Rend. Sem. Mat. Univ. Politec. Torino
Vol. 72, 1–2 (2014), 57 – 71

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# FLOW IN SOILS WITH HYSTERESIS

**Abstract.** Hysteresis in single–porosity flow through homogeneous unsaturated medium and in double–porosity model of Darcian flows through two distinct pore systems both treated as homogeneous media is here illustrated. Darcy's law and equation of continuity lead to nonlinear diffusion equation for single–porosity flow and to two coupled systems of nonlinear partial differential equations for double–porosity model. The effects of hysteresis in the relation between pressure and water content are represented by Preisach hysteresis operator. The existence results of the initial–boundary–value problems for both models with soil–moisture Preisach hysteresis term are summarized here.

## 1. Introduction

Unsaturated fluid flow through porous media is an important topic in hydrology, agronomy and soil physics. Fluid movement in unsaturated soil is subject to hysteresis. The soil–moisture hysteresis is observed in cycles of wetting–drying processes in soil. Thus hysteretic effects are in the relation between the volumetric water content and the pressure and are modeled by Preisach operator. Phenomenon of hysteresis has been investigated by different authors and for distinct applications. Already in 1930, Haines [12] had postulated that drying occurs at a higher value of pressure as the narrower section of the pores governs this process, while wetting happens at a lower pressure value originated by the wider section of the pores. This phenomenon has also been explained in these terms by using the concept of a "pore domain". The simplest version of the domain theory is the so called *independent domain theory*, where a domain consists of a unique pore of a given geometry acting independently from others, i.e., pores behave as if each one was directly connected to the outer boundary sample. This theory was conceived by Everett [7, 8, 9]. The independent domain model, which is essentially the Preisach model, was adopted by Poulovassilis [23], Mualem [20] and Parlange [22].

Concerning the single–porosity flow equation with excluded hysteresis relation, in [25], the authors proved the existence of a strong solution. There are papers devoted to this equation taking into consideration the hysteresis behaviour [2, 3, 14, 19]. In [2] existence result with hysteresis operator of Preisach type is established. Paper [3] deals with existence and asymptotic behaviour of solution. In paper [19] the wellposedness of the problem is shown. In [14] the existence of the solution to the single–porosity model of water flow with Preisach hysteresis operator was proved.

In [1], the author proved the global existence of weak solutions to the double–porosity (dual) model with mixed Dirichlet–Neumann boundary conditions. The existence and uniqueness of the solution to the dual water flow through porous media without hysteresis and with unilateral boundary conditions were obtained in [4]. In [15], the authors proved the existence of the solution to the model including dual approach, hysteresis as well as the unilateral boundary conditions.

The paper is organized as follows. In Section 2, we briefly review some basic definitions in the unsaturated flow and recall some basic results concerning the hysteresis operators. In Section 3 we introduce the equations representing single–porosity and double–porosity flows through porous media. The existence of weak solutions to the model equations are summed up in Section 4.

## 2. Preliminaries

### 2.1. Soil structure. Soil–moisture hysteresis

A soil is a porous medium which consists of particles of varying sizes. These particles join together and the spacings among them are known as pore spaces (or voids), as shown in Figure 1.

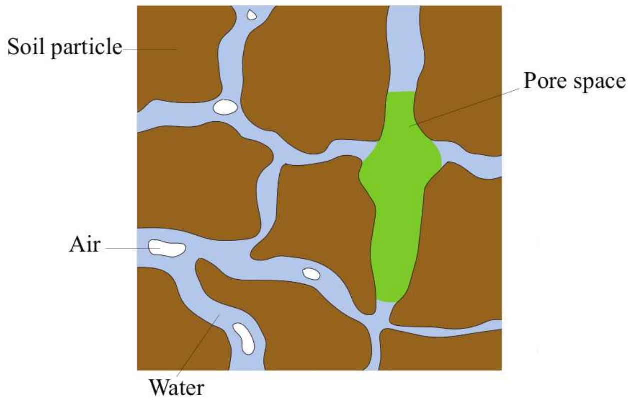


Figure 1: Schematic picture of the microstructure of a porous medium. Pore spaces can be either filled, partially filled or empty depending on the pressure head.

The soil is saturated if the pore spaces are completely filled with water. We call the water motion in this situation saturated flow. In the unsaturated case there are voids filled with air and the flow is said to be unsaturated. However, partially saturated zones may occur when all pores within them are filled with water. Then the interfaces between the saturated and unsaturated regions of the soil become free boundaries. The flow is termed saturated–unsaturated.

Water movement in unsaturated soil is subject to hysteresis, although its effects are often masked by heterogeneities. The hysteretic effect may be attributed to several factors [5, 13]:

1) the effects of nonhomogeneous pore size distribution, often referred as "ink–bottle effect",
2) entrapped air, which refers to the formation of closed air bubbles during wetting,

3) capillary condensation, which is related to adsorbed water films on the surfaces of fine–grained particles,

4) contact angle hysteresis, which is related to the difference between drying and wetting contact angles at the solid–water interface.

It was shown experimentally that there is hysteresis in the relationship between soil pressure and water content, see [12, 21, 24]. The hysteretic effect is observed in wetting–drying processes, i.e., is evident in the soil–water characteristic curve.

We can consider pore spaces as capillary tubes. The bulge in the centre of the tube can be considered as the pore. Now each pore is connected to a neighbouring pore by means of a pore throat with narrow end of the capillary tube (Figure 2). When an empty tube is placed into a water bath, water will rise up to a point above the waist of the tube until the water reaches its equilibrium state (wetting process). Likewise when a tube filled with water is placed into the same water bath, the water level will be moving down towards the waist until the water reaches its equilibrium state (drying process).

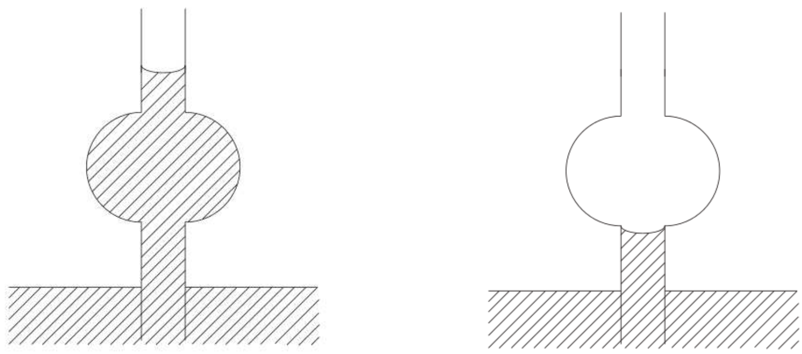

Figure 2: Simple model of the pore spaces represented by capillary tubes.

## 2.2. Hysteresis operators

### The play operator

Now we briefly recall definition and properties of the play operator, the simplest example of continuous hysteresis operator. Let $r > 0$ be a given parameter. For a given input function $u \in C([0,T])$ and initial condition $x_r^0 \in [-r,r]$, we define the output $\xi := \mathscr{P}_r[x_r^0, u] \in C([0,T]) \cap BV(0,T)$ of the play operator

$$\mathscr{P}_r : [-r,r] \times C([0,T]) \to C([0,T]) \cap BV(0,T)$$

as the solution of the variational inequality in Stieltjes integral form

$$
\begin{aligned}
&\int_0^T [u(t)-\xi(t)-y(t)]\,\mathrm{d}\xi(t) \geqslant 0, \qquad \forall y \in C([0,T]),\ \max_{0\leqslant t\leqslant T}|y(t)| \leqslant r,\\
&|u(t)-\xi(t)| \leqslant r, \quad \forall t \in [0,T],\\
&\xi(0) = u(0) - x_r^0. \qquad (1)
\end{aligned}
$$

In order to model a more complex hysteresis behavior, we consider the whole family of play operators $\mathscr{P}_r$ parametrized by $r > 0$, which can be interpreted as a memory variable. More precisely, following [16, Section II.2], we introduce the *configuration space* as well as its subspaces

$$
\Lambda := \left\{ \lambda \in W^{1,\infty}(0,\infty);\left|\frac{d\lambda(r)}{dr}\right| \leqslant 1 \quad \text{a.e. in } (0,\infty)\right\}, \tag{2}
$$

$$
\Lambda_K := \{\lambda \in \Lambda; \lambda(r) = 0 \quad \text{for } r \geqslant K\}, \quad \Lambda_0 := \bigcup_{K>0} \Lambda_K. \tag{3}
$$

The functions $\lambda \in \Lambda$ are called *memory configurations*. For a given $\lambda \in \Lambda$, we define the initial condition $x_r^0$ by formula $x_r^0 := Q_r(u(0) - \lambda(r))$, where $Q_r : \mathbb{R} \to [-r,r]$ is the projection

$$
Q_r(x) := \operatorname{sign}(x)\min\{r,|x|\} = \min\{r,\max\{-r,x\}\}.
$$

This implies that the initial configuration of the play system depends on $\lambda$ and on $u(0)$. So we can introduce the following more convenient notation

$$
p_r[\lambda,u] := \mathscr{P}_r[x_r^0,u], \tag{4}
$$

for any $\lambda \in \Lambda$, $u \in C([0,T])$ and $r > 0$.

The reason for introducing the space $\Lambda$ is that for every fixed $t \in [0,T]$ and $\lambda \in \Lambda$, the state mapping $r \to p_r[\lambda,u](t)$ belongs to $\Lambda$.

In [18], the play operator is definied in the space $G_R(0,T)$ of right–continuous regulated functions. This is the space of functions $u : [0,T] \to \mathbb{R}$ which admits the left limit $u(t_-)$ at each point $t > 0$ and the right limit $u(t_+)$ exists and coincides with $u(t)$ at each point $t \geqslant 0$. The space $G_R(0,T)$ is endowed with the norm

$$
\|u\|_{[0,T]} = \sup\{|u(\tau)|; \tau \in [0,T]\} \quad \text{for } u \in G_R(0,T), \tag{5}
$$

hence $G_R(0,T)$ is a Banach space. By [18, Theorem 2.1 and Proposition 2.4], this is Lipschitz continuous in the sense that

$$
|p_r[\lambda,u](t) - p_r[\mu,v](t)| \leqslant \max\{|\lambda(r)-\mu(r)|, \|u-v\|_{[0,T]}\} \tag{6}
$$

for any $\lambda,\mu \in \Lambda$, $u,v \in G_R(0,T)$ and $t \in [0,T]$. For step functions $u \in G_R(0,T)$ of the form

$$
u(t) = \sum_{n=1}^{m} u_m^{n-1}\chi_{[t_{n-1},t_n)}(t) + u_m^m\chi_{\{T\}}(t), \tag{7}
$$

where $0 = t_0 < t_1 < \dots t_m = T$ is a given division of $[0,T]$, we have in particular

$$p_r[\lambda,u](t) = \sum_{n=1}^{m} \xi_m^{n-1}(r)\chi_{[t_{n-1},t_n)}(t) + \xi_m^m(r)\chi_{\{T\}}(t), \tag{8}$$

where $\chi_\omega(t)$ is the characteristic function of a set $\omega \subset [0,T]$, and

$$\xi_m^0(r) = P[\lambda, u_m^0](r), \quad \xi_m^n(r) = P[\xi_m^{n-1}, u_m^n](r), \tag{9}$$

with $P : \Lambda \times \mathbb{R} \to \Lambda$ defined as

$$P[\lambda, v](r) = \max\{v - r, \min\{v + r, \lambda(r)\}\}. \tag{10}$$

### The Preisach operator

Now we briefly recall definition and some basic properties of the Preisach operator. Let us introduce the Preisach half–plane, defined as

$$\mathbb{R}_+^2 := \{(r,v) \in \mathbb{R}^2 : r > 0\} \tag{11}$$

and assume that a function $\psi \in L^1_{loc}(\mathbb{R}^2_+)$ (the Preisach density) is given with the following property.

ASSUMPTION 1. There exist $\beta_1 \in L^1_{loc}(0,\infty)$, such that

$$0 \leqslant \psi(r,v) \leqslant \beta_1(r) \quad \text{for a.e. } (r,v) \in \mathbb{R}^2_+.$$

We put

$$b_1(K) := \int_0^K \beta_1(r)\,\mathrm{d}r \quad \text{for } K > 0, \quad g(r,v) := \int_0^v \psi(r,x)\,\mathrm{d}x \quad \text{for } (r,v) \in \mathbb{R}^2_+ \tag{12}$$

and define the Preisach operator as follows.

DEFINITION 1. *Let $\psi \in L^1_{loc}(\mathbb{R}^2_+)$ be given and let $g$ be as in* (12). *Then the Preisach operator $\mathscr{W} : \Lambda_0 \times G_R(0,T) \to G_R(0,T)$ generated by the function $g$ is defined by the formula*

$$\mathscr{W}[\lambda,u](t) := \int_0^\infty g(r, p_r[\lambda,u](t))\,\mathrm{d}r = \int_0^\infty \int_0^{p_r[\lambda,u](t)} \psi(r,x)\,\mathrm{d}x\,\mathrm{d}r \tag{13}$$

*for $\lambda \in \Lambda_0$, $u \in G_R(0,T)$ and $t \in [0,T]$.*

As a counterpart of [16, Section II.3, Proposition 3.11], we have the following

PROPOSITION 1. *Let Assumption 1 be satisfied and let $K > 0$ be given. Then for every $\lambda, \mu \in \Lambda_K$ and $u, v \in G_R(0,T)$ such that $\|u\|_{[0,T]}, \|v\|_{[0,T]} \leqslant K$, the Preisach operator (13) satisfies*

$$\|\mathscr{W}[\lambda,u] - \mathscr{W}[\mu,v]\|_{[0,T]} \leqslant \int_0^K |\lambda(r) - \mu(r)|\beta_1(r)\,\mathrm{d}r + b_1(K)\|u - v\|_{[0,T]} \quad \forall t \in [0,T].$$

We finally quote the Hilpert inequality which will be used to establish the uniqueness of the solution.

PROPOSITION 2. *Let $\mathscr{W}$ be a Preisach operator (13) satisfying Assumption 1. For given $u_1, u_2 \in W^{1,1}(0,T)$, $\lambda_1, \lambda_2 \in \Lambda_0$ put $\xi_r^i := p_r[\lambda, u]$, $w_i := \mathscr{W}[\lambda_i, u_i] = \int_0^\infty g(r, \xi_r^i)\,\mathrm{d}r$, $i = 1,2$. Then for a.e. $t \in (0,T)$ we have*

$$\frac{d}{dt}(w_1(t) - w_2(t))\; H(u_1(t) - u_2(t)) \geqslant \frac{d}{dt}\int_0^\infty (g(r, \xi_r^1(t)) - g(r, \xi_r^2(t))^+\,\mathrm{d}r, \tag{14}$$

*where $H$ is the Heaviside function.*

In our equation, both the input function and the initial memory configuration depend on the space variable $x$. If $\lambda(x,\cdot)$ belongs to $\Lambda_0$ and $u(x,\cdot)$ belongs to $C([0,T])$ for (almost) every $x$, then we can define

$$\mathscr{W}[\lambda, u](x,t) := \int_0^\infty g(r, p_r[\lambda(x,\cdot), u(x,\cdot)](t))\,\mathrm{d}r. \tag{15}$$

In the following we will often write $\mathscr{W}(u)$ instead of $\mathscr{W}[\lambda, u]$ for brevity or when $\lambda$ is clear from the context.

We conclude this subsection with the convexification of the Preisach operator, i.e., that in a certain region, the convexity of the loops is satisfied (see [16, Section II.4, Proposition 4.22]).

Let $R > 0$ be fixed, set

$$\mathscr{D}_R := \{(r,v) \in \mathbb{R}_+^2 : |v| + r \leqslant R\}.$$

In addition to Assumption 1 we prescribe the following conditions.

ASSUMPTION 2.

1. $\frac{\partial\psi}{\partial v} \in L^\infty_{loc}(\mathbb{R}_+^2)$;
2. $A_R := \inf\{\psi(r,v); (r,v) \in \mathscr{D}_R\} > 0$.

Furthermore, denote

$$C_R := \sup\left\{\left|\frac{\partial}{\partial v}\psi(r,v)\right|; (r,v) \in \mathscr{D}_R\right\}.$$

Taking possibly a smaller $R > 0$, if necessary, we may assume that

$$K_R := \frac{1}{2}A_R - RC_R > 0. \tag{16}$$

We modify the density $\psi$ outside $\mathscr{D}_R$ and set

$$\psi_R(r,v) = \begin{cases} \psi(r,v) & \text{if} \quad (r,v) \in \mathscr{D}_R, \\ \psi(r, -R+r) & \text{if} \quad v < -R+r,\ r \leqslant R, \\ \psi(r, R-r) & \text{if} \quad v > R-r,\ r \leqslant R, \\ \psi(R,0) & \text{if} \quad r > R. \end{cases} \tag{17}$$

We define the convexified Preisach operator $\mathscr{W}_R$ by the formula

$$\mathscr{W}_R[\lambda,u](t) = \int_0^\infty \int_0^{p_r[\lambda,u](t)} \psi_R(r,v)\,\mathrm{d}v\,\mathrm{d}r \tag{18}$$

for $\lambda \in \Lambda_0$ and $u \in W^{1,1}(0,T)$. It has the property that all increasing trajectories of $\mathscr{W}_R$ are convex and all decreasing trajectories are concave, see [6]. This plays an important role in higher order energy inequalities.

### 2.3. Kirchhoff transformation

We apply the *Kirchhoff transformation*:

$$\mathscr{K} : p \mapsto u := \int_0^p \widetilde{k}(s)\mathrm{d}s.$$

Since $\widetilde{k}(s)$ is positive, this transformation is one–to–one with $\mathscr{K}^{-1}$ Lipschitz continuous. We introduce a new variable, $u := \mathscr{K}(p)$ and define

$$\nabla u = \nabla \mathscr{K}(p) = \widetilde{k}(p)\nabla p, \tag{19}$$

$$k(u) = \widetilde{k}(\mathscr{K}^{-1}(u)), \tag{20}$$

$$\theta = \widetilde{\mathscr{W}}[\lambda, \mathscr{K}^{-1}(u)]. \tag{21}$$

REMARK 1. By [17, Theorem 4.17], the mapping $u \mapsto \widetilde{\mathscr{W}}[\lambda, \mathscr{K}^{-1}(u)]$ is again a Preisach operator, $\mathscr{W}[\lambda,\cdot] = \widetilde{\mathscr{W}}[\lambda, \mathscr{K}^{-1}(\cdot)]$.

## 3. Model formulation

We assume that the porous medium is rigid, homogeneous and isotropic, that the fluid (water) is inviscid and incompressible. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n = 1, 2$ or 3, with a Lipschitz boundary $\partial\Omega$, representing the region occupied by the porous medium, see Figure 3. The boundary of $\Omega$ is divided into three parts, namely $\Gamma_1$ the impervious part, $\Gamma_2$ the part in contact with water and $\Gamma_3$ the part in contact with open air. For a positive $T$ we denote $Q = \Omega \times (0,T)$, $S_1 = \Gamma_1 \times (0,T)$, $S_2 = \Gamma_2 \times (0,T)$, $S_3 = \Gamma_3 \times (0,T)$, $S_T = \partial\Omega \times (0,T)$.

### 3.1. Single–porosity model

The law by which water flow through porous media can be described was found by Darcy experimentally. The law yields the following relation between the flux $q$ of water inside the porous medium, pressure $p$ and hydraulic conductivity $\widetilde{k}$

$$q = -\widetilde{k}\nabla(p + \rho g z), \tag{22}$$

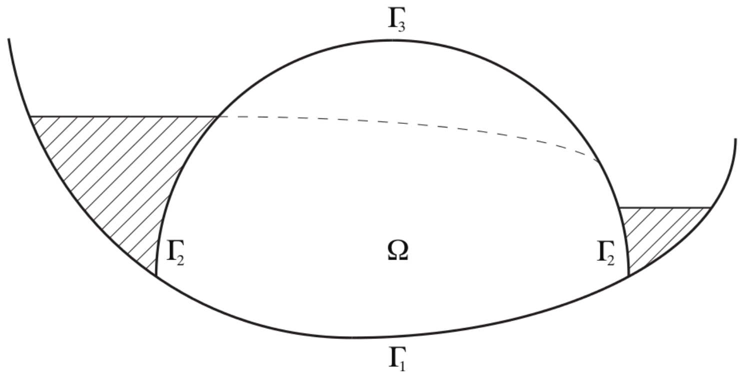


Figure 3: A porous dam with two reservoirs.

where $z$ is upward oriented unit vertical vector, $g$ is the gravity acceleration and $\rho$ is the density of the water. Combining the Darcy's law with the equation of continuity, we obtain the equation

$$\partial_t \theta - \nabla \cdot [\widetilde{k} \nabla (p + \rho g z)] = f, \tag{23}$$

where $f$ is water source term ($f > 0$), or sink term ($f < 0$) and $\theta$ is volumetric water content (or simply soil moisture). The lowest value that $\theta$ can take is $\theta_r$, the residual moisture content, which is the quantity that remains in a soil after any drainage imposed by the gravitational forced has ceased, and $\theta_s$ is the saturation volumetric moisture content. When the soil matrix is perfectly dried then $\theta = 0$, when the matrix is fully saturated with water, then $\theta = \theta_s < 1$ and finally for in–between states the matrix contains both air and water. Hence $\theta$ is bounded between $\theta_r$ and $\theta_s$, i.e., $0 < \theta_r \leqslant \theta \leqslant \theta_s < 1$.

The constitutive relation between the volumetric water content $\theta$ and the pressure $p$ is typically represented by a relation of the form

$$\theta(x,t) \in h(p)(x,t), \tag{24}$$

where $h : \mathbb{R} \to [0,1]$ is a maximal monotone graph as in Figure 4.

The relation (24) is oversimplified. Porous media exhibit hysteresis for cycle of soil wetting–drying process. Hysteretic behaviour means that at any point $x$ from the flow domain and any instant $t$, the soil moisture depends not only on the pressure, but also on the initial value of the soil moisture and on the previous evolution of the pressure at the same point.

Instead of relation (24) the hysteretic relation between volumetric water content $\theta$ and the pressure $p$ is taken into account, i.e., $\theta(x,t) = \mathscr{W}(p)(x,t)$, see Figure 5. Hysteresis is here represented by the Preisach operator.

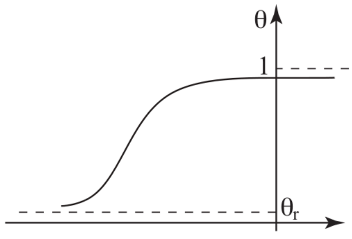


Figure 4: Water content versus pressure constitutive relation without hysteresis.

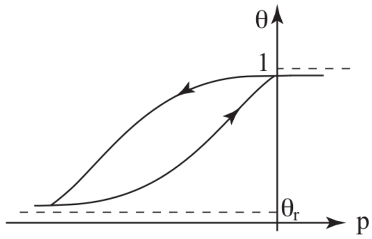


Figure 5: Water content versus pressure constitutive relation with hysteresis.

Suitable boundary conditions to equation (23) on the three boundary sets are considered:

$$-\widetilde{k}\nabla(p+\rho gz)\cdot\nu = 0 \qquad \text{on } S_1, \tag{25}$$

$$p = \widehat{p} \qquad \text{on } S_2, \tag{26}$$

$$\left\{\begin{array}{r} p \leqslant 0 \\ -\widetilde{k}\nabla(p+\rho gz)\cdot\nu \geqslant 0 \\ p[-\widetilde{k}\nabla(p+\rho gz)]\cdot\nu = 0 \end{array}\right. \qquad \text{on } S_3. \tag{27}$$

Here, $\nu$ denotes the outward normal unit vector. The condition (25) means that there is no flux through the impervious part. This condition may be replaced by a nonzero flux condition, for example $-\widetilde{k}\nabla(p+\rho gz)\cdot\nu = \mu(p-\widetilde{p})$, where $\mu > 0$ is constant and $\widetilde{p}$ is prescribed outer pressure. Hence, if the outer pressure $\widetilde{p}$ is higher than the inner pressure $p$, the fluid flows in and vice versa. The condition (26) is Dirichlet boundary condition, i.e., the case where $p$ is prescribed equal to $\widehat{p}$ and $\widehat{p}$ is non–negative function defined on $S_2$. If $\widehat{p} = 0$, i.e., $p = 0$, the boundary is considered to be fully saturated. The condition (27) says that because of capillary force the pressure $p$ is negative. In this case there is no flow across this part of the boundary. On the other hand where the pressure vanishes on $S_3$ water can only flow outward.

### 3.2. Dual porosity model

Flow in structured porous media is frequently described using dual porosity models. Such an approach assumes that the medium consists of two distinct pore homogeneous systems with separate hydraulic properties, the network of fractures and the matrix pore system. Variably saturated flow is considered for both, the fractures and the matrix pore system. The transfer of water across the fracture–matrix interface is described macroscopically using a first–order coupling term [10]. Darcian water flow in the dual porosity medium is governed by the following system of equations [10, 11]

$$\left\{\begin{array}{rcl} (\theta_1)_t &=& \Big(\widetilde{k}_1(p_1)(p_1)_z\Big)_z + [\widetilde{k}_1(p_1)]_z - \alpha_w \frac{(p_1-p_2)}{\omega} + q_1, \\ (\theta_2)_t &=& \Big(\widetilde{k}_2(p_2)(p_2)_z\Big)_z + [\widetilde{k}_2(p_2)]_z + \alpha_w \frac{(p_1-p_2)}{1-\omega} + q_2, \end{array}\right. \tag{28}$$

where the subscript 1 and 2, respectively, denotes the subsystem of fractures and matrix blocks, respectively, $\omega$ is a volume fraction, $q_1$ and $q_2$ are sink terms, $\alpha_w$ is the first order mass transfer coefficient, $\widetilde{k}_1$ and $\widetilde{k}_2$ are unsaturated hydraulic conductivities, $z$ is a position coordinate measured vertically upwards. In (28) we assume a hysteretic relation described by the Preisach operator, i.e. $\theta_i(z,t) = \mathscr{W}_i[\lambda_i, p_i](z,t)$, where $i = 1,2$. Boundary conditions are determined analogously to the case of single–porosity model, i.e, boundary conditions of Dirichlet type ($z = 0$)

$$p_i = 0 \tag{29}$$

in the case of the fully–saturated boundary (water table). In the case of unsaturated–saturated flows, the unilateral boundary conditions ($z = \ell$, $\ell > 0$) are prescribed

$$\left\{ \begin{aligned} p_i &\leqslant 0, \\ \widetilde{k}_i(p_i)(p_i + z)_z &\leqslant 0, \\ p_i \left[ \widetilde{k}_i(p_i)(p_i + z)_z \right] &= 0. \end{aligned} \right. \tag{30}$$

If the boundary is unsaturated, (30) yields a no flow boundary condition, and in the saturated case it acts as the Dirichlet boundary of a zero pressure head.

## 4. Existence results

### 4.1. Single–porosity flow

Applying the Kirchhoff transformation to the equation (23) without sink term and using notation (19)–(21), we obtain the following equation

$$\partial_t \theta = \nabla \cdot [\nabla u + k(u)\rho g z]. \tag{31}$$

The paper [14] discusses equation (31) without gravity term, i.e,

$$\partial_t \theta = \nabla \cdot \nabla u, \tag{32}$$

and with the nonzero flux boundary condition

$$\nabla u \cdot \nu = \widetilde{u}, \tag{33}$$

on $\partial\Omega$, $\widetilde{u} \in L^\infty(\partial\Omega \times (0,T))$ is a given outer pressure.
The following problem is solved.

PROBLEM 1. Let us consider a Preisach hysteresis operator $w := \mathscr{W}[\lambda, u]$ and let $u_0 \in L^2(\Omega)$, $\lambda : \Omega \to \Lambda$ be given initial data. We search for a function $u$ such that $u(x,0) = u_0(x)$ a.e. in $\Omega$ and for any $\phi \in H^1(\Omega)$, and for a.e. $t \in (0,T)$ we have

$$\int_\Omega \frac{\partial w}{\partial t} \phi \, dx + \int_\Omega \nabla u \nabla \phi \, dx = \int_{\partial\Omega} \widetilde{u} \phi \, d\sigma \tag{34}$$

The result is stated as follows.

THEOREM 1. *Let us assume operator $\mathscr{W}$ be the Preisach hysteresis operator introduced in (15) and satisfying Assumptions 1 and 2. And let $R > 0$ be fixed as in Subsection 2.2. Let $K \in [0,R]$ and $\lambda : \Omega \to \Lambda_K$ be given. Moreover $\widetilde{u} \in L^\infty(\partial\Omega \times (0,T))$, $\widetilde{u_t} \in L^2(\partial\Omega \times (0,T))$, $u_0 \in H^1(\Omega)$, $w_0 \in L^2(\Omega)$ and compatibility condition*

$$\int_\Omega \nabla u_0(x) \nabla \phi(x) \mathrm{d}x - \int_{\partial\Omega} \widetilde{u}(x,0)\phi(x) \mathrm{d}\sigma = 0$$

*holds for every $\phi \in H^1(\Omega)$. Set $\alpha := \max\{\|u_0\|_{H^1(\Omega)}, \|\widetilde{u}\|_{L^\infty(\partial\Omega\times(0,T))}, \|\widetilde{u_t}\|_{L^2(\partial\Omega\times(0,T))}\}$. Then there exists a constant $\beta > 0$ such that if $\alpha \leqslant \beta$, then Problem 1 has a unique solution such that*

$$u \in C^0(Q),$$
$$u_t \in L^2(0,T;V_*),$$

*where $V_* := \{u \in V : \int_\Omega u = 0\}$ is the space of functions with null average in $\Omega$.*

*Proof*. Existence of a solution is proved via time discretization, derivation of a priori estimates and using suitable energy inequalities, see [14]. To prove uniqueness we suppose that Problem 1 has two solutions $u_1$, $u_2$. We write equation (34) first for $u_1$, then for $u_2$. We substract the two equations and test by $\phi = H_m(u_1 - u_2)$, where $H_m$ is an approximation of the Heaviside function defined as

$$H_m(\varepsilon) = \begin{cases} 1, & \varepsilon \geqslant m, \\ \frac{\varepsilon}{m}, & 0 < \varepsilon < m, \\ 0, & \varepsilon \leqslant 0. \end{cases}$$

We obtain

$$\int_\Omega \partial_t(w_1 - w_2) H_m(u_1 - u_2) \, \mathrm{d}x + \int_\Omega \nabla(u_1 - u_2) \nabla H_m(u_1 - u_2) \, \mathrm{d}x = 0.$$

Since $H_m$ is nondecreasing, we have

$$\int_\Omega \nabla(u_1 - u_2) \nabla H_m(u_1 - u_2) \, \mathrm{d}x \geqslant 0,$$

thus

$$\int_\Omega \partial_t(w_1 - w_2) H_m(u_1 - u_2) \, \mathrm{d}x \leqslant 0.$$

Now let us pass to the limit for $m \to 0$ to obtain

$$\int_\Omega \partial_t(w_1 - w_2) H(u_1 - u_2) \, \mathrm{d}x \leqslant 0.$$

By Proposition 2 we have

$$\frac{d}{dt} \int_\Omega \int_0^\infty (g(r, \xi_r^1(t)) - g(r, \xi_r^2(t))^+ \, \mathrm{d}r \, \mathrm{d}x \leqslant 0.$$

Interchanging the roles of $u_1$ and $u_2$ we conclude that

$$0 \geqslant \frac{d}{dt}\int_{\Omega}\int_0^{\infty}(g(r,\xi_r^1(t)) - g(r,\xi_r^2(t))^+ \,\mathrm{d}r\,\mathrm{d}x + \frac{d}{dt}\int_{\Omega}\int_0^{\infty}(g(r,\xi_r^1(t)) - g(r,\xi_r^2(t))^- \,\mathrm{d}r\,\mathrm{d}x$$
$$= \frac{d}{dt}\int_{\Omega}\int_0^{\infty}|g(r,\xi_r^1(t)) - g(r,\xi_r^2(t)|\,\mathrm{d}r\,\mathrm{d}x.$$

Thus, the uniqueness of the weak solution follows, i.e., $u_1 = u_2$. □

### 4.2. Dual porosity flow

Let $T > 0$ and $\ell > 0$ be the fixed values, $\Omega = (0,\ell)$, $Q = \Omega \times (0,T)$. Applying the Kirchhoff transformation to the system (28)–(30), the resulting system we are going to solve consists of the following equations ($i = 1,2$):

$$(w_i)_t - (u_i)_{zz} - [k_i(u_i)]_z = F_i(\mathbf{u}) + q_i \qquad \text{in } Q, \tag{35}$$
$$u_i(0,t) = 0 \qquad \text{in } (0,T) \tag{36}$$

and

$$\left\{\begin{array}{r} u_i \leqslant 0 \\ (u_i)_z + k_i(u_i) \leqslant 0 \\ u_i\,[(u_i)_z + k_i(u_i)] = 0 \end{array}\right|_{z=\ell} \qquad \text{in } (0,T), \tag{37}$$

where

$$k_i(u_i) = \widetilde{k}_i(\kappa_i^{-1}(u_i)), \tag{38}$$
$$F_1(u_1,u_2) = -\alpha_w \frac{\kappa_2^{-1}(u_2) - \kappa_1^{-1}(u_1)}{\omega}, \tag{39}$$
$$F_2(u_1,u_2) = \alpha_w \frac{\kappa_2^{-1}(u_2) - \kappa_1^{-1}(u_1)}{1-\omega}, \tag{40}$$
$$w_i = \widetilde{\mathscr{W}_i}[\lambda_i, \kappa_i^{-1}(u_i)] \tag{41}$$

and $\mathbf{u} = [u_1,u_2]$. Here we suppose that all functions in (35)–(37) are smooth enough. We set $c_k$ the Lipschitz constant of $\mathbf{k} = [k_1,k_2]$ and $c_F$ the Lipschitz constant of $\mathbf{F} = [F_1,F_2]$. Let us define the closed and convex set

$$\mathscr{K} := \left\{\mathbf{v} \in \mathbb{V};\ v_j(\ell) \leqslant 0,\ j = 1,2\right\}, \tag{42}$$

where $\mathbb{V}$ be a closure of the space $\left\{\mathbf{v} \in C^\infty(\overline{\Omega})^2;\ \mathbf{v}(0) = \mathbf{0}\right\}$ in the norm of $W^{1,2}(\Omega)^2$.

The following existence result was stated and proved in [15]:

DEFINITION 2. *A vector function* $\mathbf{u} \in L^2((0,T);\mathscr{K})$*, such that* $\mathbf{u}_t \in L^2((0,T);\mathbb{V})$*, is a variational solution to the system* (35)–(37) *iff*

$$\int_0^T \langle \mathbf{w}_t, \varphi - \mathbf{u} \rangle \mathrm{d}t + \int_Q \mathbf{u}_z \cdot (\varphi - \mathbf{u})_z \, \mathrm{d}Q + \sum_{i=1}^{2} \int_Q k_i(u_i)(\varphi_i - u_i)_z \, \mathrm{d}Q \tag{43}$$
$$\geqslant \int_Q \mathbf{F}(\mathbf{u}) \cdot (\varphi - \mathbf{u}) \, \mathrm{d}Q + \int_Q \mathbf{q}(z,t) \cdot (\varphi - \mathbf{u}) \, \mathrm{d}Q$$

*holds for all* $\varphi \in L^2((0,T);\mathscr{K})$, $\mathbf{u}(0) = \mathbf{u}^0$ *and* $\mathbf{w}(0) = \mathbf{w}^0$ *a.e. in* $\Omega$.

THEOREM 2. *Let us assume* $w_i(z,t) = \mathscr{W}_i[\lambda_i, h_i](z,t)$ *is the Preisach hysteresis operator of the form (15) and satisfying Assumptions 1 and 2. And let* $R > 0$ *be fixed as in Subsection 2.2. Let* $K \in [0,R]$ *and* $\lambda : \Omega \to \Lambda_K$ *be given. Moreover* $q_1, q_2 \in W^{1,2}(Q)$, $\mathbf{u}_0 \in \mathbb{V}$ *and the following compatibility condition*

$$\int_\Omega \mathbf{u}_z^0(z) \cdot (\mathbf{v}(z))_z \, \mathrm{d}z + \sum_{i=1}^{2} \int_\Omega k_i(u_i^0(z))(v_i(z))_z \, \mathrm{d}z \tag{44}$$
$$- \int_\Omega \mathbf{F}(\mathbf{u}^0(z)) \cdot \mathbf{v}(z) \, \mathrm{d}z + \int_\Omega \mathbf{q}(z,0) \cdot \mathbf{v}(z) \, \mathrm{d}z = 0$$

*holds for every* $\mathbf{v} \in \mathbb{V}$. *Set* $\gamma := \max\{\|\mathbf{u}^0\|_{\mathbb{V}}, \|\mathbf{q}\|_{L^2(Q)^2}, \|\mathbf{q}_t\|_{L^2(Q)^2}, c_F, c_K\}$. *Then there exists* $\gamma_1 > 0$*, such that provided* $\gamma \leqslant \gamma_1$ *there exists the variational solution* $\mathbf{u} \in L^2((0,T);\mathscr{K})$ *to the system* (35)–(37)*, such that* $\mathbf{u}_t \in L^2((0,T);\mathbb{V})$.

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**AMS Subject Classification: 76S05, 47J40**

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*Lavoro pervenuto in redazione il 29.01.2015.*