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ON HEINZER-ROITMAN RINGS

Abstract. In this paper, we study commutative rings in which every regular maximal ideal is finitely generated provided some of its power is 2-generated. This notion is raised by Heinzer and Roitman in an integral domain [19, Question 3.1]. Roitman shows that coherent domains satisfy this property in 2001. We investigate the transfer of this notion to direct products, trivial ring extensions, pullbacks, and the amalgamation of rings. Our results generate new families of examples of non-coherent rings (with zerodivisors) satisfy this condition.

Keywords: *HR*-ring, power of maximal ideal, trivial rings extension, amalgamated algebra along an ideal.

1. Introduction

All rings in this paper are commutative with unity. First, we consider the following question: Suppose that some power M^n of the maximal ideal M of a ring R is finitely generated. Does it follow that M is finitely generated?

This question is raised by Robert Gilmer in [14, page 74] and was mentioned in a talk given by Robert Gilmer at the AMS meeting in Auburn, Alabama in November 1971 in an integral domain. It is also listed, for the case of a quasilocal integrally closed domain, as Problem 8 in the questions list on pages 174-176 in the 1973 Notices of the AMS from the problem session organized by Graham Evans at the January 1973 AMS meeting in Dallas.

In 2000, W. Heinzer and M. Roitman consider the following question in an integral domain (see [19, Question 3.1]):

Question 1: Suppose that some power M^n of the regular maximal ideal M of a ring R is 2-generated. Does it follow that M is finitely generated ?

A ring R is coherent if every finitely generated ideal of R is finitely presented; equivalently, if $(0 : a)$ and $I \cap J$ are finitely generated for every $a \in R$ and any two finitely generated ideals I and J of R. Examples of coherent rings are Noetherian rings, Boolean algebras, von Neumann regular rings, and Prüfer/semihereditary rings. For instance see [12, 21].

The answer to Question 1 for integral domains is negative, but it is positive for quasilocal integral domains (see [8, 28]). Also, recall that Roitman shows that Question 1 hold in every coherent domain (see [27, Theorem 1.8]). At this point, we make the following definition:

DEFINITION 1. *A commutative ring R is called a Heinzer-Roitman ring (briefly a HR-ring) if every regular maximal ideal m of R is finitely generated provided that some power of m is* 2*-generated.*

Let A be a ring and E an $A-$module. The trivial ring extension of A by E (also called idealization of E over A) is the ring $R := A \propto E$ whose underlying additive group is $A \times E$ with multiplication given by $(a,e)(a',e') = (aa', ae' + ea')$.

Trivial ring extensions have been studied extensively. Considerable work, part of is summarized in Glaz's book [12] and Huckaba's book [20], has been concerned with trivial ring extension. These extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. See for instance [2, 12, 20, 21].

Let T be a domain and let K be a field which is a retract of T, that is $T := K + M$ where M is a maximal ideal of T. Each subring D of K determines a subring $R := D + M$ of T. This construction arises frequently in algebra, especially in connection with counterexamples. The original of $D + M$ construction involved a valuation domain T with $K := T/M$, where M is the maximal ideal of T and $K \subset T$. A throughout account of results about $D + M$ construction can be find in [3, 4, 12].

Let A and B be two rings with unity, J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

called *the amalgamation of A and B along J with respect to f*. See for instance [6, 7].

In this paper, we investigate the transfer of this notion to direct products, trivial ring extensions, pullbacks, and the amalgamation of rings. Our results generate new families of examples of non-coherent rings (with zerodivisors) satisfy this condition.

2. Main results

First, we will construct a wide class of *HR*-rings.

REMARK 1. Any total ring of quotients is a *HR*-ring.

Proof. Straightforward. □

Now, we study the transfer of *HR*-property to direct product of rings.

THEOREM 1. *Let $(R_i)_{i=1,\dots,n}$ be a family of commutative rings. Then $R = \prod_{i=1}^{i=n} R_i$ is a HR-ring if and only if so is R_i for each $i = 1,\dots,n$.*

We need the following Lemma before proving Theorem 2.2.

LEMMA 1. *[23, Lemma 2.5]*
Let $(R_i)_{i=1,2}$ be a family of rings and E_i an R_i-module for $i = 1,2$. Then $E_1 \prod E_2$ is a finitely generated (resp., n-generated) $R_1 \prod R_2$-module if and only if E_i is a finitely generated (resp., n-generated) R_i-module for $i = 1,2$.

Proof of Theorem 2.2.
By induction on n, it suffices to prove the assertion for $n = 2$. Assume that $R_1 \prod R_2$ is a HR-ring and let M_1 be a regular maximal ideal of R_1 such that M_1^n is a 2-generated ideal of R_1 for some positive integer n. Then, $M := M_1 \times R_2$ is a regular maximal ideal of $R_1 \prod R_2$ and $M^n := M_1^n \times R_2^n$ is a 2-generated ideal of $R_1 \prod R_2$ by Lemma 2.3. Hence, $M := M_1 \times R_2$ is a finitely generated ideal of $R_1 \prod R_2$ since $R_1 \prod R_2$ is a HR-ring and so M_1 is a finitely generated ideal of R_1 by Lemma 2.3. Therefore, R_1 is a HR-ring.
The same argument shows that R_2 is a HR-ring.
Conversely, assume that R_1 and R_2 are HR-rings and let M be a regular maximal ideal of $R_1 \prod R_2$ such that M^n is 2-generated ideal of $R_1 \prod R_2$ for some positive integer n. Since $M := R_1 \prod M_2$ or $M := M_1 \prod R_2$, where M_i is a maximal ideal of R_i for $i = 1,2$, the conclusion follows easily as the above argument and from Lemma 2.3. □

Now, we study the transfer of the HR-property to trivial ring extension.

THEOREM 2. *Let A be a ring, E an $A-$module and $R := A \propto E$ be the trivial ring extension of A by E. Then:*

1. *Assume that A is an integral domain which is not a field, $K = qf(A)$ the quotient field of A, and E is a $K-$vector space. Then, R is a HR-ring if and only if so is A.*
2. *Assume that (A,M) is a local ring, E is a non-zero $A-$module with $ME = 0$. Then, R is a HR-ring.*

We need the following lemma before proving this Theorem 2.

LEMMA 2. *Let A be a ring, E an $A-$module, $R := A \propto E$ be the trivial ring extension of A by E, I be an ideal of A and F be a submodule of E such that $IE \subseteq F$. Then,*

1. *$(I \propto F)^n = I^n \propto (I^{n-1}F)$ for every positive integer n.*

2. *If I and F are finitely generated, then $I \propto F$ is a finitely generated ideal of R.*

3. *Assume that A is an integral domain which is not a field, $K = qf(A)$, E is a $K-$vector space, and let I be a nonzero ideal of A. Then $I \propto E$ is a finitely generated (resp., n-generated) ideal of R if and only if I is a finitely generated (resp., n-generated) ideal of A.*

Proof. (1) It is a particular case of [20, Theorem 25.1].
(2) Assume that $I := \sum_{i=1}^{i=n} Ax_i$ is a finitely generated ideal of A and $F := \sum_{i=1}^{i=m} Ae_i$ is a finitely generated A-module. Then, it is clear that $I \propto F = \sum_{i=1}^{i=n} R(x_i, 0) + \sum_{i=1}^{i=m} R(0, e_i)$, as desired.
(3) By [21, Lemma 3.3] and this completes the proof.

□

Proof of Theorem 2.4.

(1) Assume that A is an integral domain which is not a field, $K = qf(A)$, and E is a $K-$vector space. Assume that R is a HR-ring and let m be a nonzero maximal ideal of A such that m^n is a 2-generated ideal of A for some positive integer n. Hence, $(m \propto E)^n = m^n \propto E$ is a 2-generated ideal of R by Lemma 2.5(3) and so $m \propto E$ is a finitely generated ideal of R since R is a HR-ring and $m \propto E$ is a regular maximal ideal of R. Therefore, m is a finitely generated ideal of A and A is a HR-ring, as desired.

Conversely, assume that A is a HR-ring and let $M := m \propto E$ be a regular maximal ideal of R such that $(m \propto E)^n$ is a 2-generated ideal of R for some positive integer n, where m is a maximal ideal of A. Since $(m \propto E)^n = m^n \propto E$ is a 2-generated ideal of R, then m^n is a 2-generated ideal of A and so m is finitely generated since A is a HR-ring. Therefore, $M := m \propto E$ is a finitely generated ideal of R by Lemma 2.5(3), as desired.

(2) Assume that (A, M) is a local ring, E is a non-zero $A-$module with $ME = 0$. Then, R is a HR-ring by Remark 1 since it is a total ring of quotients (by [21, Proof of Theorem 2.6(1)]) and this completes the proof of Theorem 2. □

Theorem 2 enriches the literature with original examples of non-coherent HR-rings.

EXAMPLE 1. Let A be a coherent domain which is not a field, $K := qf(A)$, and let $R := A \propto K$ be the trivial ring extension of A by K. Then:

1. R is a HR-ring by Theorem 2(3).

2. R is not coherent by [21, Theorem 2.8(1)].

The following Theorem develops a result on the transfer of the HR-property to pullbacks, specially $D+M$-constructions.

THEOREM 3. *Let $T := K+M$ be a local domain, where K is a field and M is the unique maximal ideal of T; and $R := D+M$, where D is a subring of K. Then:*

1. *Assume that D is not a field. Then, R is a HR-ring if and only if so is D.*
2. *Assume that D is a field with $[K : D] = \infty$. Then, R is a HR-ring.*
3. *Assume that D is a field with $[K : D] < \infty$. Then, R is a HR-ring provided so is T.*

We need the following lemmas before proving Theorem 2.7.
Remark that a more general form of Lemma 2.8 below is proved by Cahen in [5, Lemma 3, page 507].

LEMMA 3. *Let T and R be as in Theorem 2.7. Then, every maximal ideal of R contains M.*

Proof. By [5, Lemma 3, page 507]. □

LEMMA 4. *Let T, D, K, M, and R be as in Theorem 2.7. Assume that D is not a field or D is a field and $[K : D] = \infty$. Then, for every positive integer n, M^n is never a finitely generated ideal of R.*

Proof. By [4, Lemma 1].

□

Proof of Theorem 2.7.
1) Assume that D is not a field. Then, any maximal ideal P of R has the form $P := P_0 + M$ ($=P_0R$) by Lemma 2.8, where P_0 is a nonzero maximal ideal of D (since D is not a field).
Assume that R is a HR-ring and let P_0 be a nonzero maximal ideal of D such that P_0^n is 2-generated for some positive integer n. Then, $P := P_0 + M$ ($=P_0R$) is a maximal ideal of R and $P^n := (P_0+M)^n = P_0^n + M = P_0^nR$ is a 2-generated ideal of R (since $P_0M = P_0KM = KM = M$). Hence, $P := P_0 + M$ is a finitely generated ideal of R since R is a HR-ring and so P_0 is a finitely generated ideal of D. Therefore, D is a HR-ring.
Conversely, assume that D is a HR-ring and let $P := P_0 + M$ ($=P_0R$) be a maximal ideal of R such that $P^n := (P_0+M)^n = P_0^n + M$ is a 2-generated ideal of R. Hence, P_0^n is a 2-generated ideal of D and so P_0 is finitely generated since D is a HR-ring. Therefore, $P := P_0R$ is a finitely generated ideal of R, as desired.

2) Assume that D is a field and $[K : D] = \infty$. Then, M is the only maximal ideal of R by Lemma 2.8. On the other hand, M^n is never a finitely generated ideal of R by Lemma 2.9. Therefore, R is a HR-ring.

3) Assume that D is a field with $[K : D] < \infty$. Then, M is the only maximal ideal of R by Lemma 2.8.
Assume that T is a HR-ring and assume that M^n is a 2-generated ideal of R for some positive integer n. Then, M^n is a 2-generated ideal of T and so M is a finitely generated ideal of T since T is a HR-ring. Therefore, M is a finitely generated ideal of R since T is a finitely generated R-module (by [10, Corollary (1.5)(4)]), as desired. And this completes the proof of Theorem 2.7. □

Theorem 2.7 enriches the literature with original examples of non-coherent HR-rings.

EXAMPLE 2. Let $R := \mathbb{Z} + X\mathbb{R}[[X]]$ and $T := \mathbb{R}[[X]]$. Then:

1. R is a HR-ring by Theorem 2.7(1).
2. R is not coherent by [12, Theorem 5.2.3].

The last Theorem develops a result on the transfer of the P-property to amalgamation of rings $A \bowtie^f J$.

THEOREM 4. *Let A be an integral domain, B be a ring, $f : A \to B$ be a ring homomorphism and J be a proper ideal of B such that J is a finitely generated ideal of $f(A) + J$ and $J \subseteq Rad(B)$. Then, $A \bowtie^f J$ is a HR-ring provided so is A.*

We need the following lemmas before proving this Theorem 2.11.

LEMMA 5. *Let (A,B) be a pair of rings, $f : A \to B$ be a ring homomorphism and J be a proper ideal of B such that $J \subseteq Rad(B)$. Then, $Max(A \bowtie^f J) = \{m \bowtie^f J / m \in Max(A)\}$.*

Proof. By [9, Proposition 2.5(5)]. □

LEMMA 6. *Let (A,B) be a pair of rings, $f : A \to B$ be a ring homomorphism and J be a proper ideal of B. Assume that J is a finitely generated ideal of $f(A) + J$ and let I be a finitely generated ideal of A. Then, $I \bowtie^f J$ is a finitely generated ideal of $A \bowtie^f J$.*

Proof. Assume that $I := \sum_{i=1}^{i=n} Ax_i$ is a finitely generated ideal of A, where $x_i \in I$ for all $i \in \{1, ...n\}$ and $J := \sum_{i=1}^{i=m} (f(A)+J)e_i$ is a finitely generated ideal of $f(A)+J$, where $e_i \in J$ for all $i \in \{1, ...m\}$. It is clear that $I \bowtie^f J = \sum_{i=1}^{i=n} (A \bowtie^f J)(x_i, f(x_i)) + \sum_{i=1}^{i=m} (A \bowtie^f J)(0, e_i)$, as desired. □

Proof of Theorem 2.11.
Assume that $J \subseteq Rad(B)$, J is a finitely generated ideal of $f(A)+J$ and let $M := m \bowtie^f J$ be a regular maximal ideal of $A \bowtie^f J$ (by Lemma 2.12) such that $M^n (:= (m \bowtie^f J)^n)$ is a 2-generated ideal of $A \bowtie^f J$ for some positive integer n. Hence, m^n is a 2-generated ideal of A and so m is finitely generated since A is a HR-domain. Therefore, $M := m \bowtie^f J$ is a finitely generated ideal of $A \bowtie^f J$ by Lemma 2.13, and this completes the proof of Theorem 2.11. □

Theorem 2.11 enriches the literature with original examples of non-coherent HR-rings.

EXAMPLE 3. Let $f : A \to B$ be a ring homomorphism, where A is a non-coherent HR-domain, and let J be a finitely generated proper ideal of $f(A)+J$. Then:

1. $A \bowtie^f J$ is a HR-ring by Theorem 2.11.

2. $A \bowtie^f J$ is a non-coherent ring by [9, Proposition 4.14(1)].

Finally, we show that the hypothesis "J is a finitely generated ideal of $f(A)+J$" cannot be removed in Theorem 2.11(1).

EXAMPLE 4. Let $A := K$ be a field, $B := K \propto E$ be the trivial ring extension of K by E (where E is a K-vector space with infinite rank), $J := 0 \propto E$ is the Jacobson radical of B, and $f : A \to B$ such that $f(a) = (a, 0)$. Then:

1. $A \bowtie^f J$ is a non-HR-ring since $(0 \bowtie^f J)^2 = 0$, $0 \bowtie^f J$ is a non finitely generated ideal of $A \bowtie^f J$, and $0 \bowtie^f J$ is a maximal ideal of $A \bowtie^f J$.

2. A is a HR-ring since it is a field.

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