Rend. Sem. Mat. Univ. Politec. Torino
Vol. 71, 3–4 (2013), 401 – 420

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JETS OF SINGULAR FOLIATIONS*

Abstract. Given a singular foliation satisfying locally everywhere the Frobenius condition, even at the singularities, we show how to construct its global sheaves of jets. Our construction is purely formal, and thus applicable in a variety of contexts.

1. Introduction

Let M be a complex manifold of complex dimension m. A holomorphic foliation $\mathcal{L}$ of dimension n of M is a decomposition of M in complex submanifolds, called leaves, of dimension n. Also, locally the leaves must pile up nicely, like the fibers of a holomorphic map. In other words, for each point p of M there must exist an open neighborhood U and a holomorphic submersion $\varphi\colon U \to V$ to an open subset $V \subseteq \mathbf{C}^{m-n}$ such that the fibers of φ are the intersection of the leaves with U. We say that φ defines $\mathcal{L}$ on U.

A holomorphic foliation $\mathcal{L}$ of M induces a vector subbundle of the tangent bundle T_M of M: for each $p \in M$, the vector subspace of $T_{M,p}$ is the tangent space at p of the unique leaf passing through p. Thinking in dual terms, $\mathcal{L}$ induces a surjection $w\colon T_M^* \to E$ from the bundle of holomorphic 1-forms to a holomorphic rank-n vector bundle E. The bundle E, also denoted by $T_{\mathcal{L}}^*$, is regarded as the bundle of 1-forms of $\mathcal{L}$.

Not all surjections $w\colon T_M^* \to E$ to a holomorphic vector bundle E arise from foliations. The necessary and sufficient condition for this is given by the Frobenius Theorem: locally at each point p of M, choose a trivialization of E, and consider the vector fields $X_1, \dots, X_n$ induced by w; if their Lie brackets $[X_i, X_j]$ can be expressed as sums $\sum_\ell g_\ell X_\ell$, where the g_ℓ are holomorphic functions on a neighborhood of p, then w arises from a foliation.

The surjection w can be seen, locally on an open subset $U \subseteq M$ for which there is a submersion $\varphi\colon U \to V \subseteq \mathbf{C}^{m-n}$ defining $\mathcal{L}$, as the natural map $T_U^* \to T_{U/V}^*$ from the bundle of 1-forms on U to the bundle of relative 1-forms on U over V. Also, on such a U, we may consider the natural surjection $J_U^q \to J_{U/V}^q$ from the bundle J_U^q of q-jets (or principal parts of order q) on U to the bundle $J_{U/V}^q$ of relative jets of φ, for each integer $q \geq 0$. These patch to form surjections $w^q\colon J_M^q \to J_{\mathcal{L}}^q$ to bundles $J_{\mathcal{L}}^q$ that can be regarded as the bundles of q-jets of the foliation.

But what happens if all the data are algebraic? More precisely, assume M is algebraic, E and w are algebraic, and there is an algebraic trivialization of E at each $p \in M$ such that the resulting vector fields $X_1, \dots, X_n$ are involutive, i.e. satisfy $[X_i, X_j] =$

*Supported by CNPq, Proc. 478625/03-0 and 301117/04-7, FAPERJ, Proc. E-26/170.418/00, and CNPq/FAPERJ, Proc. E-26/171.174/03.

$\sum_\ell g_\ell^{i,j} X_\ell$ for $g_\ell^{i,j}$ algebraic. Are the bundles $J^m_{\mathcal{L}}$ algebraic? In principle, they are just holomorphic, since the local submersions φ from which they arise are just holomorphic, constructed by means of the implicit function theorem.

Moreover, it is rare for a projective manifold to admit interesting foliations. For this reason, one has started to study *singular foliations*, in a variety of ways. For instance, a singular foliation of M of dimension n may be defined to be a map $w\colon T^*_M \to E$ to a rank-n holomorphic vector bundle E which, on a dense open subset $M^0 \subseteq M$, arises from a foliation $\mathcal{L}$. We still regard E as the bundle of 1-forms of the foliation. But the bundles of jets $J^m_{\mathcal{L}}$ are, in principle, only defined on M^0. Under which conditions do they extend to M?

In the present paper, we will show that if all the data are algebraic, then the bundles $J^m_{\mathcal{L}}$ are algebraic. Furthermore, if the same Frobenius' conditions, appropriately formulated, are verified at each point of $M - M^0$, then the bundles $J^m_{\mathcal{L}}$ extend to the whole M. For the proofs, we will completely bypass Frobenius Theorem, giving an entirely formal construction of the bundles of jets that applies in many categories, for instance that of differentiable manifolds, or of schemes over any base. Furthermore, not only will we consider maps to bundles E, but also to sheaves of modules, locally free or not, obtaining thus sheaves of jets.

Our construction of the sheaves of jets is by "iteration", so will only apply in characteristic zero. For an approach in positive characteristic, albeit limited in scope, see [1] or [10]. From now on, all rings are assumed to be $\mathbf{Q}$-algebras.

Here is what we do. Let X be a topological space, $\mathcal{O}_B$ a sheaf of $\mathbf{Q}$-algebras and $\mathcal{O}_X$ a sheaf of $\mathcal{O}_B$-algebras. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules and $D\colon \mathcal{O}_X \to \mathcal{F}$ an $\mathcal{O}_B$-derivation. We may think of $\mathcal{O}_X$ as the sheaf of "functions" on X and of $\mathcal{O}_B$ as the sheaf of "constant functions." For each integer $i \geq 0$, let $\mathcal{T}^i(\mathcal{F})$ be the tensor product over $\mathcal{O}_X$ of i copies of $\mathcal{F}$, and denote by $\mathcal{S}^i(\mathcal{F})$ and $\mathcal{A}^i(\mathcal{F})$ its symmetric and exterior quotients. Let $\mathcal{S}(\mathcal{F})$ be the direct product of all the $\mathcal{S}_i(\mathcal{F})$ for all integers $i \geq 0$, with its natural graded $\mathcal{O}_X$-algebra structure. We may think of $\mathcal{S}(\mathcal{F})$ as the sheaf of "formal power series" on the sections of $\mathcal{F}$.

We define a D-connection to be a map of $\mathcal{O}_B$-modules $\gamma\colon \mathcal{F} \to \mathcal{T}^2(\mathcal{F})$ satisfying

$$\gamma(am) = D(a) \otimes m + a\gamma(m)$$

for all local sections a of $\mathcal{O}_X$ and m of $\mathcal{F}$. We will see in Construction 2 how γ can be used to iterate D to obtain a sequence of maps $T_i : \mathcal{F} \to \mathcal{S}^i(\mathcal{F}) \otimes \mathcal{F}$, for $i = 0, 1, \dots$, where $T_0 := \mathrm{id}_{\mathcal{F}}$, $T_1 = \gamma$, and the T_i satisfy properties similar to that of a connection, i.e. Equations 3. We call any sequence of maps $T = (T_0, T_1, \dots)$ with these properties, for any D-connection γ, an extended D-connection.

From an extended D-connection T we get a map $h\colon \mathcal{O}_X \to \mathcal{S}(\mathcal{F})$ of $\mathcal{O}_B$-algebras, with $h_0 := \mathrm{id}_{\mathcal{O}_X}$ and $h_1 := D$, by letting $h_i(a)$ be the class in $\mathcal{S}^i(\mathcal{F})$ of $(1/i)T_{i-1}D(a)$ for each local section a of $\mathcal{O}_X$; see Construction 3. We may think of h as a way of computing "Taylor series" of functions on X. In this way, $\mathcal{S}(\mathcal{F})$ may be regarded as a sheaf of jets. We call such an h an iterated Hasse derivation.

However, D-connections are usually local gadgets. So, to be able to patch the local maps h, we need D-connections to be canonical. But there is nothing canonical about γ: for every O_X-linear map $\nu\colon \mathcal{F} \to \mathcal{T}^2(\mathcal{F})$, the sum $\gamma+\nu$ is also a D-connection! So we study special D-connections.

A D-connection γ is called flat if the image of $\gamma D(O_X)$ in $\mathcal{A}^2(\mathcal{F})$ is zero. When such γ exists, we say that D is integrable. Our main result, Theorem 5, shows that, given a flat D-connection γ, there is an extended D-connection T with $T_1=\gamma$, which is also flat, meaning that $T_iD(O_X)$ lies in the subsheaf of $\mathcal{S}^i(\mathcal{F})\otimes\mathcal{F}$ generated locally by

$$\sum_{\ell=1}^{i} m_1m_2\cdots m_{\ell-1}m_{\ell+1}m_{\ell+2}\cdots m_i\otimes m_\ell$$

for all local sections $m_1,\dots,m_i$ of $\mathcal{F}$, for each $i\geq 1$.

Now, our Proposition 1 says that a flat, extended D-connection is “comparable” to any other extended D-connection. So, Theorem 5 and Proposition 1 can be coupled to yield that all iterated Hasse derivations are equivalent; see Corollary 1. More precisely, if h and h' are iterated Hasse derivations, there is an O_X-algebra automorphism $\phi\colon \mathcal{S}(\mathcal{F})\to\mathcal{S}(\mathcal{F})$ such that $h'=\phi h$. Furthermore, the degree-i part of this ϕ is zero if $i<0$ and the identity if $i=0$.

Now, for the patching we also need the automorphisms ϕ to be canonical, so that cocycle conditions are satisfied. For this, we make a technical assumption on D, that bounds its singularities, and holds in all applications we know of; see Corollary 1 and the remark thereafter.

Finally, assume that D is locally integrable, and has bounded singularities, in the sense alluded to above. The patching of the local iterated Hasse derivations and the O_X-algebra automorphisms comparing them is straightforward. We obtain an O_X-algebra $\mathcal{J}$ and a map $h\colon O_X\to\mathcal{J}$ of O_B-algebras. Also, since the local O_X-algebra automorphisms do not decrease degrees, and their degree-0 parts are the identity, $\mathcal{J}$ comes naturally with a filtration by O_X-algebra quotients $\mathcal{J}^q$, for $q=0,1,\dots$, and natural exact sequences

$$0\to\mathcal{S}^q(\mathcal{F})\to\mathcal{J}^q\to\mathcal{J}^{q-1}\to 0$$

for each $q>0$; see Construction 6. We say that $\mathcal{J}^q$ is the sheaf of q-jets of D.

How does this formal construction fit with the geometric setting? If M is a holomorphic manifold, let (X,O_X) be the ringed space where X is the underlying topological space of M and O_X is its sheaf of holomorphic functions. Let O_B be the sheaf of constant complex functions. A map of vector bundles $w\colon T^*_M\to E$ corresponds to a derivation $D\colon O_X\to\mathcal{F}$, where $\mathcal{F}$ is the sheaf of holomorphic sections of E. To say that w arises from a foliation on an open subset $U\subseteq M$ is equivalent to say that $\mathcal{F}|_U=O_UD(O_U)$ and $D|_U$ is locally integrable on U; see Example 2. Now, assume that D is locally integrable on the whole X, and that $D(O_X)$ generates $\mathcal{F}$ as an O_X-module on a dense open subset $M^0\subseteq M$. Then there exists a sheaf of q-jets $\mathcal{J}^q$ on X, as explained above. Also, w defines a foliation $\mathcal{L}$ on M^0, and $\mathcal{J}^q|_{M^0}$ is the sheaf of sections of the bundle of q-jets $J^q_{\mathcal{L}}$; see Example 4.

Bundles of jets associated to foliations or derivations were considered by a number of people. In algebraic geometry, to my knowledge, the first was Letterio Gatto, who in his thesis [6] constructed jets from a family of stable curves $f\colon X\to\mathcal{S}$ and its canonical derivation $O_X \to \omega_{X/S}$, where $\omega_{X/S}$ is the relative dualizing sheaf. Afterwards, in [1], jets were constructed for more general families, of local complete intersection curves, over any base and in any characteristic.

Also, Dan Laksov and Anders Thorup constructed bundles of jets in a series of articles in different setups; see [8], [9] and [10]. In characteristic zero, their more general work is [9]. Actually, in [9], Laksov and Thorup construct larger sheaves of "jets", that are naturally noncommutative. The true generalization of the sheaf of jets is what they call "symmetric" jets. They show that the sheaf of (symmetric) jets is uniquely defined when $\mathcal{F}$ is free and has an O_X-basis under which D can be expressed using commuting derivations of O_X. As we have observed above, the uniqueness of the definition is important for patching. However, the commutativity is stronger than Frobenius' conditions, at least in the algebraic category — in the analytic category, at nonsingular points, the local existence of commuting derivations follows from the existence of the foliation. The present work arose from the feeling that the Frobenius' conditions should be enough to construct sheaves of jets.

There have already been applications of the sheaves of jets associated to a foliation or a derivation. They were used by Gatto [7], and Gatto and myself [3] in enumerative aplications. They were used by myself in understanding limits of ramification points [2], and of Weierstrass points, with Nivaldo Medeiros, [4] and [5]. They were also used by Jorge Vitório Pereira in the study of foliations of the projective space [12].

Finally, we will see an example where the integrability condition holds and $\mathcal{F}$ is not locally free; see Example 3. That will be the example of the canonical derivation on a special non-Gorenstein curve, arguably the simplest non-Gorenstein unibranch curve there is, whose complete local ring at the singular point is of the form $\mathbf{C}[[t^3,t^4,t^5]]$, as a subring of $\mathbf{C}[[t]]$. It could be that the integrability condition holds for the canonical derivation on any curve, Gorenstein or not. If so, the sheaf of jets might be used to define Weierstrass points on non-Gorenstein integral curves, and show that these points are limits of Weierstrass points on nearby curves, in the way done by Robert Lax and Carl Widland for Gorenstein curves; see [11] and [13]. However, this problem will not be pursued here.

Here is a layout of the article. In Section 2, we define connections, extended connections and iterated Hasse derivations, and explain a few preliminary constructions. In Section 3, we define integrable derivations and flat (extended) connections, and show that a flat, extended connection is comparable with any other extended connection. Finally, in Section 4, we show that, if a derivation D is integrable, then flat, extended connections exist, and all iterated Hasse derivations are equivalent; then we construct the sheaves of jets for locally integrable derivations.

Throughout the paper, X will stand for a topological space, O_B for a sheaf of $\mathbf{Q}$-algebras, O_X for a sheaf of O_B-algebras, $\mathcal{F}$ for a sheaf of O_X-modules and $D\colon O_X \to \mathcal{F}$ for an O_B-derivation.

This work started as a joint work with Letterio Gatto. However, he felt he did not contribute to it as much as he wished. Though a few discussions with him were vital to how this work came to be, and though I feel that this could be classified as a joint work, I had to respect his decision to not coauthor it. Anyway, being the only thing left for me to do, I thank him for his great contributions.

2. Derivations and connections

Recall the notation for X, $\mathcal{O}_B$, $\mathcal{O}_X$, $\mathcal{F}$ and D.

CONSTRUCTION 1. (*Tensor operations*) We denote by

$$\mathcal{T}(\mathcal{F}) := \prod_{n=0}^{\infty} \mathcal{T}^n(\mathcal{F}), \quad \mathcal{S}(\mathcal{F}) := \prod_{n=0}^{\infty} \mathcal{S}^n(\mathcal{F}), \quad \mathcal{A}(\mathcal{F}) := \prod_{n=0}^{\infty} \mathcal{A}^n(\mathcal{F})$$

the *formal tensor*, *symmetric* and *exterior* graded sheaf of $\mathcal{O}_X$-algebras of $\mathcal{F}$, respectively. (Note that we take the direct product and not the direct sum.)

Set $\mathcal{R}^0(\mathcal{F}) := \mathcal{O}_X$. Also, set $\mathcal{R}^n(\mathcal{F}) := \mathcal{S}^{n-1}(\mathcal{F}) \otimes \mathcal{F}$ for each integer $n \geq 1$, and

$$\mathcal{R}(\mathcal{F}) := \prod_{n=0}^{\infty} \mathcal{R}^n(\mathcal{F}).$$

Then $\mathcal{R}(\mathcal{F})$ is a graded $\mathcal{O}_X$-algebra quotient of $\mathcal{T}(\mathcal{F})$, in a natural way.

As usual, we let $\mathcal{T}_+(\mathcal{F}), \mathcal{S}_+(\mathcal{F}), \mathcal{A}_+(\mathcal{F})$ and $\mathcal{R}_+(\mathcal{F})$ denote the ideals generated by formal sums with zero constant terms in each of the indicated $\mathcal{O}_X$-algebras.

We view $\mathcal{T}(\mathcal{F})$ as a sheaf of algebras, with the (noncommutative) product induced by tensor product. So, given local sections $m_1, \ldots, m_n$ of $\mathcal{F}$, we let $m_1 \cdots m_n$ denote their product in $\mathcal{T}^n(\mathcal{F})$. Also, we view $\mathcal{S}(\mathcal{F})$, $\mathcal{A}(\mathcal{F})$ and $\mathcal{R}(\mathcal{F})$ as sheaves of $\mathcal{T}(\mathcal{F})$-algebras, and $\mathcal{S}(\mathcal{F})$ as a sheaf of $\mathcal{R}(\mathcal{F})$-algebras, under the natural quotient maps. So, given a local section ω of $\mathcal{T}^n(\mathcal{F})$ (resp. $\mathcal{R}^n(\mathcal{F})$), we will use the same symbol ω to denote its image in $\mathcal{S}^n(\mathcal{F})$, $\mathcal{A}^n(\mathcal{F})$ or $\mathcal{R}^n(\mathcal{F})$ (resp. $\mathcal{S}^n(\mathcal{F})$). These simplifications should not lead to confusion, and will clean the notation enormously.

Define the *switch operator* $\sigma\colon \mathcal{R}_+(\mathcal{F}) \to \mathcal{R}_+(\mathcal{F})$ as the homogeneous $\mathcal{O}_X$-linear map of degree 0 given by $\sigma|_{\mathcal{F}} := 0$, and on each $\mathcal{R}^n(\mathcal{F})$, for $n \geq 2$, by the formula:

$$\sigma(m_1 \cdots m_n) = \sum_{i=1}^{n-1} m_n m_{n-1} \cdots m_{n-i+2} m_{n-i+1} m_1 m_2 \cdots m_{n-i-1} m_{n-i}$$

for all local sections $m_1, \ldots, m_n$ of $\mathcal{F}$. The reader may check that σ is actually well-defined on $\mathcal{R}_+(\mathcal{F})$, and not only on $\mathcal{T}_+(\mathcal{F})$.

Let $\sigma^\star := 1 + \sigma$. Notice that $\sigma^\star$ factors through $\mathcal{S}_+(\mathcal{F})$. For each integer $n \geq 1$, let $\mathcal{K}^n(\mathcal{F}) := \sigma^\star(\mathcal{R}^n(\mathcal{F}))$. Then $\mathcal{K}^n(\mathcal{F})$ is also the kernel of $n - \sigma^\star$. In particular, $\mathcal{K}^2(\mathcal{F})$ is the kernel of the surjection $\mathcal{T}^2(\mathcal{F}) \to \mathcal{A}^2(\mathcal{F})$. Indeed, that $\mathcal{K}^n(\mathcal{F})$ is in the kernel of $n - \sigma^\star$ follows from the equality

$$\sigma^\star \sigma^\star|_{\mathcal{R}^n(\mathcal{F})} = n\sigma^\star|_{\mathcal{R}^n(\mathcal{F})},$$

a fact checked locally. And if ω is a local section of $\mathcal{R}^n(\mathcal{F})$ such that $(n - \sigma^\star)(\omega) = 0$, then $\omega = \sigma^\star((1/n)\omega)$, and thus ω is a local section of $\mathcal{K}^n(\mathcal{F})$.

Put

$$\mathcal{K}_+(\mathcal{F}) := \prod_{n=1}^{\infty} \mathcal{K}^n(\mathcal{F}).$$

DEFINITION 1. *A* Hasse derivation of $\mathcal{F}$ *is a map of O_B-algebras*

$$h = (h_0, h_1, \ldots) \colon O_X \longrightarrow \prod_{i=0}^{\infty} \mathcal{S}^i(\mathcal{F})$$

with $h_0 = id_{O_X}$. We say that h extends *D if $h_1 = D$.*

If $h = (h_0, h_1, \ldots)$ is a Hasse derivation of $\mathcal{F}$, then $h_1 \colon O_X \to \mathcal{F}$ is an O_B-derivation of $\mathcal{F}$. Conversely, given D, we may ask when there is a Hasse derivation $h = (h_0, h_1, \ldots)$ of $\mathcal{F}$ extending D. We will see in Construction 3 that such h exists when there is a D-connection.

DEFINITION 2. *A D*-connection *is a map of O_B-modules*

$$\gamma \colon \mathcal{F} \to \mathcal{T}^2(\mathcal{F})$$

satisfying

$$\gamma(am) = D(a)m + a\gamma(m) \tag{1}$$

for each local sections a of O_X and m of $\mathcal{F}$.

EXAMPLE 1. There may not exist a D-connection. For instance, assume X is the union of two transversal lines in the plane, or $X = \mathrm{Spec}(\mathbf{C}[x,y]/(xy))$. Assume $\mathcal{F} = \Omega^1_X$, the sheaf of differentials, and $D \colon O_X \to \Omega^1_X$ is the universal $\mathbf{C}$-derivation. The sheaf Ω^1_X is generated by $D(x)$ and $D(y)$, and the sheaf of relations is generated by the single relation $yD(x) + xD(y) = 0$. In particular, $D(x)$ and $D(y)$ are $\mathbf{C}$-linearly independent at the node. Suppose there were a D-connection $\gamma \colon \Omega^1_X \to \mathcal{T}^2(\Omega^1_X)$. Then

$$0 = \gamma(yD(x) + xD(y)) = D(y)D(x) + D(x)D(y) + y\gamma(D(x)) + x\gamma(D(y)).$$

However, $D(x)D(x)$, $D(x)D(y)$, $D(y)D(x)$ and $D(y)D(y)$ are linearly independent sections of $\mathcal{T}^2(\Omega^1_X)$ at the node. Hence the above relation is not possible.

When a D-connection γ exists, it is not unique, since for every O_X-linear map $\lambda \colon \mathcal{F} \to \mathcal{T}^2(\mathcal{F})$, the sum $\gamma + \lambda$ is a D-connection. However, these are all the D-connections.

A D-connection allows us to iterate D to a Hasse derivation, as we will explain in Construction 3. First, we will see how to extend a connection.

CONSTRUCTION 2. (*Extending connections*) Let $\gamma \colon \mathcal{F} \to \mathcal{T}^2(\mathcal{F})$ be a D-connection. Define a homogeneous map of degree 1 of O_B-modules,

$$\nabla \colon \mathcal{R}_+(\mathcal{F}) \to \mathcal{R}_+(\mathcal{F}), \tag{1}$$

given on each graded part $\mathcal{R}^n(\mathcal{F})$ by

$$\nabla(m_1\cdots m_n) := \sum_{i=1}^{n} m_1\cdots m_{i-1}\gamma(m_i)m_{i+1}\cdots m_n \tag{2}$$

for each local sections $m_1,\ldots,m_n$ of $\mathcal{F}$.

At first, it seems ∇ would be a well-defined map from $\mathcal{T}_+(\mathcal{F})$ to $\mathcal{T}_+(\mathcal{F})$. This would indeed be true, were γ a map of $\mathcal{O}_X$-modules. But γ is not! To check that ∇, as given above, is well-defined, we need to check the following three properties for all local sections $m_1,\ldots,m_i,m_i',\ldots,m_n'$ of $\mathcal{F}$ and a of $\mathcal{O}_X$, and each permutation τ of $\{1,\ldots,n-1\}$:

1. For each $i=1,\ldots,n$,
$$\nabla(m_1\cdots(m_i+m_i')\cdots m_n) = \nabla(m_1\cdots m_i\cdots m_n)+\nabla(m_1\cdots m_i'\cdots m_n).$$
2. For each $i,j=1,\ldots,n$,
$$\nabla(m_1\cdots(am_i)\cdots m_n) = \nabla(m_1\cdots(am_j)\cdots m_n).$$
3. $\nabla(m_1\cdots m_{n-1}m_n) = \nabla(m_{\tau(1)}\cdots m_{\tau(n-1)}m_n)$.

The first (multilinearity) and third (symmetry) properties are left for the reader to check. The second property is the key to why ∇ must take values in $\mathcal{R}_+(\mathcal{F})$, so let us check it: from the definition of ∇, and using $\gamma(am_i) = D(a)m_i + a\gamma(m_i)$, we get

$$\nabla(m_1\cdots(am_i)\cdots m_n) = m_1\cdots m_{i-1}D(a)m_i\cdots m_n + a\sum_{j=1}^{n} m_1\cdots\gamma(m_j)\cdots m_n.$$

So $\nabla(m_1\cdots(am_i)\cdots m_n)$ would depend on i, were ∇ to take values in $\mathcal{T}_+(\mathcal{F})$. But instead, ∇ takes values in $\mathcal{R}_+(\mathcal{F})$, and hence

$$m_1\cdots m_{i-1}D(a)m_i\cdots m_n = D(a)m_1\cdots m_{i-1}m_i\cdots m_n.$$

So the second property (scalar multiplication) is checked, and the three properties imply that ∇ is well-defined. Also, we proved the formula

$$\nabla(a\omega) = D(a)\omega + a\nabla(\omega)$$

for all local sections ω of $\mathcal{R}_+(\mathcal{F})$ and a of $\mathcal{O}_X$.

Now, for each integer $n\geq 1$, put

$$T_n := \frac{1}{n!}\nabla^n|_{\mathcal{F}} : \mathcal{F}\to\mathcal{R}^{n+1}(\mathcal{F}).$$

Also, set $T_0 := \mathrm{id}_{\mathcal{F}}$. Then $T_n = (1/n)\nabla T_{n-1}$ for each $n\geq 1$. Furthermore, for each integer $i\geq 1$, and each local sections a of $\mathcal{O}_X$ and m of $\mathcal{F}$,

$$T_i(am) = aT_i(m) + \sum_{j=1}^{i}\frac{1}{j}T_{j-1}D(a)T_{i-j}(m). \tag{3}$$

Indeed, Formula (3) holds for $i=1$, because $T_1=\gamma$ and γ is a D-connection. And if, by induction, Formula (3) holds for a certain $i\geq 1$, then

$$
\begin{aligned}
T_{i+1}(am) =& \frac{1}{(i+1)}\nabla T_i(am)\\
=& \frac{1}{(i+1)}\nabla\Big(aT_i(m)+\sum_{j=1}^{i}\frac{1}{j}T_{j-1}D(a)T_{i-j}(m)\Big)\\
=& \frac{1}{(i+1)}\Big(a\nabla T_i(m)+D(a)T_i(m)\Big)\\
&+\frac{1}{(i+1)}\sum_{j=1}^{i}\frac{1}{j}\Big(\nabla T_{j-1}D(a)T_{i-j}(m)+T_{j-1}D(a)\nabla T_{i-j}(m)\Big)\\
=& aT_{i+1}(m)+\frac{1}{(i+1)}D(a)T_i(m)\\
&+\frac{1}{(i+1)}\sum_{j=1}^{i}\Big(T_jD(a)T_{i-j}(m)+\frac{(i+1-j)}{j}T_{j-1}D(a)T_{i+1-j}(m)\Big)\\
=& aT_{i+1}(m)+\frac{1}{(i+1)}D(a)T_i(m)\\
&+\sum_{j=2}^{i}\Big(\frac{1}{j}T_{j-1}D(a)T_{i+1-j}(m)\Big)+\frac{1}{(i+1)}T_iD(a)m+\frac{i}{(i+1)}D(a)T_i(m)\\
=& aT_{i+1}(m)+\sum_{j=1}^{i+1}\frac{1}{j}T_{j-1}D(a)T_{i+1-j}(m).
\end{aligned}
$$

The maps T_n form an extended D-connection, according to the definition below.

DEFINITION 3. *Let n be a positive integer or $n:=\infty$. A map of $\mathcal{O}_B$-modules*

$$T=(T_0,T_1,T_2,\dots)\colon \mathcal{F}\longrightarrow\prod_{i=0}^{n}\mathcal{R}^{i+1}(\mathcal{F})$$

is called an extended D-connection *if $T_0=id_{\mathcal{F}}$ and Formula* (3) *holds for each $i\geq 1$ and all local sections a of $\mathcal{O}_X$ and m of $\mathcal{F}$.*

If nothing is noted otherwise, extended D-connections are assumed *full*, that is, with $n:=\infty$.

CONSTRUCTION 3. (*Hasse derivations extending D*) Let $T\colon \mathcal{F}\to\mathcal{R}_+(\mathcal{F})$ be an extended D-connection. Notice that the map $T_1\colon \mathcal{F}\to\mathcal{F}\otimes\mathcal{F}$ is a D-connection. However, T need not arise from T_1 as in Construction 2.

Put $h_0:=\mathrm{id}_{\mathcal{O}_X}$, and for each integer $i\geq 1$ let $h_i\colon \mathcal{O}_X\to\mathcal{S}^i(\mathcal{F})$ be the $\mathcal{O}_B$-linear map given by $h_i(a):=(1/i)T_{i-1}D(a)$ for each local section a of $\mathcal{O}_X$. Then

$$h:=(h_0,h_1,h_2,\dots)\colon \mathcal{O}_X\longrightarrow\prod_{i=0}^{\infty}\mathcal{S}^i(\mathcal{F})$$

is a Hasse derivation of $\mathcal{F}$ extending D. Indeed, clearly $h_1 = D$. Now, if a and b are local sections of $\mathcal{O}_X$, and $i \geq 1$, then

$$\begin{aligned}
h_i(ab) =&\frac{1}{i}T_{i-1}\Big(aD(b)+bD(a)\Big)\\
=&\frac{1}{i}\Big(aT_{i-1}D(b)+\sum_{j=1}^{i-1}\frac{1}{j}T_{j-1}D(a)T_{i-1-j}D(b)\Big)\\
+&\frac{1}{i}\Big(bT_{i-1}D(a)+\sum_{j=1}^{i-1}\frac{1}{j}T_{j-1}D(b)T_{i-1-j}D(a)\Big)\\
=&ah_i(b)+bh_i(a)\\
+&\frac{1}{i}\sum_{j=1}^{i-1}\Big(\frac{1}{j}T_{j-1}D(a)T_{i-1-j}D(b)+\frac{1}{i-j}T_{i-1-j}D(b)T_{j-1}D(a)\Big)\\
=&ah_i(b)+bh_i(a)+\sum_{j=1}^{i-1}\Big(\frac{1}{j(i-j)}T_{j-1}D(a)T_{i-1-j}D(b)\Big)\\
=&\sum_{j=0}^{i}h_j(a)h_{i-j}(b),
\end{aligned}$$

where in the fourth equality we used that the computation is done in $\mathcal{S}^i(\mathcal{F})$.

DEFINITION 4. *Let h be a Hasse derivation extending D. We say that h is* iterated *if there is an extended D-connection T such that $h_i = (1/i)T_{i-1}D$ for each $i \geq 1$.*

3. Flat connections and integrable derivations

Recall the notation for X, $\mathcal{O}_B$, $\mathcal{O}_X$, $\mathcal{F}$ and D.

DEFINITION 5. *A D-connection $\gamma\colon \mathcal{F} \to \mathcal{T}^2(\mathcal{F})$ is called* flat *if $\gamma D(\mathcal{O}_X) \subseteq \mathcal{K}^2(\mathcal{F})$. We say that D is* integrable *if there exists a flat D-connection.*

EXAMPLE 2. Assume that $\mathcal{F}$ is the free sheaf of $\mathcal{O}_X$-modules with basis $e_1,\dots,e_n$. Then $D = D_1e_1 + \cdots + D_ne_n$, where the D_i are $\mathcal{O}_B$-derivations of $\mathcal{O}_X$. Conversely, a n-tuple $(D_1,\dots,D_n)$ of $\mathcal{O}_B$-derivations of $\mathcal{O}_X$ defines an $\mathcal{O}_B$-derivation of $\mathcal{F}$.

There is a natural D-connection $\gamma\colon \mathcal{F} \to \mathcal{T}^2(\mathcal{F})$, satisfying

$$\gamma(\sum_{i=1}^{n}a_ie_i) = \sum_{i=1}^{n}\sum_{j=1}^{n}D_j(a_i)e_je_i$$

for all local sections $a_1,\dots,a_n$ of $\mathcal{O}_X$. Any other D-connection is of the form $\gamma+\nu$, where $\nu\colon \mathcal{F} \to \mathcal{T}^2(\mathcal{F})$ is a map of $\mathcal{O}_X$-modules. The map ν is defined by global sections $c_\ell^{j,i}$ of $\mathcal{O}_X$, for $1 \leq i, j, \ell \leq n$, satisfying

$$\nu(e_\ell) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_\ell^{j,i} e_j e_i.$$

To say that $(\gamma + \nu)D(a) = 0$ in $\mathcal{A}^2(\mathcal{F})$ for a local section a of $\mathcal{O}_X$ is to say that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} D_j D_i(a) e_j e_i + \sum_{\ell=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} c_\ell^{j,i} D_\ell(a) e_j e_i = 0$$

in $\mathcal{A}^2(\mathcal{F})$ or, equivalently,

$$D_j D_i(a) - D_i D_j(a) = \sum_{\ell=1}^{n} (c_\ell^{i,j} - c_\ell^{j,i}) D_\ell(a)$$

for all distinct i and j. In other words, D is integrable if and only if the collection $\{D_1, \dots, D_n\}$ is *involutive*, i.e., if and only if there are sections $b_\ell^{j,i}$ of $\mathcal{O}_X$ such that

$$[D_j, D_i] = \sum_{\ell=1}^{n} b_\ell^{j,i} D_\ell$$

for all distinct i and j.

Let T be the extended D-connection of Construction 2, derived from γ. From the definition of γ we have $T_q(e_i) = 0$ for each integer $q > 0$ and each $i = 1, \dots, n$. Suppose $\gamma D = 0$ in $\mathcal{A}^2(\mathcal{F})$, or in other words $[D_j, D_i] = 0$ for all i and j. Then the Hasse derivation h associated to T satisfies

$$h_q(a) = \sum_{j_1=1}^{n} \cdots \sum_{j_q=1}^{n} \frac{D_{j_1} \cdots D_{j_q}(a)}{q!} e_{j_1} \cdots e_{j_q} \tag{1}$$

for each integer $q > 0$ and each local section a of $\mathcal{F}$.

EXAMPLE 3. It is not necessary that $\mathcal{F}$ be locally free for a flat D-connection to exist. For instance, assume $X = \mathrm{Spec}(\mathbf{C}[t^3, t^4, t^5])$. Viewed as a sheaf of regular meromorphic differentials, Rosenlicht-style, the dualizing sheaf ω_X is generated by dt/t^2 and dt/t^3. Assume $\mathcal{F} = \omega_X$. Let $\eta_1 := dt/t^2$ and $\eta_2 := dt/t^3$. The following relations generate all relations η_1 and η_2 satisfy with coefficients in $\mathcal{O}_X$:

$$t^3\eta_1 = t^4\eta_2, \quad t^4\eta_1 = t^5\eta_2, \quad t^5\eta_1 = t^6\eta_2. \tag{1}$$

Assume $D \colon \mathcal{O}_X \to \omega_X$ is the composition of the universal derivation with the canonical map $\Omega_X^1 \to \omega_X$. Then D satisfies

$$D(t^3) = 3t^4\eta_1 = 3t^5\eta_2, \quad D(t^4) = 4t^5\eta_1 = 4t^6\eta_2, \quad D(t^5) = 5t^6\eta_1 = 5t^7\eta_2.$$

So $D(a)\eta_i = 0$ in $\mathcal{A}^2(\omega_X)$ for each local section a of $\mathcal{O}_X$ and each $i = 1, 2$.

Define $\gamma\colon \omega_X \to \mathcal{T}^2(\omega_X)$ by letting

$$\gamma(a\eta_1 + b\eta_2) = D(a)\eta_1 + D(b)\eta_2 + 4t^3 a\eta_2\eta_2 + 3b\eta_1\eta_1$$

for all local sections a and b of $\mathcal{O}_X$. To check that γ is well defined we need only check that the values of γ on both sides of the three relations (1) agree. This is the case; for instance,

$$\gamma(t^3\eta_1) = 3t^4\eta_1\eta_1 + 4t^6\eta_2\eta_2 = 7t^4\eta_1\eta_1 = 4t^5\eta_1\eta_2 + 3t^4\eta_1\eta_1 = \gamma(t^4\eta_2).$$

Now, since $D(a)\eta_1 = D(b)\eta_2 = 0$ in $\mathcal{A}^2(\omega_X)$, we have that $\gamma = 0$ in $\mathcal{A}^2(\omega_X)$. So γ is a flat D-connection, and hence D is integrable.

DEFINITION 6. *An extended D-connection $T\colon \mathcal{F} \to \mathcal{R}_+(\mathcal{F})$ is said to be* flat *if $TD(\mathcal{O}_X) \subseteq \mathcal{K}_+(\mathcal{F})$.*

We will see in Theorem 5 that flat extended D-connections exist, when D is integrable. Also, by Proposition 1 below, any two of them are "comparable". First, a piece of notation.

CONSTRUCTION 4. (*Generating maps*) Let n be a positive integer or $n := \infty$. Let

$$\lambda = (\lambda_0, \lambda_1, \dots)\colon \mathcal{F} \longrightarrow \prod_{i=0}^{n} \mathcal{R}^{i+1}(\mathcal{F})$$

be a map of $\mathcal{O}_X$-modules. For each integer $p > 0$ and each sequence $i_1, \dots, i_p$ of non-negative indices at most equal to n, let $q := i_1 + \cdots + i_p$ and define

$$(\lambda_{i_1} \cdots \lambda_{i_p})\colon \mathcal{T}^p(\mathcal{F}) \to \mathcal{R}^{p+q}(\mathcal{F})$$

to be the map of $\mathcal{O}_X$-modules satisfying

$$(\lambda_{i_1} \cdots \lambda_{i_p})(m_1 \cdots m_p) := \lambda_{i_1}(m_1) \cdots \lambda_{i_p}(m_p)$$

for all local sections $m_1, \dots, m_p$ of $\mathcal{F}$.

The maps $(\lambda_{i_1} \cdots \lambda_{i_p})$ are not defined on $\mathcal{R}^p(\mathcal{F})$, but the sum

$$s_q(\lambda) := \sum_{i_1+\cdots+i_p=q} (i_p + 1)\big(\lambda_{i_1} \cdots \lambda_{i_p}\big)\colon \mathcal{R}^p(\mathcal{F}) \longrightarrow \mathcal{R}^{p+q}(\mathcal{F})$$

is, for all integers $p > 0$ and each integer q with $0 \leq q \leq n$. Analogously, the sum

$$\widetilde{s}_q(\lambda) := \sum_{i_1+\cdots+i_p=q} \big(\lambda_{i_1} \cdots \lambda_{i_p}\big)\colon \mathcal{S}^p(\mathcal{F}) \longrightarrow \mathcal{S}^{p+q}(\mathcal{F})$$

is well-defined, for all integers $p > 0$ and each integer q with $0 \leq q \leq n$.

Notice that, for each local section ω of $\mathcal{K}^p(\mathcal{F})$,

$$s_q(\lambda)(\omega) = \frac{p+q}{p}\widetilde{s}_q(\lambda)(\omega) \text{ in } \mathcal{S}^{p+q}(\mathcal{F}). \tag{1}$$

Indeed, locally,

$$\omega=\sum_{i=1}^{p} m_1\cdots\widehat{m_i}\cdots m_p m_i$$

for local sections $m_1,\dots,m_p$ of $\mathcal{F}$. Thus, in $\mathcal{S}^{p+q}(\mathcal{F})$,

$$\begin{aligned}
s_q(\lambda)(\omega)=&\sum_{i=1}^{p}\sum_{j_1+\cdots+j_p=q}(j_p+1)\prod_{s=1}^{i-1}\lambda_{j_s}(m_s)\prod_{s=i}^{p-1}\lambda_{j_s}(m_{s+1})\lambda_{j_p}(m_i)\\
=&\sum_{j_1+\cdots+j_p=q}\sum_{i=1}^{p}(j_i+1)\prod_{s=1}^{p}\lambda_{j_s}(m_s)\\
=&(p+q)\sum_{j_1+\cdots+j_p=q}\prod_{s=1}^{p}\lambda_{j_s}(m_s)\\
=&\frac{p+q}{p}\sum_{j_1+\cdots+j_p=q}\sum_{i=1}^{p}\prod_{s=1}^{i-1}\lambda_{j_s}(m_s)\prod_{s=i}^{p-1}\lambda_{j_s}(m_{s+1})\lambda_{j_p}(m_i)\\
=&\frac{p+q}{p}\sum_{j_1+\cdots+j_p=q}(\lambda_{j_1}\cdots\lambda_{j_p})(\omega)\\
=&\frac{p+q}{p}\widetilde{s}_q(\lambda)(\omega).
\end{aligned}$$

PROPOSITION 1. *Let n be a positive integer. Let*

$$T=(T_0,T_1,\dots,T_n)\colon\mathcal{F}\longrightarrow\prod_{i=0}^{n}\mathcal{R}^{i+1}(\mathcal{F}),$$
$$S=(S_0,S_1,\dots,S_n)\colon\mathcal{F}\longrightarrow\prod_{i=0}^{n}\mathcal{R}^{i+1}(\mathcal{F})$$

be two O_B-linear maps. Assume that $T_iD(O_X)\subseteq\mathcal{K}^{i+1}(\mathcal{F})$ for each $i=0,\dots,n-1$. Then any two of the following three statements imply the third:

1. *The map S is an extended D-connection.*
2. *The map T is an extended D-connection.*
3. *There is a (unique) map of O_X-modules*

$$\lambda=(\lambda_0,\dots,\lambda_n)\colon\mathcal{F}\longrightarrow\prod_{i=0}^{n}\mathcal{R}^{i+1}(\mathcal{F})$$

such that $\lambda_0=\mathrm{id}_{\mathcal{F}}$ and

$$S_i=\sum_{\ell=0}^{i}s_{i-\ell}(\lambda)T_\ell \tag{1}$$

for each $i=0,1,\dots,n$.

Proof. We will argue by induction. For $n = 1$, the proposition simply says that, given a D-connection T_1, a map of $\mathcal{O}_B$-modules $S_1 \colon \mathcal{F} \to \mathcal{R}^2(\mathcal{F})$ is a D-connection if and only if $S_1 - T_1$ is $\mathcal{O}_X$-linear, a fact already observed.

Suppose now that $n \geq 2$, and that the statement of the proposition is known for $n-1$ in place of n. So we may assume that $(S_0, \ldots, S_{n-1})$ and $(T_0, \ldots, T_{n-1})$ are extended D-connections, and that there exists a map of $\mathcal{O}_X$-modules

$$\lambda = (\lambda_0, \ldots, \lambda_{n-1}) \colon \mathcal{F} \longrightarrow \prod_{i=0}^{n-1} \mathcal{R}^{i+1}(\mathcal{F})$$

such that $\lambda_0 = \mathrm{id}_{\mathcal{F}}$ and Equations (1) hold for each $i < n$.

Define

$$R_n := \sum_{\ell=1}^{n-1} s_{n-\ell}(\lambda) T_\ell.$$

We need only show that, for each local sections a of $\mathcal{O}_X$ and m of $\mathcal{F}$,

$$R_n(am) - aR_n(m) + \sum_{z=0}^{n-1} \frac{T_{n-z-1}D(a)}{n-z} T_z(m) = \sum_{z=0}^{n-1} \frac{S_{n-z-1}D(a)}{n-z} S_z(m). \tag{2}$$

Indeed, if S and T are D-connections, Formula (2) implies that $S_n - R_n - T_n$ is $\mathcal{O}_X$-linear. So, setting $\lambda_n := (1/(n+1))(S_n - R_n - T_n)$, Equation (1) holds for $i = n$ as well. Conversely, if there is an $\mathcal{O}_X$-linear map $\lambda_n \colon \mathcal{F} \to \mathcal{R}^{n+1}(\mathcal{F})$ such that Equation (1) holds for $i = n$, then $(n+1)\lambda_n + R_n = S_n - T_n$. So, from Formula (2) we see that S is a D-connection if and only if T is.

Now, on the one hand, since $(T_0, \ldots, T_{n-1})$ is an extended D-connection,

$$\begin{aligned} R_n(am) - aR_n(m) &= \sum_{\ell=1}^{n-1} \sum_{j_0+\cdots+j_\ell=n-\ell} (j_\ell+1)\big(\lambda_{j_0}\cdots\lambda_{j_\ell}\big)\Big(T_\ell(am) - aT_\ell(m)\Big) \\ &= \sum_{\ell=1}^{n-1} \sum_{j_0+\cdots+j_\ell=n-\ell} \sum_{p=0}^{\ell-1} \frac{j_\ell+1}{\ell-p}\big(\lambda_{j_0}\cdots\lambda_{j_\ell}\big)\Big(T_{\ell-1-p}D(a)T_p(m)\Big). \end{aligned}$$

Thus the left-hand side of Formula (2) is equal to

$$\sum_{\ell=1}^{n} \sum_{j_0+\cdots+j_\ell=n-\ell} \sum_{p=0}^{\ell-1} \frac{j_\ell+1}{\ell-p}\big(\lambda_{j_0}\cdots\lambda_{j_\ell}\big)\Big(T_{\ell-1-p}D(a)T_p(m)\Big). \tag{3}$$

On the other hand, using Equations (1) for $i < n$, the right-hand side of (2) becomes

$$\sum_{z=0}^{n-1} \frac{1}{n-z}\left(\sum_{k=0}^{n-z-1} \sum_{j_0+\cdots+j_k=n-z-1-k} (j_k+1)\big(\lambda_{j_0}\cdots\lambda_{j_k}\big)T_kD(a)\right)\left(\sum_{p=0}^{z} \omega_{z-p}\right),$$

where

$$\omega_\ell := \sum_{j'_0+\cdots+j'_p=\ell} (j'_p+1)\big(\lambda_{j'_0}\cdots\lambda_{j'_p}\big)T_p(m)$$

for each $\ell = 0, \ldots, n-1$. Now, for each $z = 0, \ldots, n-1$ and $k = 0, \ldots, n-z-1$, using that $T_k D(a)$ is a local section of $\mathcal{K}^{k+1}(\mathcal{F})$, Formula (1) yields the following equation in $\mathcal{S}^{n-z}(\mathcal{F})$:

$$\sum (j_k+1)(\lambda_{j_0} \cdots \lambda_{j_k}) T_k D(a) = \sum \frac{n-z}{k+1} (\lambda_{j_0} \cdots \lambda_{j_k}) T_k D(a),$$

where the sum on both sides runs over the $(k+1)$-tuples $(j_0, \ldots, j_k)$ such that $j_0 + \cdots + j_k = n-z-1-k$. Thus, introducing $\ell := k+p+1$, the right-hand side of (2) becomes

$$\sum_{\ell=1}^{n} \sum_{p=0}^{\ell-1} \frac{1}{\ell - p} \sum_{z=p}^{n-\ell+p} \sum_{j_0+\cdots+j_{\ell-p-1}=n-z-\ell+p} (\lambda_{j_0} \cdots \lambda_{j_{\ell-p-1}}) T_{\ell-p-1} D(a) \omega_{z-p},$$

whence, introducing $u := z - p$, equal to

$$\sum_{\ell=1}^{n} \sum_{p=0}^{\ell-1} \frac{1}{\ell - p} \sum_{u=0}^{n-\ell} \sum_{j_0+\cdots+j_{\ell-p-1}=n-\ell-u} (\lambda_{j_0} \cdots \lambda_{j_{\ell-p-1}}) T_{\ell-p-1} D(a) \omega_u,$$

which is equal to (3). □

4. Jets

Recall the notation for X, $\mathcal{O}_B$, $\mathcal{O}_X$, $\mathcal{F}$ and D.

THEOREM 5. *If D is integrable, then there exists a flat, extended D-connection.*

Proof. Since D is integrable, there is a flat D-connection $T_1 \colon \mathcal{F} \to \mathcal{T}^2(\mathcal{F})$. Set $T_0 := \mathrm{id}_{\mathcal{F}}$. Suppose, by induction, that for an integer $n \geq 2$ we have constructed an extended D-connection

$$T = (T_0, T_1, \ldots, T_{n-1}) \colon \mathcal{F} \longrightarrow \prod_{i=0}^{n-1} \mathcal{R}^{i+1}(\mathcal{F}).$$

We will also suppose the maps T_i satisfy one additional property, Equations (2), after we make a definition.

For each $j = 0, \ldots, n-1$, define a map of $\mathcal{O}_B$-modules $T'_j \colon \mathcal{R}^2(\mathcal{F}) \to \mathcal{R}^{j+2}(\mathcal{F})$ by letting

$$T'_j(m_1 m_2) := \sum_{i=0}^{j} T_i(m_1) T_{j-i}(m_2)$$

for all local sections m_1 and m_2 of $\mathcal{F}$. To check that T'_j is well defined, we need only check that $\sum_i T_i(am_1) T_{j-i}(m_2) = \sum_i T_i(m_1) T_{j-i}(am_2)$ for each local section a of $\mathcal{O}_X$. In fact, using (3), we see that both sides are equal to

$$\sum_{i=0}^{j} a T_i(m_1) T_{j-i}(m_2) + \sum_{\ell=1}^{j} \sum_{i=0}^{j-\ell} \frac{1}{\ell} T_{\ell-1} D(a) T_i(m_1) T_{j-\ell-i}(m_2).$$

Furthermore, we see from this computation that

$$T'_j(a\omega) = aT'_j(\omega) + \sum_{i=1}^{j} \frac{1}{j+1-i} T_{j-i} D(a) T'_{i-1}(\omega)$$

for all local sections ω of $\mathcal{R}^2(\mathcal{F})$ and a of $\mathcal{O}_X$.

Also, notice that $\sigma^\star T'_{i-1}(1-\sigma) = 0$. Indeed, for local sections m_1 and m_2 of $\mathcal{F}$,

$$\begin{aligned}
\sigma^\star T'_{i-1}(1-\sigma)(m_1 m_2) =& \sigma^\star T'_{i-1}(m_1 m_2 - m_2 m_1) \\
=& \sigma^\star \Big(\sum_{j=0}^{i-1} \Big(T_j(m_1) T_{i-1-j}(m_2) - T_{i-1-j}(m_2) T_j(m_1) \Big) \Big) \\
=& \sum_{j=0}^{i-1} \Big(\sigma^\star \big(T_j(m_1) T_{i-1-j}(m_2) \big) - \sigma^\star \big(T_{i-1-j}(m_2) T_j(m_1) \big) \Big),
\end{aligned}$$

which is zero because $\sigma^\star(\omega_1\omega_2) = \sigma^\star(\omega_2\omega_1)$ for all local sections ω_1 and ω_2 of $\mathcal{R}_+(\mathcal{F})$. Then

$$T'_{i-1}(1-\sigma) = \frac{i-\sigma}{i+1} T'_{i-1}(1-\sigma). \tag{1}$$

Now, suppose that

$$(i-\sigma)\big((i+1)T_i - T'_{i-1}(1-\sigma)T_1\big) = 0 \tag{2}$$

for each $i = 1, \ldots, n-1$. (Notice that Equation (2) holds automatically for $i = 1$, because $(1-\sigma)\sigma^\star(\mathcal{R}^2(\mathcal{F})) = 0$.) Also, from Equation (2), and the flatness of T_1, we get $T_i D(\mathcal{O}_X) \subseteq \mathcal{K}^{i+1}(\mathcal{F})$ for each $i = 1, \ldots, n-1$.

Let

$$T := \frac{1}{n+1} T'_{n-1}(1-\sigma) T_1.$$

Then

$$(n-\sigma)\Big(T(am) - aT(m) - \sum_{i=1}^{n} \frac{1}{i} T_{i-1} D(a) T_{n-i}(m)\Big) = 0 \tag{3}$$

for all local sections m of $\mathcal{F}$ and a of $\mathcal{O}_X$. Indeed,

$$\begin{aligned}
(n-\sigma)T(am) =& \frac{n-\sigma}{n+1}T'_{n-1}(1-\sigma)T_1(am)\\
=& \frac{n-\sigma}{n+1}\Big(T'_{n-1}\Big(D(a)m-mD(a)+a(1-\sigma)T_1(m)\Big)\Big)\\
=& \frac{n-\sigma}{n+1}\Big(T_{n-1}D(a)m+\sum_{i=1}^{n-1}T_{n-1-i}D(a)T_i(m)\Big)\\
-& \frac{n-\sigma}{n+1}\Big(mT_{n-1}D(a)+\sum_{i=1}^{n-1}T_i(m)T_{n-1-i}D(a)\Big)\\
+& \frac{n-\sigma}{n+1}\Big(aT'_{n-1}(1-\sigma)T_1(m)\Big)\\
+& \frac{n-\sigma}{n+1}\Big(\sum_{i=1}^{n-1}T_{n-i-1}D(a)T'_{i-1}\frac{(1-\sigma)}{n-i}T_1(m)\Big).
\end{aligned}$$

Now, first observe that, for each $i=0,\dots,n-1$, we have $(n-i-\sigma^\star)T_{n-1-i}D(a)=0$, since $T_{n-1-i}D(a)$ is a local section of $\mathcal{K}^{n-i}(\mathcal{F})$. Then

$$\begin{aligned}
(n-\sigma)\Big(T_i(m)T_{n-1-i}D(a)\Big) =& (n-\sigma)\Big(T_i(m)\frac{\sigma^\star}{n-i}T_{n-1-i}D(a)\Big)\\
=& -\frac{n-\sigma}{n-i}\Big(T_{n-1-i}D(a)\sigma^\star T_i(m)\Big).
\end{aligned}$$

Also, from (1) and (2),

$$\begin{aligned}
T_{n-i-1}D(a)T'_{i-1}(1-\sigma)T_1(m) =& T_{n-i-1}D(a)\frac{i-\sigma}{i+1}T'_{i-1}(1-\sigma)T_1(m)\\
=& T_{n-i-1}D(a)(i-\sigma)T_i(m)
\end{aligned}$$

for each $i=1,\dots,n-1$. So

$$\frac{n-\sigma}{n+1}\Big(T_{n-1}D(a)m-mT_{n-1}D(a)\Big)=\frac{n-\sigma}{n}\Big(T_{n-1}D(a)m\Big),$$

and, for each $i=1,\dots,n-1$,

$$\frac{n-\sigma}{n+1}\Big(T_{n-1-i}D(a)T_i(m)-T_i(m)T_{n-1-i}D(a)+T_{n-i-1}D(a)T'_{i-1}\frac{1-\sigma}{n-i}T_1(m)\Big)$$

is equal to

$$\frac{n-\sigma}{n+1}\Big(T_{n-1-i}D(a)\big(1+\frac{1+\sigma}{n-i}+\frac{i-\sigma}{n-i}\big)T_i(m)\Big),$$

whence equal to

$$\frac{n-\sigma}{n-i}\Big(T_{n-1-i}D(a)T_i(m)\Big).$$

Applying these equalities in the above expression for $(n-\sigma)T(am)$ we get Equation (3).

We want to show that there exists a map of $\mathcal{O}_B$-modules $T_n\colon \mathcal{F}\to\mathcal{R}^{n+1}(\mathcal{F})$ such that (2) holds for $i=n$, and such that $(T_0,\dots,T_n)$ is an extended D-connection. First we claim that there exists a map of $\mathcal{O}_B$-modules $T_n\colon \mathcal{F}\to\mathcal{R}^{n+1}(\mathcal{F})$ such that $(T_0,\dots,T_n)$ is an extended D-connection. Indeed, from T_1 construct an extended D-connection $(S_0,\dots,S_n)$ by iteration, as described in Construction 2. By Proposition 1, there is a map of $\mathcal{O}_X$-modules

$$\lambda=(\lambda_0,\lambda_1,\dots,\lambda_{n-1})\colon \mathcal{F}\longrightarrow\prod_{i=0}^{n-1}\mathcal{R}^{i+1}(\mathcal{F})$$

such that $\lambda_0=\mathrm{id}_{\mathcal{F}}$ and such that Equations (1) hold for $i=0,\dots,n-1$. Now, just set $\lambda_n:=0$ in Equation (1) for $i=n$, and let it define T_n. Then, by Proposition 1, the map $(T_0,\dots,T_n)$ is an extended D-connection.

The above map T_n does not necessarily make (2) hold for $i=n$. So, rename it by U. At any rate, since $(T_0,\dots,T_{n-1},U)$ is an extended D-connection, it follows from Equation (3) that $(n-\sigma)(U-T)$ is $\mathcal{O}_X$-linear. Set

$$T_n:=U-\frac{(n-\sigma)}{n+1}(U-T).$$

Then T_n differs from U by an $\mathcal{O}_X$-linear map, and thus $(T_0,\dots,T_n)$ is an extended D-connection. Now,

$$(n-\sigma)T_n=(n-\sigma)U-\frac{(n-\sigma)^2}{n+1}(U-T).$$

Since

$$(n-\sigma)^2|_{\mathcal{R}^{n+1}(\mathcal{F})}=(n+1)(n-\sigma)|_{\mathcal{R}^{n+1}(\mathcal{F})},$$

we get $(n-\sigma)T_n=(n-\sigma)T$. So (2) holds for $i=n$.

The induction argument is complete, showing that there is an infinite extended D-connection

$$T=(T_0,T_1,\dots)\colon \mathcal{F}\longrightarrow\prod_{i=0}^{\infty}\mathcal{R}^{i+1}(\mathcal{F})$$

such that (2) holds for each $i\geq 1$, and thus $T_iD(\mathcal{O}_X)\subseteq\mathcal{K}^{i+1}(\mathcal{F})$ for each $i\geq 0$. $\square$

DEFINITION 7. *Two Hasse derivations h and h' of $\mathcal{F}$ are said to be* equivalent *if there is an $\mathcal{O}_X$-algebra automorphism ϕ of $\mathcal{S}(\mathcal{F})$ such that $\phi_0|_{\mathcal{F}}=id_{\mathcal{F}}$ and $h'=\phi h$. We say that h and h' are* canonically equivalent *when there is only one such automorphism.*

COROLLARY 1. *Let h and h' be iterated Hasse derivations of $\mathcal{F}$ extending D. If D is integrable, then h and h' are equivalent. Furthermore, if $\nu D(\mathcal{O}_X)\neq 0$ for every nonzero $\mathcal{O}_X$-linear map $\nu\colon \mathcal{F}\to\mathcal{S}(\mathcal{F})$, then h and h' are canonically equivalent.*

Proof. First, we prove the existence of an equivalence. By Theorem 5, there is a flat, extended D-connection $T = (T_0, T_1, \dots)$. We may suppose h arises from T. Let $S = (S_0, S_1, \dots)$ be an extended D-connection from which h' arises.

By Proposition 1, there is a map of $\mathcal{O}_X$-modules

$$\lambda = (\lambda_0, \lambda_1, \dots)\colon \mathcal{F} \longrightarrow \prod_{i=0}^{\infty} \mathcal{R}^{i+1}(\mathcal{F})$$

such that $\lambda_0 = \mathrm{id}_{\mathcal{F}}$ and

$$S_i = \sum_{\ell=0}^{i} s_{i-\ell}(\lambda) T_\ell \quad \text{for each } i \geq 0. \tag{1}$$

Let $\phi\colon \mathcal{S}(\mathcal{F}) \to \mathcal{S}(\mathcal{F})$ be the map of $\mathcal{O}_X$-algebras whose graded part ϕ_q of degree q satisfies $\phi_q|_{\mathcal{S}^p(\mathcal{F})} = \widetilde{s}_q(\lambda)|_{\mathcal{S}^p(\mathcal{F})}$ for all integers $p > 0$ if $q \geq 0$, and $\phi_q = 0$ if $q < 0$. Since $\lambda_0 = \mathrm{id}_{\mathcal{F}}$, the homogeneous degree-0 part ϕ_0 is the identity, and thus ϕ is an automorphism. We claim that $h' = \phi h$.

Indeed, clearly $h'_0 = (\phi h)_0$. Now, since T is flat, $T_\ell D(\mathcal{O}_X) \subseteq \mathcal{K}^{\ell+1}(\mathcal{F})$ for each $\ell \geq 0$. Thus, for each $i \geq 0$ and each local section a of $\mathcal{O}_X$, using Equations (1) and (1), the following equalities hold on $\mathcal{S}^{i+1}(\mathcal{F})$:

$$h'_{i+1}(a) = \frac{S_i D(a)}{i+1} = \sum_{\ell=0}^{i} s_{i-\ell}(\lambda) \frac{T_\ell D(a)}{i+1} = \sum_{\ell=0}^{i} \widetilde{s}_{i-\ell}(\lambda) \frac{T_\ell D(a)}{\ell+1} = \sum_{\ell=0}^{i} \phi_{i-\ell} h_{\ell+1}(a).$$

Since $\phi_{i+1}|_{\mathcal{O}_X} = 0$, we have $h'_{i+1} = (\phi h)_{i+1}$. So $h' = \phi h$.

Now, assume that $\nu D(\mathcal{O}_X) \neq 0$ for every nonzero $\mathcal{O}_X$-linear map $\nu\colon \mathcal{F} \to \mathcal{S}(\mathcal{F})$. Let ϕ be an $\mathcal{O}_X$-algebra automorphism of $\mathcal{S}(\mathcal{F})$ such that $\phi_0|_{\mathcal{F}} = \mathrm{id}_{\mathcal{F}}$ and $h' = \phi h$. To see that ϕ is unique, we just need to show that $\phi_q|_{\mathcal{F}}$ is uniquely defined for each $q \geq 0$. We do it by induction. Since $h' = \phi h$, for each $q \geq 0$ the following equality holds:

$$h'_{q+1} = \phi_q h_1 + \phi_{q-1} h_2 + \cdots + \phi_1 h_q + h_{q+1}.$$

Then $\phi_q D$ is determined by $\phi_1, \dots, \phi_{q-1}$. Since ϕ_q is $\mathcal{O}_X$-linear and takes values in $\mathcal{S}^{q+1}(\mathcal{F})$, it follows from our extra assumption that ϕ_q is determined. □

If $(X, \mathcal{O}_X)$ is a Noetherian scheme over a Noetherian $\mathbf{Q}$-scheme $(B, \mathcal{O}_B)$, if $\mathcal{F}$ is a locally free sheaf on X, and if $D\colon \mathcal{O}_X \to \mathcal{F}$ is an $\mathcal{O}_B$-derivation such that $\mathcal{F}$ is generated by $D(\mathcal{O}_X)$ at the associated points of X, then $\nu D(\mathcal{O}_X) \neq 0$ for all nonzero $\mathcal{O}_X$-linear maps $\nu\colon \mathcal{F} \to \mathcal{S}(\mathcal{F})$.

CONSTRUCTION 6. (*Jets*) Assume that $D\colon \mathcal{O}_X \to \mathcal{F}$ is locally integrable, and that the sheaf of $\mathcal{O}_X$-linear maps from $\mathcal{F}$ to $\mathcal{S}(\mathcal{F})$ sending $D(\mathcal{O}_X)$ to zero has only trivial local sections. Let $\mathcal{U}$ be the collection of open subspaces $U \subseteq X$ such that $D|_U$ is integrable. For each $U \in \mathcal{U}$, there exist iterated Hasse derivations extending $D|_U$. Let $\mathcal{C}_U$ be the collection of these Hasse derivations. By Corollary 1, for any two $h, h' \in \mathcal{C}_U$ there is a unique $\mathcal{O}_U$-algebra automorphism $\phi_{h,h'}$ of $\mathcal{S}(\mathcal{F})|_U$ such that $h' = \phi_{h,h'} h$. Now, consider the collection of all the $h \in \mathcal{C}_U$ for all $U \in \mathcal{U}$. Consider also the collection of all the $\phi_{h,h'}$ for all $U \in \mathcal{U}$ and all $h, h' \in \mathcal{C}_U$. If $h, h', h'' \in \mathcal{C}_U$, then $h'' = \phi_{h',h''}\phi_{h,h'}h$. From the uniqueness of $\phi_{h,h''}$, we get $\phi_{h',h''}\phi_{h,h'} = \phi_{h,h''}$. The cocycle condition being satisfied, the $\phi_{h,h'}$ patch the $\mathcal{S}(\mathcal{F}|_U)$ to an $\mathcal{O}_X$-algebra $\mathcal{J}$, and the h patch to a map of $\mathcal{O}_B$-algebras $\tau\colon \mathcal{O}_X \to \mathcal{J}$. Since the $\phi_{h,h'}$ do not decrease degrees, for each integer $n \geq 0$ the truncated sheaves $\prod_{i=0}^{n} \mathcal{S}^i(\mathcal{F}|_U)$ patch to an $\mathcal{O}_X$-algebra quotient $\mathcal{J}^n$ of $\mathcal{J}$. Also, since the $(\phi_{h,h'})_0$ are the identity maps, there is a natural map of $\mathcal{O}_X$-modules $\mathcal{S}^n(\mathcal{F}) \to \mathcal{J}^n$. This map is an isomorphism for $n = 0$. Also, for each integer $n > 0$, the $\mathcal{O}_X$-algebra $\mathcal{J}^{n-1}$ is a subquotient of $\mathcal{J}^n$, and there is a natural exact sequence,

$$0 \to \mathcal{S}^n(\mathcal{F}) \to \mathcal{J}^n \to \mathcal{J}^{n-1} \to 0.$$

We say that $\mathcal{J}$ is the *sheaf of jets of D*, and that $\tau\colon \mathcal{O}_X \to \mathcal{J}$ is its *Hasse derivation*. For each integer $n \geq 0$, we say that $\mathcal{J}^n$ is the *sheaf of n-jets of D*, and that the induced $\tau_n\colon \mathcal{O}_X \to \mathcal{J}^n$ is the *n-th order truncated Hasse derivation*.

EXAMPLE 4. Let M be a complex manifold of complex dimension m, and $\mathcal{L}$ a foliation of dimension n of M. Let $w\colon T_M^* \to E$ be the surjection associated to $\mathcal{L}$. Then w induces a derivation $D\colon \mathcal{O}_M \to \mathcal{F}$, where $\mathcal{F}$ is the sheaf of holomorphic sections of E. The Frobenius conditions imply that D is integrable. Also, since w is surjective, $\mathcal{F}$ is generated by $D(\mathcal{O}_M)$. Thus, applying Construction 6, we have an associated sheaf of q-jets $\mathcal{J}^q$ on M for each integer $q \geq 0$. Also we may consider the bundle of q-jets $J_{\mathcal{L}}^q$ of $\mathcal{L}$. Then $\mathcal{J}^q$ is the sheaf of holomorphic sections of $J_{\mathcal{L}}^q$.

Indeed, for each point p of M, there exist a neighborhood X of p in M, and an open embedding of X in $\mathbf{C}^n \times \mathbf{C}^{m-n}$ whose composition $\varphi_X\colon X \to \mathbf{C}^{m-n}$ with the second projection defines $\mathcal{L}$ on X. The sheaf $\mathcal{F}|_X$ is the pullback of the sheaf of 1-forms on $\mathbf{C}^n$, and the canonical vector fields on $\mathbf{C}^n$ yield a basis $e_1, \ldots, e_n$ of $\mathcal{F}|_X$ such that $D = D_1 e_1 + \cdots + D_n e_n$, where the D_i are the pullbacks of these vector fields. Since they commute, so do the D_i.

Using the D-connection γ given in Example 2, and the associated extended D-connection T given in Construction 2, we obtain an iterated Hasse derivation h_X on X extending $D|_X$, and given by Formula (1) for each integer $q > 0$ and each local section a of $\mathcal{F}$. Then the truncation in order q of h_X has exactly the same form of the canonical Hasse derivation of the sheaf of sections of the bundle of relative q-jets $J_{\varphi_X}^q$. So, patching the h_X is compatible with patching the $J_{\varphi_X}^q$. The patching of the latter yields the bundle of jets $J_{\mathcal{L}}^q$, and hence we get that $\mathcal{J}^q$ is the sheaf of sections of $J_{\mathcal{L}}^q$.

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AMS Mathematics Subject Classification: 14F10, 13N15, 37F75.

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Lavoro pervenuto in redazione il 02.07.2013.