(t)\mathbf{h}_\alpha \in (\mathcal{S})_{-\rho}(H)$ *for* $t \in [a,b]$ *and let* $t_0 \in [a,b]$.

1. $\lim_{t\to t_0} \Phi(t) = \Phi(t_0)$ *in* $(\mathcal{S})_{-\rho}(H)$ *if and only if all* $\Phi(t), t \in [a,b]$, *belong to* $\mathrm{HS}\big((\mathcal{S}_p)_\rho; H\big)$ *for some* $p \in \mathbb{N}$ *and* $\lim_{n\to\infty} \|\Phi(t) - \Phi(t_0)\|_{\mathrm{HS},p,\rho} = 0$*;*

2. $\Phi(t)$ *is differentiable at* $t_0 \in [a,b]$ *if and only if* $\dfrac{d\Phi}{dt} := \lim_{t\to t_0} \dfrac{\Phi(t)-\Phi(t_0)}{t-t_0}$ *exists in* $\mathrm{HS}\big((\mathcal{S}_p)_\rho; H\big)$ *for some* p.

Example. (H-valued cylinder Wiener process and white noise).

Let $n(\cdot,\cdot) : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ be a bijection with

$$n(i,j) \geq ij, \quad i,j \in \mathbb{N}. \tag{8}$$

Denote $\varepsilon_n := (0,0,\dots,\underset{n}{1},0,\dots)$. The sequence $\beta_j(t) = \sum_{i=1}^{\infty} \int_0^t \xi_i(s)\,ds\,\mathbf{h}_{\varepsilon_{n(i,j)}}$ is a sequence of independent Brownian motions. Then the H-valued random process

$$W(t) = \sum_{j\in\mathbb{N}} \beta_j(t) e_j = \sum_{n\in\mathbb{N}} W_{\varepsilon_n}(t)\,\mathbf{h}_{\varepsilon_n}, \quad W_{\varepsilon_n}(t) = \int_0^t e_{j(n)}\,\xi_{i(n)}(s)\,ds \in H,$$

is a cylindrical Wiener process (here $i(n), j(n) \in \mathbb{N}$ are such that $n(i(n), j(n)) = n$).

It is easy to show that $W(t) \notin (L^2)(H)$ for all $t \in \mathbb{R}$. At the same time it follows from the well known estimate $\int_0^t \xi_i(s)\,ds = O(i^{-\frac{3}{4}})$ and (8) that $\|W(t)\|^2_{\mathrm{HS},1,\rho} < \infty$. So we have $W(t) \in \mathrm{HS}\big((\mathcal{S}_1)_\rho; H\big) \subset (\mathcal{S})_{-\rho}(H)$.

Define the H-valued cylindrical white noise by

$$\mathbb{W}(t) := \sum_{i,j\in\mathbb{N}} \xi_i(t)\,(\mathbf{h}_{\varepsilon_{n(i,j)}} e_j) = \sum_{n\in\mathbb{N}} \mathbb{W}_{\varepsilon_n}(t)\,\mathbf{h}_{\varepsilon_n}, \quad \mathbb{W}_{\varepsilon_n}(t) = \xi_{i(n)}(t)\,e_{j(n)} \in H.$$

Since $\xi_i(t) = O\big(i^{-\frac{1}{4}}\big)$, we have $\|\mathbb{W}(t)\|^2_{\mathrm{HS},1,\rho} < \infty$, thus

$$\mathbb{W}(t) \in \mathrm{HS}\big((\mathcal{S}_1)_\rho; H\big) \subset (\mathcal{S})_{-\rho}(H).$$

Note that for all $t \in \mathbb{R}$ we have $\dfrac{d}{dt} W(t) = \mathbb{W}(t)$.

Let $\mathcal{E}_\theta := e^{\langle\cdot,\theta\rangle - \frac{1}{2}|\theta|_0^2}$. For any $\theta \in \mathcal{S}$ it is a random variable on $\mathcal{S}'$ belonging to $(\mathcal{S})_\rho$ for $0 \leq \rho < 1$ with $|\mathcal{E}_\theta|_{p,\rho} \leq 2^{\rho/2} \exp\Big[(1-\rho)^{\frac{2\rho-1}{1-\rho}} |\theta|_p^{\frac{2}{1-\rho}}\Big]$ (see [1]). The following expansion holds:

$$\mathcal{E}_\theta = \sum_{\alpha\in\mathcal{T}} e_\alpha \mathbf{h}_\alpha, \quad e_\alpha = \frac{1}{\alpha!} \prod_{i=1}^{\infty} (\theta, \xi_i)_0^{\alpha_i}.$$

Let $\Phi \in (\mathcal{S})_{-\rho}(H)$, $0 \leq \rho < 1$. Define *the S-transform* of Φ by

$$(S\Phi)(\theta) = \Phi[\mathcal{E}_\theta], \quad \theta \in \mathcal{S}.$$

The proof of the following characteristic theorem almost completely repeats the proof of the corresponding theorem for the $\mathbb{C}$-valued case (see, for example, [1]), and is thus omitted.

Theorem 1 *Let $\Phi \in (\mathcal{S})_{-\rho}(H)$, $0 \le \rho < 1$. Then $F = S\Phi$ satisfies the following conditions:*

(i) for any $\theta, \nu \in \mathcal{S}$ the function $F(\theta + z\nu)$ is entire analytic function of $z \in \mathbb{C}$.

(ii) There exist $K > 0, a > 0$, $p \in \mathbb{N}$, such that

$$\|F(\theta)\| \le K \exp\left[a|\theta|_p^{\frac{2}{1-\rho}}\right], \quad \theta \in \mathcal{S}. \tag{9}$$

If $F : \mathcal{S} \to H$ satisfies (i) and (ii), then there exists a unique $\Phi \in (\mathcal{S})_{-\rho}(H)$ such that $F = S\Phi$ and for any q such that $e^2\left(\frac{2a}{1-\rho}\right)^{1-\rho}\sum_{i=1}^{\infty}(2i)^{-2(q-p)} < 1$, it holds

$$\|\Phi\|_{\mathrm{HS},q,\rho} \le K\left(1 - e^2\left(\frac{2a}{1-\rho}\right)^{1-\rho}\sum_{i=1}^{\infty}(2i)^{-2(q-p)}\right)^{-1/2}.$$

Example. For the above defined cylinder white noise we have:

$$\big(S\mathbb{W}(t)\big)(\theta) = \mathbb{W}(t)\big[\mathcal{E}_\theta\big] = \sum_{i,j\in\mathbb{N}} \xi_i(t) e_j (\theta, \xi_{n(i,j)})_0 .$$

Let H_1 and H_2 be separable Hilbert spaces. Since the space $\mathrm{HS}(H_1;H_2)$ of Hilbert–Schmidt operators acting from H_1 to H_2 is a separable Hilbert space, we can consider the space $(\mathcal{S})_{-\rho}\big(\mathrm{HS}(H_1;H_2)\big)$ of $\mathrm{HS}(H_1;H_2)$-valued generalized random variables over $(\mathcal{S})_\rho$. For S-transforms of any $\Psi \in (\mathcal{S})_{-\rho}\big(\mathrm{HS}(H_1;H_2)\big)$ and $\Phi \in (\mathcal{S})_{-\rho}(H_1)$, $F(\theta) = S\Psi(\theta)S\Phi(\theta) \in H_2$ is well defined for any $\theta \in \mathcal{S}$. Since $S\Psi(\theta)$ and $S\Phi(\theta)$ satisfy conditions (i) and (ii) of Theorem 1, for any $\theta, \nu \in \mathcal{S}$ the function $F(\theta + z\nu)$ is an entire analytic function of $z \in \mathbb{C}$ and

$$\|S\Psi(\theta)S\Phi(\theta)\|_{H_2} \le \|S\Psi(\theta)\|_{\mathrm{HS}(H_1;H_2)}\|S\Phi(\theta)\|_{H_1} \le K_1K_2\exp\left[(a_1+a_2)|\theta|_p^{\frac{2}{1-\rho}}\right],$$

where K_1, K_2, a_1, a_2 are the constants from condition (ii) of Theorem 1 for Ψ and Φ correspondingly (we can obviously suppose the constant p in these conditions to be the same). It follows that F is an S-transform of a unique generalized random variable $\Theta \in (\mathcal{S})_{-\rho}(H_2)$. This justifies the following definition.

Let $\Psi \in (\mathcal{S})_{-\rho}\big(\mathrm{HS}(H_1;H_2)\big)$, $\Phi \in (\mathcal{S})_{-\rho}(H_1)$. We will call $\Theta \in (\mathcal{S})_{-\rho}(H_2)$ such that $S\Theta = S\Psi S\Phi$ the *Wick product* of Ψ and Φ and denote it $\Psi \diamond \Phi$.

Let $Q \in \mathrm{HS}(H)$, $H_Q = Q^{\frac{1}{2}}(H)$ with scalar product $(u,v)_{H_Q} = (Q^{-\frac{1}{2}}u, Q^{-\frac{1}{2}}v)$. For the H-valued cylindrical white noise, from the estimate

$$\|\mathbb{W}_{\varepsilon_{n(i,j)}}\|_{H_Q}^2 (2\mathbb{N})^{-2p\varepsilon_{n(i,j)}} = \frac{|\xi_i(t)|^2}{\sigma_j^2\big(2n(i,j)\big)^{2p}} \le \frac{|\xi_i(t)|^2}{\sigma_j^2\big(2ij\big)^{2p}} = O\big(\sigma_j^{-2} i^{-2p-\frac{1}{2}} j^{-2p}\big),$$

it follows

Proposition 6 *For any* $Q=\sum_{j=1}^{\infty}\sigma_j^2(e_j\otimes e_j)\in \mathrm{HS}(H;H)^*$, *(i.e.* $\sum_{j=1}^{\infty}\sigma_j^2<\infty$*), if*

$$\sum_{j=1}^{\infty}\sigma_j^{-2}j^{-2p}<\infty \text{ for some } p\in\mathbb{N}, \tag{10}$$

then $\mathbb{W}(t)\in(\mathcal{S})_{-\rho}(H_Q)$ *for all* $t\in\mathbb{R}$ *and any* $\rho\in[0;1]$.

It follows from proposition 6 that if Q satisfies (10), then for any stochastic process $\Psi(t)$ with values in $(\mathcal{S})_{-\rho}\big(\mathrm{HS}(H_Q;H)\big)$ the $(\mathcal{S})_{-1}(H)$-valued random process $\Psi(t)\diamond\mathbb{W}(t)$ is well defined.

We will call an $(\mathcal{S})_{-\rho}\big(\mathrm{HS}(H_Q;H)\big)$-valued random process $\Psi(t)$ *Hitsuda–Skorohod integrable* on $[0;T]$, if $\Psi(t)\diamond\mathbb{W}(t)$ is integrable on $[0;T]$ as an $(\mathcal{S})_{-1}(H)$-valued function and will call $\int_0^T\Psi(t)\diamond\mathbb{W}(t)\,dt$ the Hitsuda–Skorohod integral of $\Psi(t)$.

The Hitsuda–Skorohod integral is a generalization of the Ito integral $\int_0^T\Psi(t)\,dW(t)$ with respect to the cylindrical Wiener process. Namely, if $\Psi(t)\in(L^2)\big(\mathrm{HS}(H_Q;H)\big)$ for all $t\in[0;T]$, $\Psi(t)$ is adapted to the filtration generated by $W(t)$ and

$$\int_0^T\|\Psi(t)\|^2_{(L^2)(\mathrm{HS}(H_Q;H))}dt<\infty,$$

then

$$\int_0^T\Psi(t)\diamond\mathbb{W}(t)\,dt=\int_0^T\Psi(t)\,dW(t)$$

Let H_1 and H_2 be separable Hilbert spaces. For $A\in\mathcal{L}(H_1,H_2)$ define

$$A\Phi:=\sum_{\alpha\in\mathcal{T}}A\Phi_\alpha\mathbf{h}_\alpha,\ \text{for}\ \ \Phi=\sum_{\alpha\in\mathcal{T}}\Phi_\alpha\mathbf{h}_\alpha\in(\mathcal{S})_{-\rho}(H_1). \tag{11}$$

(See the proof in [5]). Defined in such a way A is a linear continuous operator with values in $(\mathcal{S})_{-\rho}(H_2)$. If A is not bounded, define $(\mathrm{dom}A)$ as the set of all $\sum_{\alpha\in\mathcal{T}}\Phi_\alpha\mathbf{h}_\alpha\in(\mathcal{S})_{-\rho}(H_1)$ such that $\Phi_\alpha\in\mathrm{dom}A$ for any $\alpha\in\mathcal{T}$ and $\sum_{\alpha\in\mathcal{T}}(\alpha!)^{1-\rho}\frac{\|A\Phi_\alpha\|^2_{H_2}}{(2\mathbb{N})^{2p\alpha}}<\infty$ for some $p\in\mathbb{N}$. Then (11) defines a linear operator on $(\mathrm{dom}A)$ with values in $(\mathcal{S})_{-\rho}(H_2)$. It is easy to verify that it is closed if A is a closed operator from H_1 to H_2, and to prove the next proposition.

Proposition 7 *Let* $A:H_1\to H_2$ *be linear and closed. For any* $\Phi\in(\mathrm{dom}A)\subseteq(\mathcal{S})_{-\rho}(H_1)$ *we have* $\big[S\Phi\big](h)\in\mathrm{dom}A\subseteq H_1$ *and* $\big[SA\Phi\big](h)=A\big[S\Phi\big](h),\ h\in\mathcal{S}$.

3. The Cauchy problem for a linear operator-differential equation with multiplicative noise

Consider the Cauchy problem (5) with a linear closed operator A acting in H, $B(\cdot)\in\mathcal{L}\big(H,\mathcal{L}(H)\big)$, $\Phi\in(\mathrm{dom}A)\subseteq(\mathcal{S})_{-\rho}(H)$. We obtain it by substituting the Hitsuda–Skorohod integral for the Ito one in equation (4) and differentiating both sides of the

*For $v\in V$, $u\in U$, where V and U are Hilbert spaces, we denote by $v\otimes u$ the operator from U to V, defined by $(v\otimes u)h:=v(u,h)_U$.

equation with respect to t. Note that if Q is a nuclear operator in H satisfying (10), then since $B\big(X(t)\big) \in (\mathcal{S})_{-\rho}\big(\mathrm{HS}(H_Q;H)\big)$ for any $X(t) \in (\mathcal{S})_{-\rho}(H)$, the Wick product in (5) is well defined. Our main result is the following theorem.

Theorem 2 *Let A be the generator of a C_0-semigroup in H, B be such that for each $y \in H$*

(BI) $\ker B(\cdot)y = \{0\}$*;*

(BII) $B(\mathrm{dom}A)y \subseteq \mathrm{dom}A$*;*

(BIII) The operator $C(\cdot)y : H \to \mathcal{L}(H)$, defined by $C(x)y := AB(x)y - B(Ax)y$ for $x \in$ $\mathrm{dom}A$, is bounded.

Then for any $\Phi \in (\mathrm{dom}A) \subseteq (\mathcal{S})_{-0}(H)$ the problem (5) has a unique solution in the space $(\mathcal{S})_{-0}(H)$.

Proof. Note that by the uniform boundedness principle it follows from (BIII) that there exists $M_{AB} > 0$ such that

$$\|C(x)y\| \le M_{AB}\|x\|\,\|y\|\,, \quad x \in \mathrm{dom}A\,,\ y \in H\,. \tag{12}$$

Applying S-transform to (5) we obtain the next Cauchy problem:

$$\frac{d}{dt}\hat{X}(t,\theta) = A\hat{X}(t,\theta) + B\big(\hat{X}(t,\theta)\big)\hat{\mathbb{W}}(t,\theta)\,,\ t \ge 0\,, \quad \hat{X}(0,\theta) = \hat{\Phi}(\theta)\,, \quad \theta \in \mathcal{S}\,, \tag{13}$$

where $\hat{X}(t,\theta) = S[X(t)](\theta)$, $\hat{\mathbb{W}}(t,\theta) = S[\mathbb{W}(t)](\theta)$, $\hat{\Phi}(\theta) = S\Phi(\theta)$.

We first prove the uniqueness of solution. Note that if $\hat{X}(\cdot,\theta)$ is a solution of (13) for some $\theta \in \mathcal{S}$, it satisfies the equation

$$\hat{X}(t,\theta) = U(t)\hat{\Phi}(\theta) + \int_0^t U(t-s)B(\hat{X}(s,\theta))\hat{\mathbb{W}}(s,\theta)\,ds\,, \quad t \ge 0\,.$$

Thus it is sufficient to prove that equation

$$\hat{X}(t,\theta) - \int_0^t U(t-s)B(\hat{X}(s,\theta))\hat{\mathbb{W}}(s,\theta)\,ds = 0, \quad t \ge 0 \tag{14}$$

has the only solution $\hat{X}(t,\theta) \equiv 0$ for any $\theta \in \mathcal{S}$, where $\{U(t)\,,\, t \ge 0\}$ is the C_0-semigroup generated by A with $M > 0$, $a \in \mathbb{R}$ such that

$$\|U(t)\| \le Me^{at}\,, \quad t \ge 0\,. \tag{15}$$

This can be proved using the Volterra equations technique and the fact that $\hat{\mathbb{W}}(s,\theta)$ is an infinitely differentiable $\mathbb{H}$-valued function of s and thus is bounded on any segment of $\mathbb{R}$.

To prove existence of solution consider the series

$$T(t,\theta) = \sum_{k=0}^{\infty} T_k(t,\theta)\,, \quad \theta \in \mathcal{S}\,, \tag{16}$$

where operators $T_k(t,\theta), t \geq 0,\ k = 0,1,2,\ldots$ are defined as follows:

$$T_0(t,\theta) = U(t), \quad T_k(t,\theta)x = \int_0^t U(t-s)B\big(T_{k-1}(s,\theta)x\big)\hat{\mathbb{W}}(s,\theta)\,ds,\ x \in H.$$

Proving first for $t \geq 0, \theta \in \mathcal{S}, k \in \mathbb{N} \cup \{0\}$ and $\Phi \in (\mathrm{dom}A)$ the estimates

$$\|T_k(t,\theta)\|_{\mathcal{L}(H)} \leq M^{k+1}\|B\|^k e^{at}|\theta|_0^k \sqrt{\frac{t^k}{k!}}\,, \tag{17}$$

$$\|AT_k(t,\theta)\hat{\Phi}(\theta)\| \leq M^{k+1}\|B\|^{k-1}|\theta|_0^k e^{at}\sqrt{\frac{t^k}{k!}}\left(\|B\|\|A\hat{\Phi}(\theta)\| + kM_{AB}\|\hat{\Phi}(\theta)\|\right), \tag{18}$$

where $M > 0$ and $a \in \mathbb{R}$ are constants from (15), $\|B\| = \|B\|_{\mathcal{L}(H,\mathcal{L}(H))}$, M_{AB} is from (12), we obtain by (17) for any $n,m \in \mathbb{N}$

$$\begin{aligned} \sum_{k=n}^{n+m}\|T_k(t,\theta)\| &\leq Me^{at}\sum_{k=n}^{n+m}\frac{\left(M\sqrt{2}\|B\||\theta|_0\sqrt{t}\right)^k}{\sqrt{k!}}\cdot\frac{1}{\sqrt{2^k}} \leq \\ &\leq Me^{at}\left(\sum_{k=n}^{n+m}\frac{\left(2M^2\|B\|^2|\theta|_0^2 t\right)^k}{k!}\right)^{1/2}\left(\sum_{k=n}^{n+m}\frac{1}{2^k}\right)^{1/2}. \end{aligned} \tag{19}$$

Hence (16) is absolutely convergent to $T(t,\theta)$ in $\mathcal{L}(H)$ for any $t \geq 0, \theta \in \mathcal{S}$.

For any $\Phi \in (\mathrm{dom}A)$, by Proposition 7 and properties of C_0-semigroups we obtain: $T_0(t,\theta)\hat{\Phi}(\theta) \in \mathrm{dom}A$ for all $t \geq 0$ and $\theta \in \mathcal{S}$. It follows from (BII) that $B\big(\mathrm{dom}A\big)\hat{W}(t,\theta) \subseteq \mathrm{dom}A$ for all $t \geq 0$ and $\theta \in \mathcal{S}$ and by induction we obtain that $T_k(t,\theta)\hat{\Phi}(\theta) \in \mathrm{dom}A$ for all $\Phi \in (\mathrm{dom}A), k \in \mathbb{N}, t \geq 0$ and $\theta \in \mathcal{S}$. It also follows from (BII) that $B\big(T_k(s,\theta)\hat{\Phi}(\theta)\big)\hat{\mathbb{W}}(t,\theta) \in \mathrm{dom}A$. Moreover, we have

$$\frac{d}{dt}U(t-s)B\big(T_k(s,\theta)\hat{\Phi}(\theta)\big)\hat{\mathbb{W}}(t,\theta) = AU(t-s)B\big(T_k(s,\theta)\hat{\Phi}(\theta)\big)\hat{\mathbb{W}}(t,\theta),\ t \geq 0, \theta \in \mathcal{S}.$$

Thus for all $\Phi \in (\mathrm{dom}A)$ we have

$$\frac{d}{dt}T_0(t,\theta)\hat{\Phi}(\theta) = AT_0(t,\theta)\hat{\Phi}(\theta)\,, \tag{20}$$

$$\begin{aligned} \frac{d}{dt}T_k(t,\theta)\hat{\Phi}(\theta) = &\int_0^t AU(t-s)B\big(T_{k-1}(s,\theta)\hat{\Phi}(\theta)\big)\hat{\mathbb{W}}(s,\theta)\,ds + \\ &+ B\big(T_{k-1}(t,\theta)\hat{\Phi}(\theta)\big)\hat{\mathbb{W}}(t,\theta)\,. \end{aligned} \tag{21}$$

Since A is closed we can rewrite (21) as

$$\frac{d}{dt}T_k(t,\theta)\hat{\Phi}(\theta) = AT_k(t,\theta)\hat{\Phi}(\theta) + B\big(T_{k-1}(t,\theta)\hat{\Phi}(\theta)\big)\hat{\mathbb{W}}(t,\theta)\,. \tag{22}$$

Using (18) we obtain

$$\sum_{k=n+1}^{m} \|AT_k(t,\theta)\hat{\Phi}(\theta)\| \le Me^{at}\left(\sum_{k=n+1}^{m} \frac{(\sqrt{2}M\|B\||\theta|_{L^2(\mathbb{R})}\sqrt{t})^k}{\sqrt{k!}}\cdot\frac{1}{\sqrt{2^k}}\right)\|A\hat{\Phi}(\theta)\|+$$
$$+\frac{M}{\|B\|}e^{at}\left(\sum_{k=n+1}^{m} \frac{(\sqrt{2}M\|B\||\theta|_{L^2(\mathbb{R})}\sqrt{t})^k}{\sqrt{k!}}\cdot\frac{k}{\sqrt{2^k}}\right)M_{AB}\|\hat{\Phi}(\theta)\| \le$$

$$\le Me^{at}\left(\sum_{k=n+1}^{m} \frac{\left(2M^2\|B\|^2|\theta|_0^2 t\right)^k}{k!}\right)^{1/2}\cdot\left(\sum_{k=n+1}^{m}\frac{1}{2^k}\right)^{1/2}\|A\hat{\Phi}(\theta)\|+$$
$$+\frac{M}{\|B\|}e^{at}\left(\sum_{k=n+1}^{m} \frac{\left(2M^2\|B\|^2|\theta|_0^2 t\right)^k}{k!}\right)^{1/2}\cdot\left(\sum_{k=n+1}^{m}\frac{k^2}{2^k}\right)^{1/2}M_{AB}\|\hat{\Phi}(\theta)\|.$$

it follows from here that the series $\sum_{k=0}^{\infty} AT_k(t,\theta)\hat{\Phi}(\theta)$ converges in H for all $\theta\in\mathcal{S}$, $\Phi\in(\mathrm{dom}A)$. Taking sum of equalities (20) and (22) with respect to all $k\in\mathbb{N}$ we obtain in the right hand side a series converging in H for all $t\ge 0$, $\theta\in\mathcal{S}$. This proves that $\hat{X}(t,\theta)=T(t,\theta)\hat{\Phi}(\theta)$ is a solution of (13).

It follows from (19) that

$$\|T(t,\theta)\| \le \sum_{k=0}^{\infty}\|T_k(t,\theta)\| \le Me^{at}\sum_{k=0}^{\infty}\frac{\left(M\sqrt{2}\|B\||\theta|_{L^2(\mathbb{R})}\sqrt{t}\right)^k}{\sqrt{k!}}\cdot\frac{1}{\sqrt{2^k}} \le$$
$$\le Me^{at}\left(\sum_{k=0}^{\infty}\frac{\left(2M^2\|B\|^2|\theta|_0^2 t\right)^k}{k!}\right)^{1/2}\left(\sum_{k=0}^{\infty}\frac{1}{2^k}\right)^{1/2} = M\sqrt{2}e^{at}\exp\left(M^2\|B\|^2|\theta|_0^2 t\right).$$

By (9) we have $\|\hat{\Phi}(\theta)\|\le\|\Phi\|_{HS,p,0}\exp\left(|\theta|_p^2\right)$, $\theta\in\mathcal{S}$, for some $p\in\mathbb{N}$. It follows that for $t\ge 0$ we have

$$\|\hat{X}(t,\theta)\| \le M\sqrt{2}e^{at}\exp\left(\left(M^2\|B\|^2 t+1\right)|\theta|_p^2\right)\|\Phi\|_{HS,p,0}, \quad \theta\in\mathcal{S}.$$

It follows from here that for each $t\ge 0$ $\hat{X}(t,\theta)$ is an S-transform of a unique $X(t)\in(\mathcal{S})_{-0}(H)$, which is a unique solution of problem (13). □

It is easy to see that A and B defined by (2) and (3) respectively satisfy the conditions of Theorem 2. Thus the stochastic perturbation of our model problem described in introduction has a unique solution in $(\mathcal{S})_{-0}(L^2[0;1])$.

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