Rend. Sem. Mat. Univ. Politec. Torino
Vol. 72, 1–2 (2014), 3 – 16

S. Dipierro*

ASYMPTOTICS OF FRACTIONAL PERIMETER FUNCTIONALS AND RELATED PROBLEMS

Abstract. In this paper we review some recent results concerning the asymptotics of a fractional perimeter and the regularity of the corresponding minimizers. We also provide an elementary example of set with infinite s-perimeter.

1. Introduction

In [4] the notion of nonlocal perimeter was introduced. Namely, given $s \in (0,1/2)$ and a bounded open set $\Omega \subset \mathbb{R}^n$ with $C^{1,\gamma}$-boundary, for some $\gamma \in (0,1)$, the s-perimeter of a (measurable) set $E \subseteq \mathbb{R}^n$ in Ω is defined as

$$
\begin{aligned}
\operatorname{Per}_s(E;\Omega) :=\, & \mathscr{L}(E\cap\Omega,(\mathscr{C}E)\cap\Omega) \\
& + \mathscr{L}(E\cap\Omega,(\mathscr{C}E)\cap(\mathscr{C}\Omega)) + \mathscr{L}(E\cap(\mathscr{C}\Omega),(\mathscr{C}E)\cap\Omega),
\end{aligned} \tag{1}
$$

where $\mathscr{C}E = \mathbb{R}^n \setminus E$ denotes the complement of E. The interaction $\mathscr{L}$ considered in [4] is the following

$$
\mathscr{L}(A,B) := \int_A \int_B \frac{dx\,dy}{|x-y|^{n+2s}}
$$

for any measurable sets A and B.

We mention also [22, 23], where the author analyses some functionals related to the one defined in (1), also in connection with fractal dimensions.

The idea behind definition (1) is that each point in E interacts with each point in the complement of E, in such a way that the set Ω is taken into account. More precisely, in Figure 1 we can see that Ω splits E, which is the set below the line, and the complement of E into four parts: the black set $E\cap(\mathscr{C}\Omega)$, the dark gray set $E\cap\Omega$, the light gray set $(\mathscr{C}E)\cap\Omega$ and the white set $(\mathscr{C}E)\cap(\mathscr{C}\Omega)$. Then, the functional in (1) takes into account the interactions between all these sets, with the exception of the interaction between the black one and the white one. The reason for this is that in [4] the authors were interested in minimizing the functional in (1), and therefore the interaction of the two sets outside Ω is assumed as a fixed boundary datum.

Notice that, when $s \in (-\infty,0]\cup[1/2,+\infty)$, the integrals in (1) may diverge even for smooth sets. In the case $s \leqslant 0$ the problem comes from the interaction between

*Acknowledgements. The author has been supported by EPSRC grant EP/K024566/1 “Monotonicity formula methods for nonlinear PDEs”.

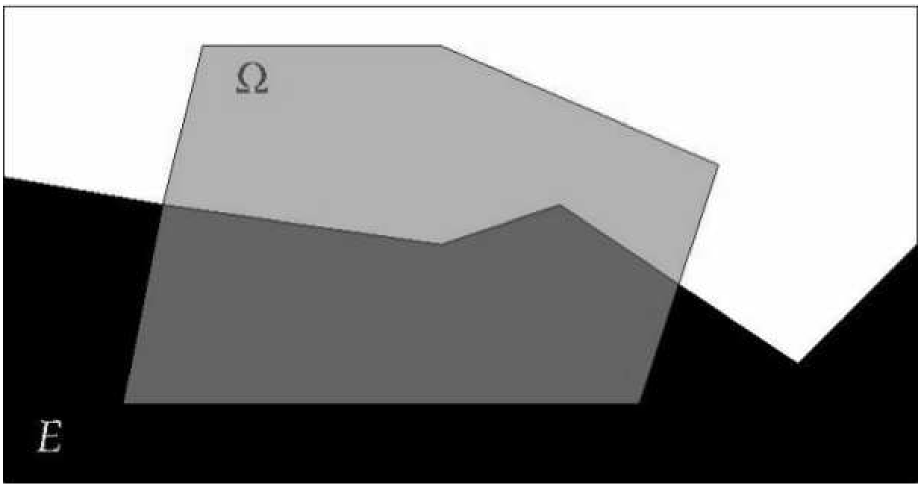

Figure 1: The sets considered in the nonlocal perimeter (1).

points x and y that are very far away from each other, while in the case $s \geqslant 1/2$ from the interaction between points x and y that are very close to each other, since the contribution to the integrals in (1) of these interactions becomes unbounded. On the contrary, when $s \in (0,1/2)$, the integrals in (1) are finite, for instance, if the set E is smooth, see [5].

As already mentioned, in [4] the minimization problem corresponding to (1) was introduced. That is, we say that a set E is s-minimal in Ω if

$$\mathrm{Per}_s(E,\Omega) \leqslant \mathrm{Per}_s(F,\Omega)$$

for any measurable set F that coincides with E outside Ω, i.e. $F \setminus \Omega = E \setminus \Omega$.

The existence of an s-minimizer is ensured by the following result, which is proved using the lower semicontinuity of the functional in (1) and a compactness property, see Section 3 in [4]:

THEOREM 1. *(Theorem 3.2 in [4]) Let Ω be a bounded Lipschitz domain, and $E_0 \subset \mathbb{R}^n \setminus \Omega$ be a given set. Then, there exists a set E, with $E \setminus \Omega = E_0$ such that*

$$\mathrm{Per}_s(E,\Omega) \leqslant \mathrm{Per}_s(F,\Omega)$$

for any F such that $F \setminus \Omega = E_0$.

Moreover, in [4] the authors established the Euler-Lagrange equation associated to the functional in (1): that is, if E is an s-minimizer in Ω and $x \in \Omega \cap (\partial E)$, then

$$\int_{\mathbb{R}^n} \frac{\chi_E(x+y) - \chi_{\mathscr{C}E}(x+y)}{|y|^{n+2s}}\, dy = 0. \tag{2}$$

We remark that the integral equation in (2) is the fractional counterpart of the zero mean curvature equation, that is the equation satisfied by classical minimal surfaces (i.e. surfaces that minimize the classical perimeter). Therefore, the integral in (2) can be seen as a sort of fractional mean curvature, see [1] where this notion was introduced and compared with the classical one.

From a geometric point of view, we can say that (2) means that there is a balance between a suitable average of the set E, centred at points of the boundary ∂E, and the average of its complement.

In [4], equation (2) is taken in the viscosity sense for any measurable set E, since the denominator may be singular if E is not smooth. An interesting thing is that, setting $\widetilde{\chi}_E := \chi_E - \chi_{\mathscr{C}E}$, (2) reads

$$(-\Delta)^s \widetilde{\chi}_E = 0 \quad \text{along } \partial E,$$

we refer to [20, 10] for a basic introduction to the fractional Laplace operator.

As a consequence of the Euler-Lagrange equation (2), in [4] the authors proved a comparison principle, namely if an s-minimizer is contained in a strip outside Ω then it is contained in the same strip inside Ω too.

Using this comparison principle, one can see that the half-plane in an s-minimizer in any domain Ω (see Corollary 5.3 in [4]). As far as we know, actually the half-plane is the only explicit example of s-minimizer.

Recently, the s-perimeter has attracted a lot of attention and inspired many works in different directions, both in the pure mathematical setting (see, for instance, the papers [3, 6, 19], where the problem of the regularity of the s-minimal surfaces was studied) and in view of concrete applications (such as phase transition problems with long range interactions, see [16, 17, 18, 21]).

The limits as $s \searrow 0$ and $s \nearrow 1/2$ are somehow the critical cases for the s-perimeter, due to the fact that the functional in (1) diverges as it is. Nevertheless, if suitably rescaled, these limits seem to give useful information on the problem, concerning, for instance, the regularity of the nonlocal minimal surfaces, see [6].

The paper is organized as follows. In Sections 2 and 3 we discuss the asymptotics of the s-perimeter when $s \nearrow 1/2$ and $s \searrow 0$, respectively. In Section 4 we give an example of set E which has infinite s-perimeter for any $s \in (0, 1/2)$ (and therefore it does not make sense to talk about the asymptotics for such a set E). In Section 5 we review the state of the art concerning the regularity of s-minimal surfaces. Finally, in Section 6 we recall the Bernstein problem and some related results obtained in the nonlocal setting.

2. Asymptotic of the s-perimeter when $s \nearrow 1/2$

One of the reasons for which the functional defined in (1) is called s-perimeter relies on the asymptotic as $s \nearrow 1/2$. Indeed, one can prove that, when $s \nearrow 1/2$, the fractional perimeter, suitably renormalised, approaches the classical perimeter, as stated in the following:

THEOREM 2. *([5, 2])*

i) *Let $\alpha \in (0,1)$, $R > 0$ and $s_k \nearrow 1/2$. Suppose that E is a set with $C^{1,\alpha}$-boundary in B_R. Then,*

$$\lim_{k \nearrow +\infty} (1-2s_k)\,\mathrm{Per}_{s_k}(E, B_r) = \omega_{n-1}\,\mathrm{Per}(E, B_r) \qquad a.e.\ r \in (0,R).$$

ii) *Let $R > r > 0$, $s_k \nearrow 1/2$ and E_k be such that*

$$\sup_{k \in \mathbb{N}} (1-2s_k)\,\mathrm{Per}_{s_k}(E_k, B_R) < +\infty.$$

Then, up to subsequence,

$$\chi_{E_k} \to \chi_E \qquad in\ L^1(B_r),$$

for a suitable set E with finite perimeter in B_r.

iii) *Let $R > r > 0$, $s_k \nearrow 1/2$ and E_k be s_k-minimizers in B_R such that*

$$\chi_{E_k} \to \chi_E \qquad in\ L^1(B_R).$$

Then, E is a minimizer for the perimeter in B_r. Also, E_k approach E uniformly in B_r, that is for any $\varepsilon > 0$ there exists k_0, possibly depending on r and ε, such that if $k \geqslant k_0$ then $E_k \cap B_r$ and $B_r \setminus E_k$ are contained, respectively, in an ε-neighbourhood of E and of $\mathbb{R}^n \setminus E$.

We observe that in the first statement i) of Theorem 2 the convergence holds true for almost any ball, since pathological sets may exist. Namely, one can construct a set E whose boundary hits the boundary of a ball B_r and then has a segment that lies on it. Since in the definition of the classical perimeter one considers open balls, the part of ∂E that coincides with ∂B_r is not taken into account. On the other hand, the integrals in the definition of the s-perimeter do not "feel" the difference between open and closed balls, and therefore also the part of ∂E that coincides with ∂B_r plays a role in the interactions. This means that for such a set E and such a ball the convergence is not true. Anyway, notice that one can slightly decrease or increase the radius of the ball to obtain the convergence of the s-perimeter of E into $B_{r\pm\varepsilon}$ to its classical perimeter.

Theorem 2 was proved in [5] from a geometric point of view, while in [2] the authors proved the convergence of the fractional perimeter to the classical perimeter as $s \nearrow 1/2$ in a Γ-convergence setting. In both the cases the authors obtained the (locally uniformly) convergence of s-minimizers to classical minimizers.

3. Asymptotic of the s-perimeter when $s \searrow 0$

In order to deal with the limit as $s \searrow 0$ of the functional in (1), we recall that the fractional Sobolev space $H^s(\mathbb{R}^n)$ is defined as

$$H^s(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{n}{2}+s}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n) \right\}.$$

[1] As usual, we denote by $\omega_{n-1} := \mathscr{H}^{n-1}(S^{n-1})$ the surface of the $(n-1)$-dimensional sphere.

This space is endowed with the norm

$$\|u\|_{H^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |u(x)|^2 dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} \, dx\, dy \right)^{1/2},$$

where the term

$$[u]_{H^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} \, dx\, dy \right)^{1/2} \tag{3}$$

is the so-called Gagliardo seminorm, see [10] for a basic introduction to the fractional Sobolev spaces.

Now, a first result regarding the asymptotic of the s-perimeter when $s \searrow 0$ is the following:

THEOREM 3. *(Theorem 3 in [15]) Suppose that $u \in H^{s_0}(\mathbb{R}^n)$ for some $s_0 \in (0,1/2)$. Then,*

$$\lim_{s \searrow 0} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} \, dx\, dy = \omega_{n-1} \int_{\mathbb{R}^n} |u(x)|^2 dx. \tag{4}$$

This means that the Gagliardo seminorm of u converges, up to a multiplicative constant, to the L^2-norm when $s \searrow 0$.

A proof of Theorem 3 when u is in the Schwartz space of rapidly decaying smooth functions goes as follows (see also Remark 4.3 in [10]). For these functions the definition in (3) agrees, up to a multiplicative constant (depending on n and s), with the Fourier definition

$$[u]_{H^s(\mathbb{R}^n)} = c(n,s) \int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{u}(\xi)|^2 \, d\xi,$$

where $\widehat{u}$ denotes the Fourier transform of u, see [10]. The normalising constant $c(n,s)$ is such that

$$\lim_{s \searrow 0} c(n,s) s = \omega_{n-1},$$

see Proposition 3.4 and Corollary 4.2 in [10]. Hence, applying the Plancherel Theorem, one has

$$\begin{aligned} \lim_{s \searrow 0} s\, [u]_{H^s(\mathbb{R}^n)} &= \lim_{s \searrow 0} c(n,s)\, s \int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{u}(\xi)|^2 \, d\xi \\ &= \omega_{n-1} \int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2 \, d\xi = \omega_{n-1} \|\widehat{u}\|_{L^2(\mathbb{R}^n)} = \omega_{n-1} \|u\|_{L^2(\mathbb{R}^n)}, \end{aligned}$$

which is (4).

As a particular case, one can take $u := \chi_E$ in (4), for some set $E \subseteq \Omega$ such that $\mathrm{Per}_{s_0}(E,\Omega) < +\infty$ (and so $\chi_E \in H^{s_0}(\mathbb{R}^n)$) for some $s_0 \in (0,1/2)$. Since $E \cap (\mathscr{C}E) = \varnothing$, from (1) one has that

$$\mathrm{Per}_s(E,\Omega) = \mathscr{L}(E \cap \Omega, (\mathscr{C}E) \cap \Omega) + \mathscr{L}(E \cap \Omega, (\mathscr{C}E) \cap (\mathscr{C}\Omega)) = \mathscr{L}(E, \mathscr{C}E),$$

and so

$$
\begin{aligned}
\lim_{s\searrow 0} 2s \operatorname{Per}_s(E,\Omega) &= \lim_{s\searrow 0} 2s\,\mathscr{L}(E,\mathscr{C}E) \\
&= \lim_{s\searrow 0} 2s \int_E \int_{\mathscr{C}E} \frac{dx\,dy}{|x-y|^{n+2s}} \\
&= \lim_{s\searrow 0} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\chi_E(x)-\chi_E(y)}{|x-y|^{n+2s}}\,dx\,dy \\
&= \lim_{s\searrow 0} s\,[\chi_E]^2_{H^s(\mathbb{R}^n)} = \omega_{n-1}\|\chi_E\|^2_{L^2(\mathbb{R}^n)} = \omega_{n-1}|E|,
\end{aligned} \tag{5}
$$

where $|E|$ denotes the Lebesgue measure of E.

The general case was treated in [11]. Given a set E, possibly unbounded, the authors introduced the following parameter

$$
\widetilde{\alpha}(E) := \lim_{s\searrow 0} \frac{2s}{\omega_{n-1}} \int_{E\setminus B_1} \frac{dy}{|y|^{n+2s}}, \tag{6}
$$

called the "weighted volume of E towards infinity", and the normalized Lebesgue measure $\mathscr{M}(E) := \omega_{n-1}|E|$. Then, the result proved in [11] is the following:

THEOREM 4. *(Theorems 2.5 and 2.7 in [11]) Suppose that* $\operatorname{Per}_{s_0}(E,\Omega) < +\infty$ *for some* $s_0 \in (0,1/2)$*, and that the limit in* (6) *exists. Then,*

$$
\lim_{s\searrow 0} 2s \operatorname{Per}_s(E,\Omega) = (1-\widetilde{\alpha}(E))\mathscr{M}(E\cap\Omega) + \widetilde{\alpha}(E)\mathscr{M}(\Omega\setminus E). \tag{7}
$$

Also, if $\operatorname{Per}_{s_0}(E,\Omega) < +\infty$ *for some* $s_0 \in (0,1/2)$ *and* $|E\cap\Omega| \neq |\Omega\setminus E|$*, then the existence of the limit in* (6) *is equivalent to the existence of the limit in* (7).

Notice that $\widetilde{\alpha}(E) \in [0,1]$ and therefore the limit in (7) is a convex combination of the normalized Lebesgue measure of the sets $E\cap\Omega$ and $\Omega\setminus E$. In particular, if $E \subseteq \Omega$ then $\widetilde{\alpha}(E) = 0$ (notice that E is bounded and recall Footnote 2) and so (7) boils down to (5).

In some sense, (7) says that the s-perimeter "localizes" in Ω when $s \searrow 0$, because it takes into account only the Lebesgue measure of two sets which are contained in Ω. This is not completely true, since the parameter $\widetilde{\alpha}(E)$ in the convex combination takes into account the contribution of E coming from infinity. Hence, the nonlocal character of the s-perimeter is preserved also in the limit, by means of the parameter that interpolates the two Lebesgue measures which are set in Ω.

[2] Notice that in (6) one can take the integral over $E\setminus B_R$ for any $R>0$, instead of $E\setminus B_1$. Indeed

$$
\int_{B_R\setminus B_1} \frac{dy}{|y|^{n+2s}} = \omega_{n-1}\int_1^R \frac{\rho^{n-1}\,d\rho}{\rho^{n+2s}} = \frac{\omega_{n-1}}{2s}(1-R^{-2s}),
$$

and therefore

$$
\lim_{s\searrow 0} \frac{2s}{\omega_{n-1}} \int_{B_R\setminus B_1} \frac{dy}{|y|^{n+2s}} = 0.
$$

In Theorem 4 the assumption of the existence of the limit in (6) cannot be removed, since the limit in (6) (and hence the limit in (7)) may not exist. Indeed, in [11] the authors provide an example of set for which such limit does not exist (see also Section 1.1 in [21]). Roughly speaking, the idea is constructing a set which looks like a cone of variable opening angle: one starts with a cone of small opening in an annulus, then changes his mind and enlarges the opening of the cone in the subsequent annulus, and so on, alternating cones of small and big opening angles in the subsequent annuli (see Figure 2 where the set is grossly drawn). In this way, the parameter $\widetilde{\alpha}(E)$ "detects" the different openings and this creates an oscillation of the value of $\widetilde{\alpha}(E)$ in the different annuli. As a consequence, the limit in (6) does not exist.

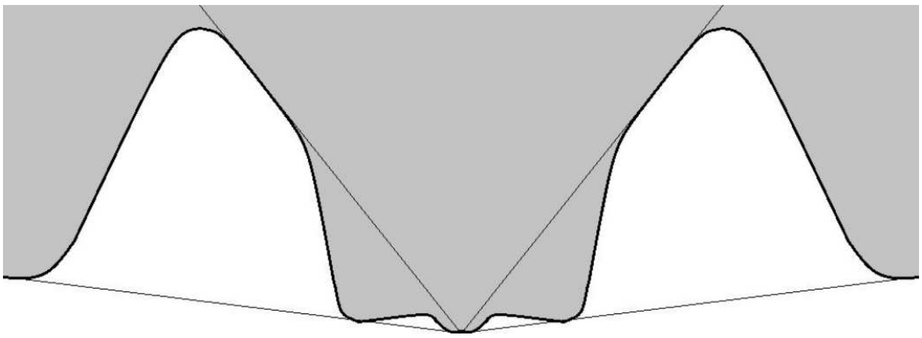

Figure 2: An example for which the limit as $s \searrow 0$ of the fractional perimeter does not exist.

In Theorem 4 it is also required that the set E has finite s_0-perimeter for some $s_0 \in (0,1/2)$. We point out that this assumption cannot be dropped in general, since there are sets that do not satisfy it (and for them the limit of the functional in (1) as $s \searrow 0$ does not make any sense), see the subsequent Section 4.

4. Example of set with infinite s-perimeter for any $s \in (0,1/2)$

In this section we give an example of sets E and Ω for which

$$\mathrm{Per}_s(E,\Omega) = +\infty \quad \text{for any } s \in (0,1/2).$$

For this, we take $n \geqslant 1$ and a real number β such that

$$0 < \beta < \frac{2s}{n-2s}. \tag{8}$$

Notice that β is well defined, since $n \geqslant 1$ and $s \in (0,1/2)$. Moreover, for any $k \geqslant 1$, we consider

$$i_k := \frac{C_\beta}{k^{\beta+1}}, \tag{9}$$

where $0 < C_\beta < 1$ is a constant depending on β, such that the following holds true

$$\sum_{k=1}^{+\infty} i_k = 1.$$

For any $k \geqslant 1$, we set

$$r_k := 1 - \sum_{j=1}^{k} i_j.$$

Thanks to (9), we have that

$$r_k = \sum_{j=k+1}^{+\infty} i_j \geqslant \frac{C_\beta}{k^\beta}, \tag{10}$$

up to relabeling C_β.

Now, for any $k \geqslant 1$, we consider the annulus A_k of thickness i_k. Notice that each A_k can be covered by the union of balls $B_{i_k/2}(x)$ of radius $i_k/2$ and centred at points lying on the sphere of radius $r_k + (i_k/2)$. Namely,

$$A_k = \bigcup_{x \in \partial B_{r_k+(i_k/2)}} B_{i_k/2}(x).$$

Since, for any $k \geqslant 1$ and $x \in \partial B_{r_k+(i_k/2)}$, the radius of the ball $B_{i_k/2}(x)$ is $i_k/2 < +\infty$, we can apply the Vitali's covering Theorem (see e.g. [12]), obtaining that, for any $k \geqslant 1$, there exists a countable subcollection of disjoint balls $B_{i_k/2}(x_j)$, $j = 1, \dots N_k$, such that

$$A_k = \bigcup_{x \in \partial B_{r_k+(i_k/2)}} B_{i_k/2}(x) \subseteq \bigcup_{j=1}^{N_k} B_{5i_k/2}(x_j). \tag{11}$$

We claim that, for any $k \geqslant 1$,

$$N_k < +\infty. \tag{12}$$

For this, notice that

$$\bigcup_{j=1}^{N_k} B_{i_k/2}(x_j) \subseteq A_k,$$

and therefore

$$c_n N_k i_k^n = N_k |B_{i_k/2}(x_j)| \leqslant |A_k|, \tag{13}$$

for a suitable positive constant c_n depending on n, where $|B_{i_k/2}(x_j)|$ and $|A_k|$ denote the Lebesgue measures of $B_{i_k/2}(x_j)$ and A_k, respectively. Since $|A_k|$ is finite, (13) implies (12).

Moreover, we claim that

$$N_k \geqslant C \left(\frac{r_k}{i_k} \right)^{n-1}, \tag{14}$$

for some constant $C > 0$ (only depending on n). To show this, we use the Binomial Theorem and (11) to get

$$|B_1| \, r_k^{n-1} i_k \leqslant |B_1| \, [(r_k + i_k)^n - r_k^n] = |A_k| \leqslant \sum_{j=1}^{N_k} |B_{5i_k/2}(x_j)| = C_1 \, N_k \, i_k^n,$$

for some $C_1 > 0$ depending on the dimension, and this implies (14).

Notice that from (9), (10) and (14) we have

$$N_k \geqslant C \, k^{n-1}, \tag{15}$$

up to renaming the constant C.

Let us make the following observation. We consider the unit ball B_1 and a smooth non-empty set $S_1 \subset B_1$. Notice that any smooth set would do the job, we will take a smiley face for typographical convenience in Figure 3. Since the boundary of S_1 is smooth and both S_1 and $B_1 \setminus S_1$ are not empty, from Lemma 11 in [5] we obtain that

$$0 < C_* := \int_{S_1} \int_{B_1 \setminus S_1} \frac{dx\, dy}{|x-y|^{n+2s}} < +\infty.$$

Hence, setting B_ρ the ball of radius ρ and S_ρ the scaled version of S_1 by a factor ρ, for any $\rho > 0$, we have

$$\begin{aligned} \int_{S_\rho} \int_{B_\rho \setminus S_\rho} \frac{dx\, dy}{|x-y|^{n+2s}} &= \int_{S_1} \int_{B_1 \setminus S_1} \frac{\rho^{2n} d\widetilde{x} d\widetilde{y}}{\rho^{n+2s} |\widetilde{x} - \widetilde{y}|^{n+2s}} \\ &= \rho^{n-2s} \int_{S_1} \int_{B_1 \setminus S_1} \frac{d\widetilde{x} d\widetilde{y}}{|\widetilde{x} - \widetilde{y}|^{n+2s}} \\ &= C_* \, \rho^{n-2s}, \end{aligned} \tag{16}$$

where we have used the change of variables $\widetilde{x} = x/\rho$ and $\widetilde{y} = y/\rho$.

Now, in each annulus A_k we have N_k disjoint balls of radius $i_k/2$, and in each of these balls we insert a smiley face S_k^j, $j = 1, \ldots, N_k$. We define the set E as the union of these smiley faces, that is

$$E := \bigcup_{k=1}^{+\infty} \bigcup_{j=1}^{N_k} S_k^j$$

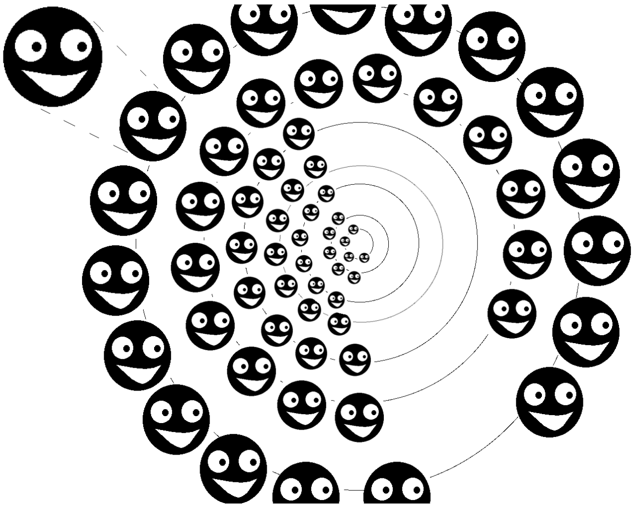

Figure 3: The set E with the smiley faces.

(as depicted in Figure 3). Notice that $E \subset B_1$, and therefore, using (16), (9), (15) and (8), we get

$$
\begin{aligned}
\mathrm{Per}_s(E,B_1) &= \mathscr{L}(E,\mathscr{C}E) \\
&\geqslant \sum_{k=1}^{+\infty}\sum_{j=1}^{N_k} \mathscr{L}(S_k^j, B_{i_k/2}(x_j)\setminus S_k^j) \\
&= \sum_{k=1}^{+\infty}\sum_{j=1}^{N_k} \int_{S_k^j}\int_{B_{i_k/2}(x_j)\setminus S_k^j} \frac{dx\,dy}{|x-y|^{n+2s}} \\
&= C_2 \sum_{k=1}^{+\infty} N_k i_k^{n-2s} \\
&\geqslant C_3 \sum_{k=1}^{+\infty} \frac{k^{n-1}}{k^{(1+\beta)(n-2s)}} \\
&= C_3 \sum_{k=1}^{+\infty} \frac{1}{k^{\beta(n-2s)-2s+1}} \\
&= +\infty,
\end{aligned}
$$

for suitable positive constants C_2 and C_3. This gives the desired result.

We mention that in [11] an example in one dimension of a set with infinite s-perimeter for any $s \in (0,1/2)$ was constructed (see Subsection 3.10 there).

5. Regularity of s-minimal surfaces

One of the main problems considered in [4] is the one of the regularity of the s-minimizers:

THEOREM 5. *(Theorem 2.4 in [4]) If E is an s-minimizer in B_1, then $\partial E \cap B_{1/2}$ is a $C^{1,\alpha}$-hypersurface around each of its points, possibly except a closed set Σ of finite $(n-2)$-Hausdorff dimension.*

We observe that the boundary of E is supposed to have dimension $n-1$, and so the fact that Σ can be at most an $(n-2)$-dimensional object implies that it is negligible inside ∂E. Hence, Theorem 5 says that if E is an s-minimizer then its boundary is smooth at "most of its points".

Anyway, as far as we know, there are no examples of s-minimizers with a non-empty singular set. One may be tempted to say that, for instance, the classical cone in the plane

$$\mathscr{C} := \{(x,y) \in \mathbb{R}^2 : xy > 0\}$$

is an s-minimizer, since it satisfies the Euler-Lagrange equation (2). But in fact, there is an original idea of L. Caffarelli (explained in Section 1.2 of [21]) which shows that the cone $\mathscr{C}$ is not s-minimal. Notice that this says that the s-minimality implies, but is not equivalent to, the Euler-Lagrange equation (2).

Even though the regularity obtained in Theorem 5 is only $C^{1,\alpha}$, in [3] it was proved by non trivial bootstrap arguments that one can improve it towards C^∞.

Now, concerning the regularity of minimizers of the functional in (1), at the moment, there are only other two results in two different directions: the first result says that one can recover the classical minimal surface theory in any dimensions but only when s is sufficiently close to $1/2$, while the second result is valid for any s in the range $(0,1/2)$ but only in the plane.

In particular, starting from Theorem 2, in [6] the authors proved the following:

THEOREM 6. *([6]) For any $n \in \mathbb{N}$ there exists $\varepsilon_n \in (0,1/2]$ such that if $s \in ((1/2)-\varepsilon_n, 1/2)$ then s-minimal surfaces are "as regular as the classical minimal surfaces", that is*

- *if $n \leqslant 7$, then any s-minimal surface is locally C^∞,*
- *if $n = 8$, then any s-minimal surface is locally C^∞ except, at most, at countably many isolated points,*
- *if $n > 8$, then any s-minimal set is locally C^∞ outside a closed set $\Sigma \subset \partial E$ with finite $(n-8)$-Hausdorff dimension.*

Although there are no examples of singular sets in any dimension and for any $s \in (0,1/2)$, as far as we know, Theorem 6 seems to be the only improvement of Theorem 5 about the regularity of s-minimal surfaces valid in any dimension (even if only for s

close to $1/2$). On the other hand, the proof of Theorem 6 is based on a compactness argument, and therefore the value of the quantity ε_n is not explicit (we only know that it is a universal constant that depends only on the dimension).

As we already said, the proof of Theorem 6 relies on the asymptotic of the functional in (1) as $s \nearrow 1/2$. Although the proof is very technical and delicate, morally, the idea is that, since the s-perimeter converges to the classical perimeter as $s \nearrow 1/2$, the s-minimal surfaces inherit the regularity of the classical minimal surfaces when s is sufficiently close to 1/2. For this, some care is needed in order to obtain uniform regularity estimates in s, which can pass to the limit.

Concerning the second result that we mentioned about the regularity of s-minimal surfaces, it is contained in [19] and states the following:

THEOREM 7. *([19]) Let* $n = 2$. *Suppose that* $R > r > 0$ *and that* E *is an* s*-minimal set in* B_R*, then* $(\partial E) \cap B_r$ *is a* C^∞*-curve.*

Also, if E *is an* s*-minimal set in* B_ρ *for every* $\rho > 0$*, then* ∂E *is a straight line.*

Theorem 7 says that in the plane any s-minimal surface is smooth, and this is exactly what happens in the classical case. In particular, as a byproduct, one obtains an improvement of Theorem 5: the singular set Σ has finite $(n-3)$-Hausdorff dimension, instead of $n-2$. Still, we do not know if this is optimal, see Theorem 6.

A consequence of the above result is also that in the plane global s-minimal sets (i.e. s-minimal sets in any ball) are straight lines.

Now, it seems very difficult to obtain further information from the asymptotic as $s \searrow 0$, since for s close to 0 the minimizers of the s-perimeter seem to be somehow related to the minimizers of the Lebesgue measure (see Theorem 4), which can have very wild boundary.

On the other hand, in [7] the authors proved that when s is close to 0 all symmetric cones are unstable if the dimension $n \leqslant 6$ and stable if $n = 7$. This tells us that, when $n \leqslant 6$, a symmetric cone is not an s-minimizer, but, still, we are not able to conclude that the singular set Σ in Theorem 5 is empty, since the example of a non-smooth s-minimizer can be a non symmetric cone.

We also mention a very recent paper [8], where the authors constructed an example of surface that satisfies the equation in (2) for s sufficiently close to $1/2$, that is the so-called "nonlocal catenoid" (see Theorem 1 there).

6. The Bernstein problem

The regularity theory for minimal surfaces is related to the Bernstein problem: if the graph of a function on $\mathbb{R}^n$ is a minimal surface in $\mathbb{R}^{n+1}$, is it true that the function is affine?

In the classical case, the result is true in dimension $n \leqslant 8$ and false when $n \geqslant 9$, see e.g. [13].

In the nonlocal setting, there is only a very recent paper [14], where the authors were able to extend a result of De Giorgi for minimal surfaces (see [9]), showing that Bernstein's Theorem holds true in dimension $n+1$ if there are no singular s-minimal cones in $\mathbb{R}^n$.

For this, we recall that a s-minimal surface E is a "s-minimal graph" if it can be written as a global graph in some direction. Namely, up to rotation, there exists a function $u : \mathbb{R}^n \to \mathbb{R}$ such that

$$E := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : t < u(x)\}.$$

Then, we have the following:

THEOREM 8. *(Theorem 1.2 in [14]) Let* $E := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : t < u(x)\}$ *be a s-minimal graph, and suppose that there are no singular s-minimal cones in* $\mathbb{R}^n$. *Then u is an affine function*.

This means that if the only s-minimal cone in $\mathbb{R}^n$ is the half-space and E is a s-minimal graph, then E is a half-space.

If $n = 1$ in Theorem 8 we obtain a particular case of Theorem 7.

If $n = 2$, since there are not s-minimal cones in the plane (see Theorem 7), as a byproduct of the general result in Theorem 8 one obtains that s-minimal graphs have C^∞-boundary in $\mathbb{R}^3$. Unfortunately, this is not enough to further improve Theorem 5, that is we cannot say that the singular set Σ has $(n-4)$-Hausdorff dimension, because of the assumption that E is a s-minimal graph.

On the other hand, combining Theorems 6 and 8 we have that, when s is sufficiently close tp $1/2$, Bernstein Theorem holds true up to dimension 8.

References

[1] N. ABATANGELO, E. VALDINOCI, *A notion of nonlocal curvature*, Numer. Funct. Anal. Optim., 35 (2014), no. 7-9, 793–815.

[2] L. AMBROSIO, G. DE PHILIPPIS, L. MARTINAZZI, *Γ-convergence of nonlocal perimeter functionals*, Manuscripta Math. **134** (2011), 377–403.

[3] B. BARRIOS, A. FIGALLI, E. VALDINOCI, *Bootstrap regularity for integro-differential operators and its application to nonlocal minimal surfaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 13 (2014), no. 3, 609–639.

[4] L. CAFFARELLI, J.-M. ROQUEJOFFRE, O. SAVIN, *Nonlocal minimal surfaces*, Comm. Pure Appl. Math. **63** (2010), 1111–1144.

[5] L. CAFFARELLI, E. VALDINOCI, *Uniform estimates and limiting arguments for nonlocal minimal surfaces*, Calc. Var. Partial Differential Equations **41** (2011), no. 1–2, 203–240.

[6] L. CAFFARELLI, E. VALDINOCI, *Regularity properties of nonlocal minimal surfaces via limiting arguments*, Adv. Math. **248** (2013), 843–871.

[7] J. DÁVILA, M. DEL PINO, J. WEI, *Nonlocal minimal Lawson cones*, Preprint (2013), `http://arxiv.org/abs/1302.0276`

[8] J. DÁVILA, M. DEL PINO, J. WEI, *Nonlocal s-minimal surfaces and Lawson cones*, Preprint (2014), `http://arxiv.org/pdf/1402.4173.pdf`

[9] E. De Giorgi, *Una estensione del teorema di Bernstein*, Ann. Scuola Norm. Sup. Pisa (3) **19** (1965), 79–85.

[10] E. Di Nezza, G. Palatucci, E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. math. **136** (2012), no. 5, 521–573.

[11] S. Dipierro, A. Figalli, G. Palatucci, E. Valdinoci, *Asymptotics of the s-perimeter as $s \searrow 0$*, Discrete Contin. Dyn. Syst. **33** (2013), no. 7, 2777–2790.

[12] L. C. Evans, R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC press (1992), pp. 27.

[13] A. Farina, E. Valdinoci, *The state of the art for a conjecture of De Giorgi and related problems*, Recent progress on reaction-diffusion systems and viscosity solutions, World Sci. Publ., Hackensack, NJ (2009).

[14] A. Figalli, E. Valdinoci, *Regularity and Bernstein-type results for nonlocal minimal surfaces*, J. Reine Angew. Math., DOI 10.1515/crelle-2015-0006.

[15] V. Maz'ya, T. Shaposhnikova, *On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces*, J. Funct. Anal. **195** (2002), 230–238.

[16] G. Palatucci, O. Savin, E. Valdinoci, *Local and global minimizers for a variational energy involving a fractional norm*, Ann. Mat. Pura Appl. (4) **192** (2013), no. 4, 673–718.

[17] O. Savin, E. Valdinoci, *Density estimates for a variational model driven by the Gagliardo norm*, J. Math. Pures Appl. (9) **101** (2014), no. 1, 1–26.

[18] O. Savin, E. Valdinoci, *Γ-convergence for nonlocal phase transitions*, Ann. Inst. H. Poincaré Anal. Non Linéaire **29** (2012), 479–500.

[19] O. Savin, E. Valdinoci, *Regularity of nonlocal minimal cones in dimension 2*, Calc. Var. Partial Differential Equations **48** (2013), no. 1–2, 33–39.

[20] L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Ph.D. Thesis, University of Texas at Austin (2005).

[21] E. Valdinoci, *A fractional framework for perimeters and phase transitions*, Milan J. Math. **81** (2013), no. 1, 1–23.

[22] A. Visintin, *Nonconvex functionals related to multiphase systems*, SIAM J. Math. Anal. **21** (1990), 1281–1304.

[23] A. Visintin, *Generalized coarea formula and fractal sets*, Japan J. Industrial Appl. Math. **8** (1991), 175–201.

AMS Subject Classification: 28A75, 49Q05, 49Q20, 60G22

Serena DIPIERRO,
Maxwell Institute for Mathematical Science and School of Mathematics
University of Edinburgh, James Clerk Maxwell Building, Peter Guthrie Tait Road
Edinburgh EH9 3FD, UNITED KINGDOM
e-mail: serena.dipierro@ed.ac.uk

Lavoro pervenuto in redazione il 29.01.2015.