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NEW EXAMPLES OF MAGNETIC MAPS INVOLVING TANGENT BUNDLES

Dedicated to Prof. Anna-Maria Pastore
with the occasion of her anniversary

Abstract. We produce new examples of magnetic maps, having as either source or target manifold the tangent bundle of a Riemannian manifold equipped with several Riemannian metrics. In particular we study when the canonical projection, a vector field and the tangent map are, respectively, magnetic maps.

1. Preliminaries

In a tentative to generalize the notion of magnetic trajectory on a Riemannian manifold, the authors define in [13] the notion of *magnetic maps*. As we see, both magnetic curves and harmonic maps can be obtained as particular situations of magnetic maps.

1.1. Magnetic maps

Let $f : N \longrightarrow M$ be a smooth map between two Riemannian manifolds (N,h) of dimension n and (M,g) of dimension m. Let ξ be a global divergence free vector field on N and ω be a 1-form on M. For the moment suppose that N is compact. The energy of f (or the Dirichlet integral of f) is known as

$$E(f) = \frac{1}{2}\int_N |df|^2 dv_h,$$

where dv_h denotes the volume element on N and $|df|$ is the Hilbert Schmidt norm of the differential df given in a point $p \in N$ by

$$|df_p|^2 = \sum_{i=1}^{n} g_{f(p)}\left(f_{*,p}e_i, f_{*,p}e_i\right).$$

Here $\{e_i;\ i = 1,\dots,n\}$ is an arbitrary orthonormal basis for T_pN.
A smooth map $f : (N,h) \longrightarrow (M,g)$ which is a critical point of $E(f)$ is called a *harmonic map* (see e.g. [10, 25]).
Let us now define the following functional for f associated to ξ and ω:

$$LH(f) = E(f) + \int_N \omega(df(\xi))dv_h. \tag{1}$$

Take a smooth variation $\{\mathcal{F}_\varepsilon\}_{\varepsilon\in I}$ of f, that is a smooth map $\mathcal{F} : N \times I \longrightarrow M$, such that $\mathcal{F}(p,0) = f(p)$. Here I is an open interval containing 0. For the sake of simplicity we use to write $f_\varepsilon(p) = \mathcal{F}(p,\varepsilon)$.

DEFINITION 1. The map f is called *magnetic* with respect to ξ and ω if it is a critical point of the Landau Hall integral $LH(f)$, i.e., the first variation $\frac{d}{d\varepsilon}LH(f_\varepsilon)\big|_{\varepsilon=0}$ is zero for any f_ε.

REMARK 1. In analogy to the definition of harmonic maps, one may replace "N-compact" by the condition "compact support variation".

Let (N,h), (M,g), ξ and ω as before. In [13] the authors prove the following.

THEOREM 1. *Let* $f:(N,h)\longrightarrow(M,g)$ *be a smooth map. Then* f *is a magnetic map with respect to* ξ *and* ω *if and only if it satisfies the Lorentz equation, that is*

$$\tau(f)=\phi(f_*\xi), \tag{2}$$

where $\tau(f):=\mathrm{trace}_h\nabla df$ *is the tension field of* f. *The endomorphism* ϕ, *called the Lorentz force associated to the potential 1-form* ω, *is defined by* $g(\phi(X),Y)=d\omega(X,Y)$, *for all* X,Y *tangent to* M.

Sometimes, equation (2) will be called the *magnetic equation*. Recall that on a Riemannian manifold (M,g) a *magnetic field* is defined by a closed 2-form F and the *Lorentz force* associated to F is a $(1,1)$ tensor field ϕ on M given by $g(\phi X,Y)=F(X,Y)$. The *magnetic trajectories* of F are curves γ satisfying the *Lorentz equation* $\nabla_{\gamma'}\gamma'=\phi\gamma'$. This equation is a particular case of equation (2) when N is an interval of $\mathbb{R}$ and $\xi=\frac{d}{dt}$, where t is the global coordinate on N. Magnetic curves were intensively studied in the last years by several geometers (including the authors of this article) in different ambient spaces.

REMARK 2. The Lorentz equation (2) was obtained from a variational principle assuming that the domain is compact and the 2-form F is exact. Since it has a tensorial character, one can define a magnetic map $f:(N,h)\longrightarrow(M,g)$ without the assumptions N compact and F exact (but only closed). Moreover, the condition "ξ is divergence free" will be also removed.

More precisely, let ξ be a global vector field on N, F be a magnetic field on M and ϕ the Lorentz force associated to F. Similarly to magnetic curves, we may also introduce a *strength* (i.e., a real number) in the equation. Hence, we give the following.

DEFINITION 2. We say that f is a *magnetic map* with strength $q\in\mathbb{R}$ associated to ξ and F if the Lorentz equation

$$\tau(f)=q\,\phi(f_*\xi) \tag{3}$$

is satisfied.

1.2. Tangent bundle of a Riemannian manifold

Let (M,g) be a Riemannian manifold of dimension n and $\pi:T(M)\longrightarrow M$ its tangent bundle. Denote by ∇ the Levi-Civita connection of g. For each $u\in T(M)$ we have the following decomposition of the tangent space $T_{\mathfrak{u}}T(M)$ (in $\mathfrak{u}$ at $T(M)$)

$$T_{\mathfrak{u}}T(M) = V_{\mathfrak{u}}T(M) \oplus H_{\mathfrak{u}}T(M),$$

where $V_{\mathfrak{u}}T(M) = \ker \pi_{*,\mathfrak{u}}$ is the vertical space and $H_{\mathfrak{u}}T(M)$ is the horizontal space at $\mathfrak{u}$ obtained by using ∇. A curve $\tilde{\gamma} : I \longrightarrow T(M)$, $t \longmapsto (\gamma(t), V(t))$ is *horizontal* if the vector field $V(t)$ is parallel along $\gamma = \pi \circ \tilde{\gamma}$. A vector on $T(M)$ is *horizontal* if it is tangent to a horizontal curve and *vertical* if it is tangent to a fiber. Locally, take a chart (U, x^i), $i = 1, \ldots, n = \dim(M)$ in $p \in M$, and consider the induced chart $(\pi^{-1}(U), x^i, y^i)$ on $T(M)$. If $\Gamma^k_{ij}(x)$ are the Christoffel symbols, then $\delta_i := \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^k_{ij}(x) y^j \frac{\partial}{\partial y^k}$ in $\mathfrak{u}$, for $i = 1, \ldots, n$, span $H_{\mathfrak{u}}T(M)$, while $\dot{\partial}_i := \frac{\partial}{\partial y^i}$, for $i = 1, \ldots, n$, span the vertical space $V_{\mathfrak{u}}T(M)$. We have defined the horizontal (resp. vertical) distributions HTM (resp. VTM) and the direct sum decomposition

$$TTM = HTM \oplus VTM$$

of the tangent bundle of $T(M)$. If $X \in \mathfrak{X}(M)$, denote by X^H (resp. X^V) the horizontal (resp. the vertical) lift of X to $T(M)$. See for more details [8].

Two classical examples of Riemannian metrics on $T(M)$ are well known, namely the Sasaki metric and the Cheeger-Gromoll metric, respectively. See for example [7, 12, 17, 23]. These metrics are only two possible choices inside a wide family of Riemannian metrics on $T(M)$, known as Riemannian *g-natural metrics*. A large number of papers related to this topic have been published so far, but we emphasize only few of them: [1, 3, 16, 20].

The **Sasaki metric** is defined uniquely by the following relations

$$g_S(X^H, Y^H) = g_S(X^V, Y^V) = g(X, Y) \circ \pi,\ g_S(X^H, Y^V) = 0, \tag{4}$$

for all X, Y tangent to M. We give here, for later use, the expression of the Levi-Civita connection ${}^S\nabla$ of the Sasaki metric in terms of an adapted local basis defined above:

$$\begin{cases} {}^S\nabla_{\dot{\partial}_i} \dot{\partial}_j = 0, & {}^S\nabla_{\delta_i} \delta_j = \Gamma^h_{ij} \delta_h - \frac{1}{2} R^h_{0ij} \dot{\partial}_h, \\ {}^S\nabla_{\delta_i} \dot{\partial}_j = \Gamma^h_{ij} \dot{\partial}_h + \frac{1}{2} R^h_{i0j} \delta_h, & {}^S\nabla_{\dot{\partial}_i} \delta_j = \frac{1}{2} R^h_{j0i} \delta_h, \end{cases} \tag{5}$$

where $R^h_{kij} \frac{\partial}{\partial x^h} = R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^k}$ and "0" stands for the contraction with $\mathfrak{u}$, namely $R^h_{0ij} = R^h_{kij} y^k$. See e.g. [4, Chapter 9] and [15].

The **Cheeger-Gromoll metric** is given by

$$\begin{cases} g_{CG(p,\mathfrak{u})}(X^H, Y^H) = g_p(X, Y),\ g_{CG(p,\mathfrak{u})}(X^H, Y^V) = 0, \\ g_{CG(p,\mathfrak{u})}(X^V, Y^V) = \frac{1}{1+2t} \left[g_p(X, Y) + g_p(X, \mathfrak{u}) g_p(Y, \mathfrak{u}) \right], \end{cases} \tag{6}$$

for all X, Y tangent to M at $p = \pi(\mathfrak{u})$. We have denoted by t the energy density in $(p, \mathfrak{u})$ on $T(M)$, that is $\frac{1}{2}\, g_p(\mathfrak{u}, \mathfrak{u})$. Again, we give, for later use, the expression of the

Levi-Civita connection ${}^{CG}\nabla$ of the Cheeger-Gromoll metric:

$$\begin{cases} {}^{CG}\nabla_{\dot{\partial}_i}\dot{\partial}_j = \frac{1}{1+2t}\left(g_{ij}C - g_{i0}\dot{\partial}_j - g_{j0}\dot{\partial}_i\right) + \frac{1}{(1+2t)^2}\left(g_{ij} + g_{i0}g_{j0}\right)C, \\ {}^{CG}\nabla_{\delta_i}\delta_j = \Gamma^h_{ij}\delta_h - \frac{1}{2}R^h_{0ij}\dot{\partial}_h, \quad {}^{CG}\nabla_{\dot{\partial}_i}\delta_j = \frac{1}{2(1+2t)}R^h_{j0i}\delta_h, \\ {}^{CG}\nabla_{\delta_i}\dot{\partial}_j = \Gamma^h_{ij}\dot{\partial}_h + \frac{1}{2(1+2t)}R^h_{i0j}\delta_h. \end{cases} \tag{7}$$

See e.g. [12, 24]. The vector field C is the Liouville vector field that will be defined in what follows. In fact, two global vector fields may be defined on $T(M)$:

(a) The *geodesic spray* ξ of the connection ∇ is the unique tangent vector at $(p,\mathfrak{u})$ which is horizontal and satisfies $\pi_{*,(p,\mathfrak{u})}\xi_{(p,\mathfrak{u})} = \mathfrak{u}$. As consequence, any integral curve $(x(t),y(t))$ of ξ through the point $(p,\mathfrak{u})$ obeys $\dot{x}(0) = \mathfrak{u}$ and $\nabla_{\dot{x}(0)}y = 0$. Therefore, if $\mathfrak{u}$ is a tangent vector at p to M and $\gamma : t \mapsto \gamma(t)$ is the geodesic through $p = \gamma(0)$ with $\dot{\gamma}(0) = \mathfrak{u}$, it is the projection under π of the integral curve $\tilde{\gamma} : t \mapsto \tilde{\gamma}(t)$ of ξ through $\mathfrak{u}$.

(b) The Liouville vector field C on $T(M)$ is the infinitesimal generator of the flow given by homotheties on each fiber, that is $(t,\mathfrak{u}) \in \mathbb{R} \times T_pM \mapsto e^t\mathfrak{u} \in T_pM$. The vector field C is the unique vertical vector field on $T(M)$ satisfying $K_{(p,\mathfrak{u})}C_{(p,\mathfrak{u})} = \mathfrak{u}$, where K is the connection map. See e.g. [8]. It is also called, sometimes, *the radial vector field* on $T(M)$.

2. Canonical projection $\pi : T(M) \longrightarrow M$ as magnetic map

Let (M,g) be a Riemannian manifold of dimension n and let $T(M)$ be its tangent bundle. In [2] the authors find a necessary and sufficient condition for the harmonicity of the canonical projection from $T(M)$ equipped with an arbitrary Riemannian g-natural metric to (M,g). In particular they prove that if the Riemannian g-natural metric is such that the horizontal and the vertical distributions are orthogonal, then $\pi : T(M) \longrightarrow M$ is harmonic. This is the case of the two metrics we have already mentioned, that is the Sasaki metric g_S and the Cheeger-Gromoll metric g_{CG}.

Let ξ be a global vector field on $T(M)$, F a magnetic field on M whose Lorentz force is ϕ and $q \neq 0$ an arbitrary real number. Then π is a magnetic map with strength q with respect to ξ and F if and only if $\pi_*\xi \in \ker\phi$. In particular, if ξ is vertical (as the Liouville vector field C) then π is magnetic with respect to that ξ and any magnetic field F.

In the following we will consider a nonlinear connection on $T(M)$. More precisely, let (M,g) be a Riemannian manifold of dimension n and $\pi : T(M) \longrightarrow M$ its tangent bundle, as before. The vertical subspaces (respectively the vertical distribution) on $T(M)$ depend only on the differential structures of M and $T(M)$. Thus, their definition is the same as in Section 1.2, that is at $\mathfrak{u} \in T_pM$, we have

$$V_{\mathfrak{u}}T(M) := \ker\, \pi_{*,\mathfrak{u}}. \tag{8}$$

A *horizontal distribution* HTM on $T(M)$ is a supplementary distribution to VTM, that is

$$T_{u}TM = V_{u}TM \oplus H_{u}TM.$$

A *nonlinear connection* on $T(M)$ is a vector bundle morphism $v : TTM \longrightarrow VTM$ such that $v \circ \iota = Id_{VTM}$, where $\iota : VTM \longrightarrow TTM$ is the canonical inclusion. Hence, the kernel of the morphism v is the horizontal subbundle HTM. See, for details, e.g. [6, Part I] and the references therein. In the following, the names "horizontal" and "vertical" have the obvious meaning.

It is clear that, due to (8), the restriction $\pi_{*,u} : H_{u}TM \longrightarrow T_{p}M$, where $p = \pi(u)$, is an isomorphism of vector spaces. Its inverse map is called the *horizontal lift* induced by the nonlinear connection.

A local chart (U, φ, x) on M induces on $T(M)$ a local chart $(\pi^{-1}(U), \Phi, (x,y))$ with respect to which we have a local adapted frame in HTM defined by the following vector fields

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x,y)\frac{\partial}{\partial y^j}, \text{ for } i = 1, \ldots, n. \tag{9}$$

They represent the horizontal lifts of the canonical basis $\frac{\partial}{\partial x^i}$ (defined on U). The functions N_i^j are known as the *coefficients of the nonlinear connection* defined by HTM.

Hence, for any (local) vector field $X = X^i(x)\frac{\partial}{\partial x^i}$ on M, the horizontal and the vertical lifts at u are given by:

$$X_u^H = X^i(x) \left.\frac{\delta}{\delta x^i}\right|_u, \; X_u^V = X^i(x) \left.\frac{\partial}{\partial y^i}\right|_u . \tag{10}$$

Note that the classical situation is obtained when $N_i^j(x,y) = \Gamma_{ik}^j(x)y^k$, where $\Gamma_{ik}^j(x)$ are the coefficients of the Levi-Civita connection of g.

One can define on $T(M)$ a Riemannian metric g_s, of Sasaki type, as follows

$$g_s(X^V, Y^V) = g(X,Y) \circ \pi, \; g_s(X^H, Y^V) = 0, \; g_s(X^H, Y^H) = g(X,Y) \circ \pi. \tag{11}$$

It follows that π is a Riemannian submersion.

Denote by $\sigma(\pi)$ the second fundamental form of the projection π, that is $\sigma(\pi) = \nabla d\pi$. We immediately have

$$\begin{cases} \sigma(\pi)\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \frac{1}{2}g^{hk}\left(\frac{\partial g_{ij}}{\partial x^k} - g_{il}B\Gamma_{jk}^l - g_{jl}B\Gamma_{ik}^l\right)\frac{\partial}{\partial x^h}, \\ \sigma(\pi)\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = 0, \end{cases}$$

where $B\Gamma_{jk}^l(x,y) = \frac{\partial N_k^l}{\partial y^j}$ represent the coefficients of the Berwald connection. See e.g. [6].

For an arbitrary nonlinear connection, the fibers of the projection π are not, in general, totally geodesic. Yet, in the classical case when $N_i^j(x,y) = \Gamma_{ik}^j(x)y^k$, the projection π

has all its fibers totally geodesic. However, the projection $\pi : (T(M), g_{\mathrm{s}}) \longrightarrow (M, g)$ is harmonic if and only if π has minimal fibers. See e.g. [19].

Interesting results concerning the geometry of the tangent bundle $T(M)$ equipped as before, may be obtained when the functions $N^l_k(x,y)$ are polynomials in variables y^i, for $i = 1, \ldots, n$. Therefore, let $N^l_k(x,y)$ be a polynomial of degree 2 in y of the following form:

$$N^l_k(x,y) = R^l_{skh}(x)y^s y^h + \Gamma^l_{ks}(x)y^s + T^l_{ks}(x)y^s + \Psi^l_k(x), \tag{12}$$

where R^l_{skh} (respectively T^l_{ks} and Ψ^l_k) are the coefficients of a $(1,3)$ (respectively $(1,2)$ and $(1,1)$) tensor field on M.

Compute the tension field of π. Since $\tau(\pi) = \mathrm{trace}_{g_{\mathrm{s}}}(\sigma(\pi))$ we have

$$\tau(\pi) = g^{ij}\sigma(\pi)\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \frac{1}{2}\, g^{kh}\left[T^l_{kl}(x) - \big(R^l_{lks}(x) + R^l_{skl}(x)\big)y^s\right]\frac{\partial}{\partial x^h}.$$

Let now ξ be a global vector field on $T(M)$, F a magnetic field on M whose Lorentz force is ϕ and $q \neq 0$ a real number. Then π is a magnetic map if and only if the magnetic equation (with strength q) is satisfied, that is $\tau(\pi) = q\,\phi(\pi_*\xi)$. Of course if ξ is vertical then π is magnetic if and only if it is harmonic. Take $\xi|_{\mathfrak{u}} = y^k \frac{\delta}{\delta x^k}|_{\mathfrak{u}}$, where $\mathfrak{u} = y^k \frac{\partial}{\partial x^k}|_p$, $p = \pi(\mathfrak{u})$. Then, the magnetic equation becomes

$$g^{kh}\left[T^l_{kl}(x) - \big(R^l_{lks}(x) + R^l_{skl}(x)\big)y^s\right] = 2q\phi^h_s(x)y^s,$$

which is equivalent to

$$T^l_{kl}(x) - \big(R^l_{lks}(x) + R^l_{skl}(x)\big)y^s = 2q\, F_{sk}(x)y^s,$$

for all $\mathfrak{u} \in T(M)$. Here ϕ^h_s (respectively F_{sk}) are coefficients of the Lorentz force ϕ (respectively of the magnetic field F) in the local chart on M.

It follows that

$$\begin{cases} T^l_{kl}(x) = 0, \\ R^l_{lks}(x) + R^l_{skl}(x) = 2q\, F_{sk}(x). \end{cases} \tag{13}$$

We can formulate the following result.

THEOREM 2. *Let* $\pi : (T(M), g_{\mathrm{s}}) \longrightarrow (M, g)$ *defined as before, where the nonlinear connection is defined by three tensor fields on M, namely* $R \in \mathcal{T}^1_3(M)$, $T \in \mathcal{T}^1_2(M)$ *and* $\Psi \in \mathcal{T}^1_1(M)$, *as in* (12). *Let* ξ *be the canonical horizontal vector field on* $T(M)$ *and* F *a magnetic field on M. Then* π *is a magnetic map with strength q with respect to* ξ *and* F *if and only if*

(i) *the 1-form* $\mathrm{trace}_g\big(\bullet \mapsto T(X, \bullet)\big)$ *vanishes;*

(ii) $2qF(X,Y) = \mathrm{trace}_g\big(\bullet \mapsto R(X,Y)\bullet\big) + \mathrm{trace}_g\big(\bullet \mapsto R(X,\bullet)Y\big)$.

REMARK 3. We have put $T(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = T^l_{ij}(x)\frac{\partial}{\partial x^l}$ and $R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^k} = R^l{}_{kij}(x)\frac{\partial}{\partial x^l}$.

EXAMPLE 1. If J is a skew symmetric $(1,1)$ tensor field on M and a is a 1-form on M, then $T = a \otimes J$, defined by $T(X,Y) = a(X)JY$ fulfills condition (i) of Theorem 2.

EXAMPLE 2. Suppose that R is a curvature-like tensor on M, namely it has all the symmetries as the Riemannian curvature tensor (including the first Bianchi identity). Suppose that its Ricci tensor $\rho(R)$ is skew symmetric. Then, the condition (ii) is satisfied if and only if $\rho(R)$ is a closed 2-form and $F = \frac{1}{2q}\rho(R)$.

EXAMPLE 3. We give two other situations when the condition (ii) of Theorem 2 is fulfilled:

(a) $R = 2q\, g \otimes \phi$, that is $N^l_k(x,y) = g_{kh}(x)\phi^l_s(x)y^h y^s + \Gamma^l_{ks}(x)y^s + T^l_{ks}(x)y^s + \Psi^l_k(x)$,

(b) $R = \frac{2q}{n+1}\, F \otimes I$, that is $N^l_k(x,y) = F_{kh}(x)y^h y^l + \Gamma^l_{ks}(x)y^s + T^l_{ks}(x)y^s + \Psi^l_k(x)$,

where I is the identity tensor on $T(M)$. Here F is the magnetic field on M and ϕ is the corresponding Lorentz force.

EXAMPLE 4. If V is a Killing vector field on M, consider $R = g \otimes \nabla V$.
We have $R^h_{kij} = g_{ij}\nabla_k V^h$. Therefore $R^l_{lks} = g_{ks}\nabla_l V^l = 0$ (since V is divergence free) and $R^l_{skl} = \nabla_s V_k$, where $V_k = g_{kl}V^k$. As V is Killing, we obtain that F_0 is a 2-form, where $F_0(X,Y) = g(\nabla_X V, Y)$. In case when F_0 is closed, condition (ii) is satisfied with $F = \frac{1}{2q}\, F_0$. This happens in several situations, for example when:

- M is the Euclidean n-space $\mathbb{E}^n(x^1, \ldots, x^n)$ and $V = \frac{\partial}{\partial x^j}$ (which is parallel);
- M is the Euclidean 3-space $\mathbb{E}^3(x,y,z)$ and $V = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$;
- M is a Sasakian manifold and V is the Reeb vector field; in such a case F_0 is the contact 2-form.

3. When a vector field is a magnetic map?

Let (M,g) be a compact, orientable Riemannian manifold of dimension n and $(T(M), g_S)$ its tangent bundle equipped with the Sasaki metric. On $T(M)$ we can also define an almost complex structure J_S by

$$J_S X^H = X^V,\ J_S X^V = -X^H, \text{ for all } X \in \mathfrak{X}(M). \tag{14}$$

It is known that $(T(M), g_S, J_S)$ is an almost Käherian manifold. See [8]. Hence, the Kähler 2-form $\Omega_S = g_S(J_S\,\cdot,\ \cdot)$ may be considered as a magnetic field on $T(M)$.

Let $\xi \in \mathfrak{X}(M)$ be thought as a map from (M,g) to $(T(M), g_S, J_S)$. We can compute the differential of this map, that is $\xi_{*,p} : T_pM \longrightarrow T_{(p,\xi(p))}T(M)$. If X is tangent to M we have

$$\xi_{*,p}X(p) = X^H_{\xi(p)} + (\nabla_X \xi)^V_{\xi(p)}. \tag{15}$$

We easily find the well known result: *The map* $\xi:(M,g)\longrightarrow(T(M),g_S)$ *is an isometric immersion if and only if* $\nabla\xi=0$. See e.g. [9, Proposition 2.1].

The aim of this section is to find conditions under which the vector field $\xi:M\longrightarrow T(M)$ is a magnetic map with respect to ξ itself and the magnetic field Ω_S on $T(M)$. To do this, we first compute the Hilbert-Schmidt norm $||d\xi||_g$. In a point $p\in M$, take an orthonormal frame $\{e_k\}_{k=1,\dots,n}$. We have

$$||d\xi||_g^2(p)=\sum_{k=1}^{n}\left[g_S(e_k^H,e_k^H)+g_S((\nabla_{e_k}\xi)^V,(\nabla_{e_k}\xi)^V)\right]_{\xi(p)}=n+||\nabla\xi||^2.$$

Then we find the Dirichlet energy of ξ on M, that is

$$E(\xi)=\frac{n}{2}\,\mathsf{vol}(M)+\frac{1}{2}\int_M||\nabla\xi||_g^2dv_g, \tag{16}$$

where dv_g is the volume form on M and $\mathsf{vol}(M)$ is the volume of M. The number

$$\mathcal{B}(\xi)=\int_M||\nabla\xi||_g^2dv_g \tag{17}$$

is called the *total bending* of the vector field ξ. We know that $\xi:(M,g)\longrightarrow(T(M),g_S)$ is harmonic if and only if ξ is parallel. In such a case it is an absolute minimum of the energy functional $E(\xi)$. See e.g. [14]. However, if M is not compact, tension field of ξ must be computed. In the book of Dragomir and Perrone [9], the authors write the following formula

$$\tau(\xi)=-\left\{(\mathrm{trace}_gR(\nabla_\bullet\xi,\xi)\bullet)^H+(\Delta_g\xi)^V\right\}\circ\xi. \tag{18}$$

Here Δ_g denotes the rough Laplacian on vector fields, defined by

$$\Delta_gX=-\sum_{k=1}^{n}\left[\nabla_{e_k}\nabla_{e_k}X-\nabla_{\nabla_{e_k}e_k}X\right],$$

where $\{e_k\}_{k=1,\dots,n}$ is an orthonormal frame on M.

The magnetic equation (3) writes as

$$\tau(\xi)=q\,J_S(\xi_*\xi),\ q\in\mathbb{R}. \tag{19}$$

Using (14) and (15) we get

$$J_S(\xi_*\xi)=\xi^V-(\nabla_\xi\xi)^H.$$

Now we plug this expression into (19), use (18) and then identify the horizontal and the vertical parts. We may state the following.

THEOREM 3. *Let* (M,g) *be a Riemannian manifold and* $(T(M),g_S,J_S)$ *its tangent bundle endowed with the usual almost Kählerian structure. Let* ξ *be a vector field on M. Then* ξ *is a magnetic map with strength q associated to* ξ *itself and the Kähler magnetic field* Ω_S *if and only if the following conditions hold:*

$$\mathrm{trace}_gR(\nabla_\bullet\xi,\xi)\bullet)=q\,\nabla_\xi\xi, \tag{20}$$

$$\Delta_g\xi=-q\,\xi. \tag{21}$$

COROLLARY 1. *Let $\xi : M \longrightarrow T(M)$ be a non-zero, non-harmonic magnetic map. If M is compact and oriented, then the strength q is strictly negative.*

Proof. The operator Δ_g satisfies the following identity

$$g(\Delta_g\xi,\xi) = \frac{1}{2}\Delta(||\xi||^2) + ||\nabla\xi||^2,$$

where Δ is the Beltrami Laplace operator on functions. Using (21) we obtain

$$\frac{1}{2}\Delta(||\xi||^2) + q||\xi||^2 + ||\nabla\xi||^2 = 0.$$

If $q \geq 0$ we observe that $\Delta(||\xi||^2) \leq 0$ and hence $||\xi||^2$ is a harmonic function. Then $q = 0$ and $\nabla\xi = 0$. Hence ξ is a harmonic map, which is false. Therefore, $q < 0$. □

Interesting results may be obtained in the case where the curvature tensor has a certain expression. In what follows, we will describe two such situations.

1. Suppose that the manifold M is of constant sectional curvature c. Then its curvature tensor writes as

$$R(X,Y)Z = c\big(g(Y,Z)X - g(X,Z)Y\big), \text{ for all } X,Y,Z \in \mathfrak{X}(M).$$

We successively have

$$\begin{aligned}\mathrm{trace}_g R(\nabla_\bullet\xi,\xi)\bullet &= \sum_{k=1}^{n} R(\nabla_{e_k}\xi,\xi)e_k = c\sum_{k=1}^{n}\left[g(\xi,e_k)\nabla_{e_k}\xi - g(e_k,\nabla_{e_k}\xi)\xi\right]\\ &= c\left[\nabla_\xi\xi - (\mathrm{div}\ \xi)\xi\right].\end{aligned}$$

Here div ξ states for the divergence of the vector field ξ.
Consequently, the equation (20) becomes

$$(c-q)\nabla_\xi\xi - c(\mathrm{div}\ \xi)\xi = 0. \tag{22}$$

Let us observe the following:

(i) If $c = 0$, that is M is flat (not necessarily compact) it follows that ξ is self-parallel.

(ii) If $c \neq 0$, then we have

$$\left(1 - \frac{q}{c}\right)\nabla_\xi\xi = (\mathrm{div}\ \xi)\xi.$$

Hence, for $q = c$, the vector field ξ is divergence free.

2. Suppose now that M is a Sasakian space form. We briefly explain this structure. See for more details the Blair's book [4].

A (φ,ξ,η)-*structure* on a manifold M is defined by a field φ of endomorphisms of tangent spaces, a vector field ξ and a 1-form η satisfying

$$\eta(\xi) = 1, \ \ \varphi^2 = -I + \eta\otimes\xi, \ \ \varphi\xi = 0, \ \ \eta\circ\varphi = 0.$$

If (M,φ,ξ,η) admits a compatible Riemannian metric g, namely

$$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y),\ \text{for all}\ X,Y\in\mathfrak{X}(M),$$

then M is said to have an *almost contact metric structure*, and (M,φ,ξ,η,g) is called an *almost contact metric manifold.* Consequently, ξ is unitary and $\eta(X)=g(\xi,X)$, for any $X\in\mathfrak{X}(M)$. Denoting by ∇ the Levi Civita connection associated to g, the Sasakian manifold (M,φ,ξ,η,g) is characterized by

$$(\nabla_X\varphi)Y=-g(X,Y)\xi+\eta(Y)X,\ \text{ for any }\ X,Y\in\mathfrak{X}(M).$$

As a consequence, we have

$$\nabla_X\xi=\varphi X,\ \forall X\in\mathfrak{X}(M). \tag{23}$$

A plane section Π at $p\in M^{2n+1}$ is called a φ-section if it is invariant under φ_p. The sectional curvature $k(\Pi)$ of a φ-section is called the φ-*sectional curvature* of M^{2n+1} at p. A Sasakian manifold $(M^{2n+1},\varphi,\xi,\eta,g)$ is said to be a *Sasakian space form* and denote this by $M^{2n+1}(c)$, if it has constant φ-sectional curvature c. In such a case, the curvature tensor is given by

$$\begin{aligned}R(X,Y)Z=&\tfrac{c+3}{4}\big(g(Y,Z)X-g(X,Z)Y\big)\\&+\tfrac{c-1}{4}\Big(\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\\&+g(Z,\varphi Y)\varphi X-g(Z,\varphi X)\varphi Y+2g(X,\varphi Y)\varphi Z\Big).\end{aligned} \tag{24}$$

Our aim is to study the conditions when the Reeb vector field ξ is magnetic, that is it satisfies the condition in Theorem 3.

In order to compute the left side in (20) we consider an adapted orthonormal frame $\{e_i,\varphi e_i,\xi\ :\ i=1,\dots,n\}$, where $e_i\in\ker\eta$ for all $i=1,\dots,n$. From (24) we obtain

$$R(X,\xi)Z=\eta(Z)X-g(X,Z)\xi.$$

Now, since $\nabla_\bullet\xi=\varphi\,\bullet$ we get

$$R(\nabla_{e_i}\xi,\xi)e_i=-g(\varphi e_i,e_i)\xi=0,\ \text{for all}\ i=1,\dots,n.$$

Hence the equation (20) is automatically satisfied in the light of the relation $\nabla_\xi\xi=0$. Concerning (21) we should compute $\Delta_g\xi$. We have

$$\nabla_X\nabla_Y\xi-\nabla_{\nabla_XY}\xi=\nabla_X(\varphi Y)-\varphi\nabla_XY=(\nabla_X\varphi)Y=-g(X,Y)\xi+\eta(Y)X.$$

Setting successively $X=Y=e_i$, $X=Y=\varphi e_i$ and $X=Y=\xi$ we get

$$\Delta_g\xi=2n\xi.$$

Observe that we should take $q=-2n$.

4. Magnetic maps between tangent bundles

Let (M,g) be a Riemannian manifold and $T(M)$ its tangent bundle. A $(1,1)$-tensor field L on a Riemannian manifold (M,g) defines a map $L : T(M) \longrightarrow T(M)$, by $(p,\mathfrak{u}) \mapsto (p,L_p\mathfrak{u})$, for any $p \in M$ and $\mathfrak{u} \in T_pM$. An interesting problem is to determine conditions under which L is a magnetic map with respect to the geodesic flow (resp. the Liouville vector field) and the usual magnetic field Ω_S on $T(M)$. See [11] for the study of harmonicity of endomorphism fields on a pseudo-Riemannian manifold, when the complete lift metric on $T(M)$ is considered either on the source or on the target manifold.

Let us first consider $L = I$, the identity map of $T(M)$. More precisely, we study when the map $I : (T(M),G) \longrightarrow (T(M),g_S,J_S)$ is magnetic, for some metrics G on $T(M)$.

It is well known that the identity is a harmonic map when the same metric is considered either on the source, or on the target manifold. Therefore, the case $G = g_S$ is not interesting. Let us see what happens when G is the Cheeger-Gromoll metric g_{CG}.

We compute the second fundamental form $\sigma(I)$ on the two distributions VTM and HTM, respectively, using the formulas (5) and (7) to get

$$\begin{cases} \sigma(I)\left(\dot{\partial}_i,\dot{\partial}_j\right) = -\frac{1}{1+2t}\left(g_{ij}C - g_{i0}\dot{\partial}_j - g_{j0}\dot{\partial}_i\right) - \frac{1}{(1+2t)^2}\left(g_{ij}+g_{i0}g_{j0}\right)C, \\ \sigma(I)\left(\delta_i,\delta_j\right) = 0. \end{cases}$$

Then, we obtain the tension field $\tau(I)$, computing the *trace* of $\sigma(I)$, that is

$$\tau(I) = \left((1+2t)g^{ij} - y^iy^j\right)\sigma(I)\left(\dot{\partial}_i,\dot{\partial}_j\right) = \frac{2(1-n)(1+t)}{1+2t}\,C.$$

So I is a magnetic map (with strength $q = 1-n$) with respect to the vector field $\frac{2+g(\mathfrak{u},\mathfrak{u})}{1+g(\mathfrak{u},\mathfrak{u})}\xi$ and the magnetic field Ω_S. Here $\xi_{\mathfrak{u}} = y^i\delta_i|_{\mathfrak{u}}$ denotes (as above) the geodesic flow on $T(M)$.

Suppose now that we have a nonlinear connection on the tangent bundle of a Riemannian manifold M with the coefficients $N_k^l(x,y)$ as described in Section 2. Let us write down the (local) expressions of the Levi-Civita connection ${}^{\mathsf{s}}\nabla$ of the Sasaki type metric g_{s}:

$$\begin{cases} {}^{\mathsf{s}}\nabla_{\dot{\partial}_i}\dot{\partial}_j = -\frac{1}{2}g^{hk}\left(\frac{\partial g_{ij}}{\partial x^k} - g_{il}\frac{\partial N_k^l}{\partial y^j} - g_{jl}\frac{\partial N_k^l}{\partial y^i}\right)\bar{\delta}_h, \\ {}^{\mathsf{s}}\nabla_{\dot{\partial}_i}\bar{\delta}_j = -\frac{1}{2}R_{kj}^l g_{li}g^{kh}\bar{\delta}_h + \frac{1}{2}g^{hk}\left(\frac{\partial g_{ik}}{\partial x^j} - g_{il}\frac{\partial N_j^l}{\partial y^k} - g_{lk}\frac{\partial N_j^l}{\partial y^i}\right)\dot{\partial}_h, \\ {}^{\mathsf{s}}\nabla_{\bar{\delta}_i}\dot{\partial}_j = -\frac{1}{2}R_{ki}^l g_{lj}g^{kh}\bar{\delta}_h + \frac{1}{2}g^{hk}\left(\frac{\partial g_{jk}}{\partial x^i} - g_{jl}\frac{\partial N_i^l}{\partial y^k} + g_{lk}\frac{\partial N_i^l}{\partial y^j}\right)\dot{\partial}_h, \\ {}^{\mathsf{s}}\nabla_{\bar{\delta}_i}\bar{\delta}_j = \Gamma_{ij}^h\bar{\delta}_h + \frac{1}{2}R_{ji}^h\dot{\partial}_h. \end{cases}$$

See also [18]. Here we made the following notations:

$$\dot{\partial}_i = \left(\frac{\partial}{\partial x^i}\right)^V = \frac{\partial}{\partial y^i}\,, \bar{\delta}_i = \left(\frac{\partial}{\partial x^i}\right)^H = \frac{\partial}{\partial x^i} - N_i^j(x,y)\frac{\partial}{\partial y^j} \text{ and } R_{ij}^h = \bar{\delta}_i(N_j^h) - \bar{\delta}_j(N_i^h).$$

Looking back to (12), let us consider the coefficients of the nonlinear connection be given in the following way

$$N_k^l(x,y) = \Gamma_{ks}^l(x)y^s + \Psi_k^l(x),$$

where Ψ_k^i are the coefficients of a $(1,1)$ tensor field Ψ on M. Consequently we have

$$\bar{\delta}_i = \delta_i - \Psi_i^k(x)\dot{\partial}_k.$$

Compute the second fundamental form $\sigma(I)$ restricted to the two distributions VTM and HTM, respectively. Here the horizontal distribution HTM is that corresponding to the nonlinear connection on the source manifold $(T(M), g_{\mathrm{s}})$.
We find

$$\sigma(I)\left(\dot{\partial}_i, \dot{\partial}_j\right) = 0,$$

$$\sigma(I)\left(\bar{\delta}_i, \bar{\delta}_j\right) = -\tfrac{1}{2}\left(\Psi_i^k R_{j0k}^h + \Psi_j^k R_{i0k}^h\right)\delta_h - \tfrac{1}{2}\left(R_{0ji}^h + \nabla_j \Psi_i^h + \nabla_i \Psi_j^h\right)\dot{\partial}_h.$$

Therefore, the tension field $\tau(I)$ is given by

$$\tau(I) = \mathrm{trace}_{g_{\mathrm{s}}}\sigma(I) = -g^{ij}\Psi_i^k R_{j0k}^h \delta_h - g^{ij}\nabla_i \Psi_j^h \dot{\partial}_h.$$

If $\xi = A^h(x,y)\delta_h + B^h(x,y)\dot{\partial}_h$ is a vector field on $T(M)$ then I satisfies the magnetic equation $\tau(I) = J_S\xi$ if and only if

$$A^h = -g^{ij}\nabla_i \Psi_j^h \text{ and } B^h = g^{ij}\Psi_i^k R_{j0k}^h.$$

Hence I is a magnetic map with respect to ξ and Ω_S (with strength $q = 1$) if and only if ξ is given by

$$\xi = -\Big(\mathrm{trace}_g(\nabla_\bullet \Psi)\bullet\Big)^H + \Big(\mathrm{trace}_g R(\mathrm{u}, \Psi\bullet)\bullet - \Psi\mathrm{trace}_g(\nabla_\bullet \Psi)\bullet\Big)^V.$$

Of course, the horizontal lift is considered with respect to the nonlinear connection.
As particular case we consider $\Psi_k^l = \delta_k^l$. Then I is a magnetic map if and only if ξ is given by

$$\xi = (Q\mathrm{u})^V.$$

Here Q is the Ricci operator on M defined by $g(QX,Y) = Ric(X,Y)$, where Ric is the usual Ricci tensor on M.

The study of the magnetic equation for an arbitrary endomorphism $L: T(M) \longrightarrow T(M)$, when the Sasaki metric (on both source and target) is considered, will be done in a subsequent paper.

Let us consider now an arbitrary smooth map $f: M \longrightarrow N$ between two Riemannian manifolds (M,g) and (N,h). Let $F = df: T(M) \longrightarrow T(N)$ be the differential of f, defined by

$$F(p,\mathrm{u}) = (f(p), f_{*,p}\mathrm{u}), \text{ for every } p \in M \text{ and } \mathrm{u} \in T_pM.$$

On $T(M)$ (respectively on $T(N)$) set the Sasakian metric g_S (respectively h_S). The tension field of F was computed by Sanini in [22]. In order to fix certain notations, let us briefly sketch some computations.

Let (U,x) be a local chart on M and $(\pi_M^{-1}(U),x,y)$ be the induced chart on $T(M)$, where $\pi_M : T(M) \longrightarrow M$ is the canonical projection. In the same manner define local charts on N and $T(N)$, respectively. From now on the indices i,j,k range from 1 to $m = \dim(M)$, while indices α,β,γ range from 1 to $n = \dim(N)$ and so, we distinguish geometric objects defined on M from those defined on N. We have

$$\begin{cases} F_*\dot{\partial}_i = f_i^\alpha \dot{\partial}_\alpha \\ F_*\delta_i = f_i^\alpha \delta_\alpha + \sigma(f)^\gamma(\partial_i,\partial_j)y^j\dot{\partial}_\gamma, \end{cases}$$

where $\sigma(f)$ is the second fundamental form of f, that is

$$\sigma(f)(\partial_i,\partial_j) = \left(f_{ij}^\gamma + {}^h\Gamma_{\alpha\beta}^\gamma(f(x))f_i^\alpha f_j^\beta - {}^g\Gamma_{ij}^k(x)f_k^\gamma\right)\partial_\gamma.$$

Here we set $f_i^\alpha := \frac{\partial f^\alpha}{\partial x^i}$ and $f_{ij}^\alpha := \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j}$. Note that $\sigma(f)(X,Y)$ is a section in the induced bundle $f^{-1}T(N)$ (over M), for all X, Y tangent to M and it is symmetric in X and Y.

Computing the tension field $\tau(F)$ we find

$$\tau(F) = \Big[\tau(f) + \operatorname{trace}_g {}^hR\big(df(\mathfrak{u}),\sigma(f)(\bullet,\mathfrak{u})\big)df(\bullet)\Big]^H + \big[\big(\operatorname{div}\sigma(f)\big)(\mathfrak{u})\big]^V, \tag{25}$$

where

$$\big(\operatorname{div}\sigma(f)\big)(X) = \operatorname{trace}_g\,\big(\bar{\nabla}_\bullet\sigma(f)\big)(\bullet,X).$$

Here $\bar{\nabla}$ is the induced connection in the vector bundle $S_2(T^*(M)) \otimes f^{-1}T(N)$ and it is defined by

$$\big(\bar{\nabla}_X\sigma\big)(Y,Z) = {}'\nabla_X\sigma(Y,Z) - \sigma({}^g\nabla_XY,Z) - \sigma(Y,{}^g\nabla_XZ),$$

where ${}'\nabla$ is the induced connection in the induced bundle $f^{-1}T(N)$.

Observe that div $\sigma(f)$ is a 1-form on M with values in the induced bundle $f^{-1}T(N)$.

Now we consider on $T(N)$ the magnetic field Ω_S (with the corresponding Lorentz force J_S) and let q be a real number.

If C_M denotes the Liouville vector field on $T(M)$, then the magnetic equation (for F) with respect to C_M and Ω_S may be written as $\tau(F) = qJ_SF_*C_M$. Identifying the vertical and the horizontal parts respectively, we obtain

$$\begin{cases} \big(\operatorname{div}\sigma(f)\big)(\mathfrak{u}) = 0, \\ \tau(f) + \operatorname{trace}_g {}^hR\big(df(\mathfrak{u}),\sigma(f)(\bullet,\mathfrak{u})\big)df(\bullet) - q\,df(\mathfrak{u}) = 0. \end{cases} \tag{26}$$

The left side of the second equation represents a polynomial of second order in y^i, hence all the coefficients vanish. Therefore, we get:

THEOREM 4. *Under the previous hypothesis, df is magnetic with respect to C_M and Ω_S if and only if the following conditions are satisfied:*

f *is harmonic;*
$q = 0$ *that is F is harmonic, or f is a constant map;*
$\mathrm{trace}_g\, {}^hR\big(df(X), \sigma(f)(\bullet, X)\big)df(\bullet) = 0$, *for all X tangent to M;*
$\mathrm{div}\ \sigma(f) = 0$.

If ξ_M denotes the geodesic spray on $T(M)$, then the magnetic equation (for F) with respect to ξ_M and Ω_S yields the following equations for f:

$$\begin{cases} \tau(f) + \mathrm{trace}_g\, {}^hR\big(df(u), \sigma(f)(\bullet, u)\big)df(\bullet) + q\, \sigma(f)(u,u) = 0 \\ \big(\mathrm{div}\ \sigma(f)\big)(u) = q\, df(u). \end{cases} \tag{27}$$

With the same argument as before, we obtain:

THEOREM 5. *Under the hypothesis above, df is magnetic with respect to ξ_M and Ω_S if and only if the following conditions are satisfied:*

f *is harmonic;*
$\mathrm{trace}_g\, {}^hR\big(df(X), \sigma(f)(\bullet, X)\big)df(\bullet) + q\, \sigma(f)(X,X) = 0$, *for all X tangent to M;*
$\mathrm{div}\ \sigma(f) = q\, df$.

Let us consider the case when f is an isometric immersion, namely when M is an immersed submanifold in N.

REMARK 4. If M is totally geodesic in N, that is $\sigma(f) = 0$ and $m \geq 1$, then the condition that df is magnetic with respect to ξ_M and Ω_S implies that df is harmonic.

REMARK 5. If $f : M \longrightarrow N$ is a minimal hypersurface in the real space form N, then df is magnetic with respect to ξ_M and Ω_S implies df is harmonic.

Proof. We use the Codazzi equation $(\bar{\nabla}_X\sigma)(Y,Z) - (\bar{\nabla}_Y\sigma)(X,Z) = \big({}^hR(X,Y)Z\big)^{\perp}$. Here $\bar{\nabla}$ is known as the Van der Waerden-Bortolotti connection; see e.g. [21, Chapter 3, §9]. Since N is a real space form, the right side of the above equation vanishes. When M is a hypersurface, the normal bundle is 1-dimensional; choose ν unitary and normal to M. Set $X = Z = e_i$, where $\{e_i\}_{i=1,\dots,m}$ is an orthonormal basis on M. We get

$$\big(\bar{\nabla}_{e_i}\sigma\big)(Y, e_i) - \big(\bar{\nabla}_Y\sigma\big)(e_i, e_i) = 0.$$

Summing up on $i = 1, \dots, m$ we find $\mathrm{div}\ \sigma(f)(Y) - \big[\nabla^{\perp}_Y(m\vec{H}) - 2\sigma({}^g\nabla_Y e_i, e_i)\big] = 0$.
We can choose the orthonormal basis on M consisting in eigenvectors of the shape operator. As M is minimal, i.e. the mean curvature vector $\vec{H}$ vanishes, we obtain $\mathrm{div}\ \sigma(f) = 0$.
It follows that $q = 0$ and hence df is harmonic. $\square$

THEOREM 6. *If M is a submanifold in N such that* div $\sigma(f) = q\, df$, *then either f is constant or $q = 0$.*

Proof. For X tangent to M we know that div $\sigma(f)(X)$ is a normal vector. Thus div $\sigma(f) = 0$ and $qdf = 0$. Hence the conclusion. □

THEOREM 7. *If M is a submanifold in the real space form $N(c)$ such that* $\mathrm{trace}_g\, {}^hR\big(df(X), \sigma(f)(\bullet, X)\big)df(\bullet) + q\,\sigma(f)(X,X) = 0$, *then either f is totally geodesic or $q = c$.*

Proof. Since $df(X)$ is tangent and $\sigma(Y,Z)$ is normal to f we compute

$$\sum_{k=1}^{m} {}^hR\big(df(X), \sigma(f)(e_k, X)\big)df(e_k) = -c\sum_{k=1}^{m} \sigma(f)\big(g(X, e_k)e_k, X\big) = -c\sigma(f)(X,X).$$

As $\sigma(f)$ is symmetric we immediately get the conclusion. □

As consequence of these remarks we should ask:

Can we determine hypersurfaces in a space form $N(c)$ such that df is harmonic?

Doing similar computations as before we obtain either $c = 0$ or M is totally geodesic in N. Therefore, we formulate another question, that is:

Can we determine hypersurfaces in space forms $N(c)$ satisfying div $\sigma(f) = 0$?

Straightforward computations as in Remark 5 lead to div $\sigma(f) = m\nabla^{\perp}\vec{H}$. So, $\sigma(f)$ is divergence free if and only if M has parallel mean curvature vector. In case when M is a hypersurface in N, the condition div $\sigma(f) = 0$ implies that M is a CMC hypersurface in N.

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References

[1] ABBASSI M. T. K. AND CALVARUSO G., *g-natural contact metrics on unit tangent sphere bundles*, Monatsh. Math. **151** 2 (2007), 89–109.

[2] ABBASSI M. T. K. AND CALVARUSO G., *Harmonic maps having tangent bundles with g-natural metrics as source or target*, Rend. Sem. Mat. Univ. Politec. Torino **68** 1 (2010), 37–56.

[3] ABBASSI M.T.K. AND SARIH M., *On some hereditary properties of Riemannian g-natural metrics on tangent bundles of Riemannian manifolds*, Differential Geom. Appl. **22** 1 (2005), 19–47.

[4] BLAIR D.E., *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Math. 203, Birkhäuser, Boston-Basel-Berlin 2002.

[5] BOECKX E. AND VANHECKE L., *Harmonic and minimal vector fields on tangent and unit tangent bundles*, Differential Geom. Appl. **13** (2000), 77–93.

[6] BUCATARU I. AND MIRON R., *Finsler-Lagrange geometry. Applications to dynamical systems*, Editura Academiei Romane, 2007.

[7] CHEEGER J. AND GROMOLL D., *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. **96** (1972), 413–443.

[8] DOMBROWSKI P., *On the Geometry of the Tangent Bundle*, J. Reine Angew. Math. **210** (1962), 73–88.

[9] DRAGOMIR S. AND PERRONE D., *Harmonic vector fields. Variational principles and differential geometry*, Elsevier 2012.

[10] EELLS J. AND LEMAIRE L., *Selected Topics in Harmonic Maps*, Conf. Board Math. Sci 50, American Mathematical Society, Providence, RI 1983.

[11] GARCIA-RÍO E., VANHECKE L. AND VÁSQUEZ-ABAL M.E., *Harmonic endomorphism fields*, Illinois J. Math. **41** (1997), 23–30.

[12] GUDMUNDSSON S. AND KAPPOS E., *On the Geometry of the tangent bundle with the Cheeger-Gromoll metric*, Tokyo J. Math. **25** (2002), 75–83.

[13] INOGUCHI J. AND MUNTEANU M.I., *Magnetic maps*, Int. J. Geom. Methods Mod. Phys. **11** 6 (2014), art. 1450058, (22 pages).

[14] ISHIHARA T., *Harmonic sections of tangent bundles*, J. Math. Tokushima Univ. **13** (1979), 23–27.

[15] KOWALSKI O., *Curvature of the Induced Riemannian Metric on the Tangent Bundle of a Riemannian Manifold*, J. Reine Angew. Math. **250** (1971), 124–129.

[16] KOWALSKI O. AND SEKIZAWA M., *Natural transformations of Riemannian metrics on manifolds to metrics on tangent bundles. A classification*, Bull. Tokyo Gakugei Univ. (4) **40** (1988), 1–29.

[17] MUSSO E. AND TRICERRI F., *Riemannian metrics on tangent bundles*, Ann. Mat. Pura Appl. **150** (1988), 1–19.

[18] ONICIUC C., *The Nonlinear Connections and Harmonicity*, Gen. Math. **5** (1997), 287–296.

[19] ONICIUC C., *Nonlinear connections on tangent bundle and harmonicity*, Ital. J. Pure Appl. Math. **6** (1999), 109–122.

[20] OPROIU V., *Some new geometric structures on the tangent bundle*, Publ. Math. Debrecen **55** 3–4 (1999), 261–281.

[21] POSTNIKOV M.M., *Geometry VI: Riemannian Geometry*, Encyclopaedia of Mathematical Sciences vol. 91, Springer, New-York–Berlin 2001.

[22] SANINI A., *Applicazioni armoniche tra i fibrati tangenti di varietà riemanniane*, Boll. U.M.I. **6** 2A (1983), 55–63.

[23] SASAKI S., *On the differential geometry of tangent bundles of Riemannian manifolds*, Tohoku Math. J. **10** (1958), 338–354.

[24] SEKIZAWA M., *Curvatures of tangent bundles with Cheeger-Gromoll metric*, Tokyo J. Math. **14** (1991), 407–417.

[25] URAKAWA H., *Calculus of Variations and Harmonic Maps*, Translations of Mathematical Monographs Vol. 132, American Mathematical Society, Providence, RI, 1993.

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