Geom. Struc. on Riem. Man.-Bari
Vol. 73/1, 3–4 (2015), 127 – 150

C. Medori - A. Spiro[1]

STRUCTURE EQUATIONS OF LEVI DEGENERATE CR HYPERSURFACES OF UNIFORM TYPE

Abstract. We explicitly determine the structure equations of 5-dimensional Levi 2-non-degenerate CR hypersurfaces, using our recently constructed canonical Cartan connection for this class of CR manifolds. We also give an outline of the basic properties of absolute parallelisms and Cartan connections, together with a brief discussion of the absolute parallelisms for such CR manifolds existing in the literature.

1. Introduction

Let M be a $(2n+1)$-dimensional CR hypersurface, that is a manifold endowed with a pair $(\mathcal{D}, J)$ formed by

a) a distribution $\mathcal{D} \subset TM$ of codimension 1,

b) a smooth family J of complex structures $J_x : \mathcal{D}_x \longrightarrow \mathcal{D}_x$, satisfying the integrability condition, i.e. the complex distribution $\mathcal{D}^{10} \subset T^{\mathbb{C}}M$ of the $(+i)$-eigenspaces of the J_x is involutive.

We recall that the Levi form of M is defined as follows. For $x \in M$, let ϑ be a 1-form on a neighbourhood $\mathcal{U}$ of x with $\ker \vartheta_y = \mathcal{D}_y$ at each point y of $\mathcal{U}$. The *Levi form at x* is the symmetric bilinear map

$$\mathcal{L}_x : \mathcal{D}_x \times \mathcal{D}_x \to \mathbb{R}, \qquad \mathcal{L}_x(v,w) := d\vartheta_x(v, Jw),$$

which is well known to be J_x-invariant and independent on the choice ϑ, up to a scalar multiple. If the dimension of $\ker \mathcal{L}_x$ is constant over M, we call the CR hypersurface *of uniform type*. The case $\dim\ker \mathcal{L}_x = 0$ occurs if and only if the distribution $\mathcal{D}$ is contact and in this case $(M, \mathcal{D}, J)$ is called *Levi-nondegenerate*. If $\mathcal{D}$ is of uniform type with $\dim\ker \mathcal{L}_x > 0$ at all points, we call it *uniformly Levi-degenerate*.

The simplest examples of uniformly Levi degenerate CR hypersurfaces are given by the cartesian products $\overline{M} \times S$ of a Levi-nondegenerate CR hypersurface $(\overline{M}, \overline{\mathcal{D}}, \overline{J})$ and an m-dimensional complex manifold (S, J^S). The natural CR structure of $\overline{M} \times S$ is the pair $(\mathcal{D}, J)$ defined by

$$\mathcal{D}_x := \overline{\mathcal{D}}_{\bar{x}} + T_s S, \qquad J_x := \bar{J}_{\bar{x}} \times J^S_s \text{ for all } x = (\bar{x}, s) \in \overline{M} \times S.$$

If a CR hypersurface is locally CR equivalent with a cartesian product of this kind around any point, we say that *it admits local CR straightenings*.

[1]*Acknowledgments*. This research was partially supported by the Project MIUR "Real and Complex Manifolds: Geometry, Topology and Harmonic Analysis" and by GNSAGA of INdAM.

Under appropriate uniformity assumptions on the CR structure, any uniformly Levi degenerate CR hypersurface $(M,\mathcal{D},J)$ is equipped with a nested sequence of complex distributions

$$\ldots \subset \mathcal{F}_2 \subset \mathcal{F}_1 \subset \mathcal{F}_0 \subset \mathcal{F}_{-1} \subset \mathcal{F}_{-2} = T^{\mathbb{C}}M, \tag{1}$$

in which $\mathcal{F}_{-1} := \mathcal{D}^{10}$ and all other subdistribution $\mathcal{F}_i$, $i \geq 0$, are inductively defined in a special way that implies that

$$[\mathcal{F}_i, \mathcal{F}_j] \subset \mathcal{F}_{i+j} \qquad \text{for each } i, j \geq -2$$

(here, we assume that $\mathcal{F}_{i+j} := T^{\mathbb{C}}M$ if $i+j \leq -2$). This nested sequence of distributions necessarily stabilises after a finite number of steps and, by a result of Freeman ([11]), it has the following crucial property: $(M,\mathcal{D},J)$ *admits local CR straightenings if and only if the first stabilising distribution* $\mathcal{F}_k$*, that is such that* $\mathcal{F}_{k+\ell} = \mathcal{F}_k$ *for all* $\ell \geq 0$*, is non trivial*.

The uniformly Levi degenerate CR hypersurfaces with trivial stabilising distribution $\mathcal{F}_k$ (hence, with no CR straightenings) are called *Levi* $(k+1)$*-nondegenerate*. This notion extends the concept of Levi nondegeneracy, since the Levi 1-nondegenerate hypersurfaces are precisely the Levi nondegenerate hypersurfaces in the usual sense.

The smallest dimension for a CR hypersurface to be uniformly Levi degenerate and with no CR straightenings is 5. By dimension counting, any such 5-dimensional CR hypersurfaces is *2-nondegenerate*. For conciseness, we call the CR manifolds of this kind *girdled CR manifolds*.

The class of girdled CR manifolds and the associated equivalence problem has been the main object of investigation in several recent papers. In particular, in [14] we proved the existence of a canonical Cartan connection for any girdled CR manifold, obtaining in this way a solution to the equivalence problem and a complete set of invariants for this class of CR manifolds. Independently and with preprints posted almost at the same time, Isaev and Zaitsev presented in [13] an alternative solution, hence another set of invariants, for the same equivalence problem. Isaev and Zaitsev's solution is however not corresponding to a Cartan connection. Shortly after, a third solution and another set of invariants has been given by Pocchiola in [16].

Due to this, in several occasions, the following question has been posed: *Is there a way to compare one to the other such solutions to the equivalence problem of girdled CR manifolds?*

Having this question in mind, in this paper we newly present our solution to the equivalence problem for girdled CR manifolds, in a way that allows an immediate comparison with the other existing solutions. More precisely, we first provide a quick review of the notions of equivalence problems, absolute parallelisms and Cartan connections. The intention of such overview is twofold: to fix unambiguously the meaning of all terms of our discussion and to clarify the main reasons of interests for solutions to equivalence problems coming from canonical Cartan connections. We then describe in detail the canonical Cartan connections of girdled CR manifolds introduced in [14], giving the explicit expressions of the corresponding structure equations and making manifest all curvature restrictions that characterise such connections.

2. Equivalence problems and Cartan connections

Let G be a class of geometric structures, that is of pairs $(M,\mathcal{S})$ formed by a manifold M with some geometric datum $\mathcal{S}$ of fixed type (as, for instance, a Riemannian metric g, a distribution $\mathcal{D}$, a CR structure $(\mathcal{D},J)$, etc.). Given two geometric structures $(M,\mathcal{S})$, $(M',\mathcal{S}')$ in G, the *local equivalences around points $x \in M$ and $x' \in M'$* are the local diffeomorphisms $f : \mathcal{U} \to \mathcal{U}'$ between neighbourhoods $\mathcal{U}$, $\mathcal{U}'$ of x, x', transforming $\mathcal{S}|_{\mathcal{U}}$ into $\mathcal{S}'|_{\mathcal{U}'}$. The *equivalence problem for the class* G it is the query for an algorithm that establishes when, given two points, there exists a local equivalence around such two points.

A standard approach to such problem consists in looking for constructions that give for each $(M,\mathcal{S})$ in G a unique triple $(P,(X_i),\widetilde{(\cdot)})$ made of:

i) a bundle $\pi : P \to M$ over the manifold M;

ii) an absolute parallelism (X_i) on P, i.e. an ordered N-tuples of vector fields $(X_1,\dots,X_N)$ that gives a frame at each tangent space T_uP;

iii) an operator $\widetilde{(\cdot)}$ which maps each local equivalence $f : \mathcal{U} \to M'$ into a bundle diffeomorphism $\widetilde{f} : \mathcal{V} \subset P \to P'$ that projects onto f,

such that the following holds: *a local diffeomorphism $F : \mathcal{V} \subset P \to P'$ between the bundles P, P' of two structures $(M,\mathcal{S})$, $(M',\mathcal{S}')$ in* G *maps the associated parallelisms (X_i), (X_i') one into the other if and only if $F = \widetilde{f}$ for some local equivalence f.*

Triples $(P,(X_i),\widetilde{(\cdot)})$ with this property are called *canonical absolute parallelisms for the class* G and any algorithm that provides canonical absolute parallelisms solves the equivalence problem for G in the following sense.

Any absolute parallelism (X_i) is uniquely determined by the N-tuple of its dual 1-forms $(\omega^1,\dots,\omega^N)$ and a local diffeomorphism transforms one absolute parallelism into another if and only it it transforms the corresponding dual coframe fields one into the other. We now observe that the differentials $d\omega^i$ admit unique expansions of the form $d\omega^i = \sum_{j<k} c^i_{jk}\omega^j \wedge \omega^k$. These are the so-called *structure equations of the parallelism* (X_i) and the functions c^i_{jk} are the associated *first order invariants*. Note that the invariants c^i_{jk} can be explicitly determined from the vector fields X_i by recalling that

$$c^i_{jk} = d\omega^i(X_j,X_k) = -\omega^i([X_j,X_k]). \tag{1}$$

Their differentials have the form $dc^i_{jk} = c^i_{jk|\ell_1}\omega^{\ell_1}$ and the functions $c^i_{jk|\ell_1}$ are called *second order invariants*. Their differentials $dc^i_{jk|\ell_1} = c^i_{jk|\ell_1\ell_2}\omega^{\ell_2}$ define the *third order invariants* $c^i_{jk|\ell_1\ell_2}$ and so on. By a fundamental result of Cartan and Sternberg, if appropriate constant rank conditions hold, there is an m_o such that all invariants of order $r \le m_o + 1$ give a map $F^{(m_o)} := (c^i_{jk},\, c^i_{jk|\ell_1},\, c^i_{jk|\ell_1\ell_2},\, \dots\, ,c^i_{jk|\ell_1\dots\ell_{m_o}}) : P \to \mathbb{R}^{N_o}$, which completely characterises the pair $(P,(X_i))$ up local equivalences ([24], Thm. VII.4.1; see also [12, 17, 23]). So, any question on existence of equivalences between canonical absolute parallelisms of G is in principle completely solvable by studying the invariants of the parallelisms up to some finite order. This is the reason why any algorithm that provides canonical absolute parallelisms for G is considered as a solution to the equivalence problem for this class.

Solutions of this type to the equivalence problems are usually not unique. For instance, the so-called *G-structures of finite type* admit canonical absolute parallelisms, determined via a finite number steps, each of them based on choices of certain normalising conditions ([24, 12, 25, 15, 1, 17]). Different choices lead to non-equivalent canonical absolute parallelisms, hence to distinct solutions to the same equivalence problem. Other examples are provided by the celebrated absolute parallelisms of Chern and Moser ([9]) and of Tanaka ([26, 27]) for the Levi-nondegenerate CR hypersurfaces, whose first order invariants are actually constrained by non-equivalent sets of linear equations. There exists also three distinct solutions to the equivalence problems for the elliptic and hyperbolic CR manifolds of codimension two, which have been determined in [10, 18, 20].

Amongst all canonical absolute parallelisms that one might associate with the structures of a given class, there are sometimes some special ones that correspond to Cartan connections. As we will shortly see, parallelisms of this kind have several very important additional features.

We recall that a *Cartan connection on a manifold M, modelled on a homogeneous space G/H,* is a pair (P,ϖ), formed by a principal H-bundle $\pi: P \to M$ and a $\mathfrak{g}$-valued 1-form $\varpi: TP \to \mathfrak{g} = Lie(G)$ such that:

(a) for each $y \in P$, the map $\varpi_y : T_yP \to \mathfrak{g}$ is a linear isomorphism and $(\varpi_y)^{-1}|_{\mathfrak{h}} : \mathfrak{h} \to T^V_y P$ is the standard isomorphism, given by the right action of H on P, between $\mathfrak{h} = Lie(H)$ and the tangent space $T^V_y P$ of the fiber,

(b) $R^*_h\varpi = \mathrm{Ad}_{h^{-1}} \circ \varpi$ for any $h \in H$.

Given a class of geometric structures G, a correspondence between the structures in G and Cartan connections on the underlying manifolds, is called *canonical* if there is an associated bijection between the local equivalences $f : \mathcal{U} \to \mathcal{U}'$ between manifolds M, M' of G and the local diffeomorphisms $\widetilde{f} : P|_{\mathcal{U}} \to P'|_{\mathcal{U}'}$ between the bundles of the associated Cartan connections (P,ϖ), (P',ϖ'), that satisfy the condition $\widetilde{f}^*\varpi' = \varpi$.

Note that if there is a canonical Cartan connection (P,ω) for any manifold M of G, each basis (E^o_i) for $\mathfrak{g} = Lie(G)$ determines a canonical absolute parallelism $(P,(E_i),\widetilde{(\cdot)})$, formed by the bundle P and the absolute parallelism (E_i) given by the vector fields

$$E_i|_u := \varpi_u^{-1}(E^o_i), \qquad u \in P. \tag{2}$$

Hence, *any construction of canonical Cartan connections for a class* G *automatically provides a solution to the corresponding equivalence problem.*

However, the interest for Cartan connections is by far much wider than their uses for equivalence problems. For an introduction to the variety of possible applications, see e.g. [12, 22, 6, 7, 8, 4, 19, 2, 28] and references therein.

One of the most basic reasons of interest for Cartan connections is given by the following fact: *If* (P,ϖ) *is a Cartan connection on* M *modelled on* G/H, *the associated* $\mathfrak{g}$*-valued curvature* 2*-form* $= d\varpi + \frac{1}{2}[\varpi,\varpi]$ *on* P *vanishes identically if and only if* P *is locally equivalent to the Lie group* G *and* M *is locally equivalent to the homogeneous model* G/H. This means that if the elements of a class G of geometric structures admit canonical Cartan connections modelled on a given homogeneous spaces, for each of them there exists a very informative indicator (namely, the curvature) of how it locally deviates from the homogeneous model.

From this and other facts on Cartan connections, one has also that geometric structures admitting canonical Cartan connections are equipped with distinguished families of appropriate curves or submanifolds of higher dimension, which are invariant under local equivalences and play the same role of geodesics and chains in Riemannian geometry and in geometry of Levi non-degenerate hypersurfaces, respectively (see e.g. [3, 18]). Such distinguished curves and submanifolds can be also combined and determine systems of normal coordinates, which allow to reduce several questions to geometric properties of the homogeneous models (see, for instance, [18, 19]).

At the best of our knowledge, the first methodical study on the possibilities of constructing canonical Cartan connections was done by Tanaka in [27]. There he proved the existence of canonical Cartan connections for an important family of classes of geometric structures, modelled on homogeneous spaces G/H of (semi)simple Lie groups and with parabolic isotropy subgroups $H \subset G$. His results were later extended in various senses by T. Morimoto in [15] and Čap and Schichl in [5]. For a concise review of Tanaka's results, see [1].

We conclude this short discussion of Cartan connections recalling that in [1], Alekseevsky and the second author proved that Tanaka's method of construction of Cartan connections can be considered as a derivation of a more general method of construction of absolute parallelisms, also invented by Tanaka ([25]). This second method applies to a wider range of geometric structures, called *Tanaka's structures of finite type*, and produces canonical parallelisms $(P,(X_i),\widetilde{(\cdot)})$, formed by bundles $\pi : P \to M$ that *in general are not principal bundles* and by parallelisms (X_i) that *in general are not determined by a* $\mathfrak{g}$*-valued* 1*-form* ϖ satisfying the properties of Cartan connections. Nonetheless, for a special class of Tanaka structures, modelled on homogeneous spaces G/H of a semisimple G and parabolic subgroup $H \subset G$, the general construction can be performed in such a way that it produces a bundle $\pi : P \to M$, which *is a principal* H*-bundle*, and an absolute parallelism (X_i) on P, which *is determined by a Cartan connection* ϖ (see [1] for details).

3. Cartan connections of girdled CR manifolds and corresponding structure equations

Now we focus on girdled CR manifolds $(M,\mathcal{D},J)$, i.e. on 5-dimensional CR hypersurfaces of uniform type, which are Levi 2-nondegenerate. As we already mentioned in the introduction, any such CR hypersurface is Levi degenerate and yet admits no local straightenings. The name *girdled* has been chosen to allude to such lack of straightenings.

One of the most important examples of girdled CR manifolds and, as we will shortly see, a model for these geometric structures is given by the following homogeneous manifold. Consider the bilinear form $(\cdot,\cdot)$ and the pseudo-Hermitian form $<\cdot,\cdot>$ on $\mathbb{C}^5$ defined by

$$(1) \qquad (t,s) = t^T I_{3,2} s, \qquad <t,s>=(\bar{t},s), \qquad I_{3,2} = \left(\begin{array}{c|c} I_3 & 0 \\ \hline 0 & -I_2 \end{array}\right),$$

and the corresponding semi-algebraic subset $M_o \subset \mathbb{C}P^4$ defined by

$$(2) \qquad \begin{cases} (t,t) = (t^0)^2 + (t^1)^2 + (t^2)^2 - (t^3)^2 - (t^4)^2 = 0, \\ <t,t>= |t^0|^2 + |t^1|^2 + |t^2|^2 - |t^3|^2 - |t^4|^2 = 0, \\ \Im\left(t^3 \overline{t^4}\right) > 0. \end{cases}$$

It is known (see e.g. [21]) that M_o is a $\mathrm{SO}^o_{3,2}$-homogeneous, 5-dimensional CR submanifold of $\mathbb{C}P^4$ (here, $\mathrm{SO}^o_{3,2}$ is the identity component of $\mathrm{SO}_{3,2}$) and contains $\mathsf{T}_o = M_o \cap \{\Im(t^3\overline{(t^0+t^4)}) > 0\}$ as open dense subset, which is CR equivalent to the so called *tube over the future light cone in* $\mathbb{C}^3$, i.e. to the real hypersurface

$$(3) \qquad \mathsf{T} = \{\, (z^1,z^2,z^3) \in \mathbb{C}^3 \,:\, (x^1)^2 + (x^2)^2 - (x^3)^2 = 0, x^3 > 0 \,\}.$$

It turns out that M_o is girdled and its group of CR automorphisms coincides with $\mathrm{Aut}\,(M_o) = \mathrm{SO}^o_{3,2}$. Hence, if we denote by $H \subset \mathrm{SO}^o_{3,2}$ the isotropy subgroup of $\mathrm{Aut}\,(M_o)$ at some point, M_o is CR equivalent to the homogeneous space $\mathrm{SO}^o_{3,2}/H$, equipped with an appropriate invariant girdled CR structure.

The homogeneous CR manifold $M_o = \mathrm{SO}^o_{3,2}/H$ is a modelling space, of which any girdled CR manifold can be considered as a local deformation. This is a consequence of the main theorem of our paper [14], namely

THEOREM 1. *For any 5-dimensional girdled CR manifold $(M,\mathcal{D},J)$, there exists a canonical Cartan connection (Q,ϖ), modelled on the homogeneous CR manifold $M_o = \mathrm{SO}^o_{3,2}/H$ described above.*

The proof of this theorem is constructive and provides an explicit description of the bundle $\pi : Q \longrightarrow M$ and of the $\mathfrak{so}_{3,2}$-valued 1-form ϖ (more precisely, of a collection of vector fields, by which ϖ is uniquely determined). Our construction is based on a modification of Tanaka's general scheme for building up absolute parallelisms. The fact that our collection of vector fields actually defines a Cartan connection is a consequence of an appropriate tuning of each step of the construction.

As it is shown in [1] (see also above, end of §2), even the classical Tanaka's method can be used to produce Cartan connections, provided that appropriate algebraic conditions are satisfied. Such conditions certainly occur for the *parabolic geometries* [5], i.e. the geometric structures modelled on homogeneous spaces G/H of semisimple Lie groups G with parabolic H. Since the girdled CR manifolds are modelled on a homogeneous space G/H of the semisimple Lie group $G = \mathrm{SO}^o_{3,2}$ with a *non parabolic* H, our result shows that the above conditions might occur for a wider and interesting class of homogeneous models.

As pointed out in §2, the absolute parallelism, that is determined by the canonical Cartan connection (Q, ϖ) and a basis of $\mathfrak{so}_{3,2}$, provides a solution to the equivalence problem for girdled CR manifolds. At the best of our knowledge, at the moment there are two other absolute parallelisms for girdled CR manifolds, hence two other solutions to the same problem ([13, 16]), but none of them corresponds to a Cartan connection.

In the next sections, we select a special basis for $\mathfrak{so}_{3,2}$ and we write explicitly the structure equations of the absolute parallelism corresponding to such special basis. Such explicit expressions also allow immediate comparisons with the structure equations of the parallelisms provided by the other solutions to the equivalence problem for girdled CR manifolds.

3.1. A convenient basis for $\mathfrak{so}_{3,2}$

The Lie algebra $\mathfrak{g} = \mathfrak{so}_{3,2}$ has a natural structure of graded Lie algebra, which can be explicitly described as follows. Consider a system of projective coordinates on $\mathbb{C}P^4$, in which the scalar product $(\cdot,\cdot)$ defined in (1) assumes the form

$$(t,s) = t^T I\, s \qquad \text{with} \qquad I = \left(\begin{array}{cc|c|cc} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right). \tag{4}$$

By means of these new coordinates, the Lie algebra $\mathfrak{so}_{3,2}$ of the isometries of $(\cdot,\cdot)$ can be identified with the Lie algebra of real matrices A such that $A^T I + IA = 0$, i.e., of the form

$$A = \left(\begin{array}{cc|c|cc} a_1 & a_2 & a_5 & a_7 & 0 \\ a_3 & a_4 & a_6 & 0 & -a_7 \\ \hline a_8 & a_9 & 0 & -a_6 & -a_5 \\ \hline a_{10} & 0 & -a_9 & -a_4 & -a_2 \\ 0 & -a_{10} & -a_8 & -a_3 & -a_1 \end{array}\right), \qquad \text{for some } a_i \in \mathbb{R}.$$

This shows that $\mathfrak{so}_{3,2}$ is the direct sum of the vector subspaces

$$\begin{gathered} \mathfrak{g}_{-2} =< e^o_{-2} >,\, \mathfrak{g}_{-1} =< e^o_{-1|1}, e^o_{-1|2} >,\, \mathfrak{g}_0 =< e^o_{0|1}, e^o_{0|2}, E^o_{0|1}, E^o_{0|2} >, \\ \mathfrak{g}_1 =< E^o_{1|1}, E^o_{1|2} >,\, \mathfrak{g}_2 =< E^o_2 >, \end{gathered} \tag{5}$$

spanned by the matrices

$$
(6)\qquad
\begin{gathered}
e^o_{-2}=\left(\begin{array}{cc|c|cc} 0&0&0&0&0\\ 0&0&0&0&0\\ \hline 0&0&0&0&0\\ \hline 1&0&0&0&0\\ 0&-1&0&0&0 \end{array}\right),\
e^o_{-1|1}=\left(\begin{array}{cc|c|cc} 0&0&0&0&0\\ 0&0&0&0&0\\ \hline 1&0&0&0&0\\ \hline 0&0&0&0&0\\ 0&0&-1&0&0 \end{array}\right),\
e^o_{-1|2}=\left(\begin{array}{cc|c|cc} 0&0&0&0&0\\ 0&0&0&0&0\\ \hline 0&1&0&0&0\\ \hline 0&0&-1&0&0\\ 0&0&0&0&0 \end{array}\right),\\
e^o_{0|1}=\left(\begin{array}{cc|c|cc} 1&0&0&0&0\\ 0&-1&0&0&0\\ \hline 0&0&0&0&0\\ \hline 0&0&0&1&0\\ 0&0&0&0&-1 \end{array}\right),\
e^o_{0|2}=\left(\begin{array}{cc|c|cc} 0&1&0&0&0\\ 1&0&0&0&0\\ \hline 0&0&0&0&0\\ \hline 0&0&0&0&-1\\ 0&0&0&-1&0 \end{array}\right),\\
E^o_{0|1}=\left(\begin{array}{cc|c|cc} 1&0&0&0&0\\ 0&1&0&0&0\\ \hline 0&0&0&0&0\\ \hline 0&0&0&-1&0\\ 0&0&0&0&-1 \end{array}\right),\
E^o_{0|2}=\left(\begin{array}{cc|c|cc} 0&1&0&0&0\\ -1&0&0&0&0\\ \hline 0&0&0&0&0\\ \hline 0&0&0&0&-1\\ 0&0&0&1&0 \end{array}\right),\\
E^o_{1|1}=\left(\begin{array}{cc|c|cc} 0&0&1&0&0\\ 0&0&0&0&0\\ \hline 0&0&0&0&-1\\ \hline 0&0&0&0&0\\ 0&0&0&0&0 \end{array}\right),\
E^o_{1|2}=\left(\begin{array}{cc|c|cc} 0&0&0&0&0\\ 0&0&1&0&0\\ \hline 0&0&0&-1&0\\ \hline 0&0&0&0&0\\ 0&0&0&0&0 \end{array}\right),\
E^o_{2}=\left(\begin{array}{cc|c|cc} 0&0&0&1&0\\ 0&0&0&0&-1\\ \hline 0&0&0&0&0\\ \hline 0&0&0&0&0\\ 0&0&0&0&0 \end{array}\right).
\end{gathered}
$$

We will refer to the collection $\mathcal{B}^o$ of these matrices as *standard basis* of $\mathfrak{so}_{3,2}$.
Note that the each $\mathfrak{g}_k$ in (5) is the eigenspace of the adjoint action of the *grading element* $Z := E^o_{0|1}$ with eigenvalue k, so that

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \qquad \text{for all } i, j,$$

where, by convention, we assume $\mathfrak{g}_k = \{0\}$ for any $k \notin \{-2,-1,0,2,2\}$. In other words, $\mathfrak{so}_{3,2}$ *has a natural structure of graded Lie algebra*.
We note that also the Lie algebra $\mathfrak{h} = Lie(H)$ of the isotropy subgroup $H \subset \mathrm{SO}^o_{3,2}$ at $x_o = [1 : i : 0 : 0 : 0]$ is natural graded. Indeed, it decomposes into the direct sum $\mathfrak{h} = \mathfrak{h}_0 + \mathfrak{g}_1 + \mathfrak{g}_2$ with $\mathfrak{h}_0 := \langle E_{0|1}, E_{0|2} \rangle$.

We conclude this section introducing another convenient basis for $\mathfrak{so}_{3,2}$, which is a technical modification of $\mathcal{B}^o$, more suitable for several arguments concerning the CR structure of the model space $\mathrm{SO}^o_{3,2} \cdot x_o \simeq \mathrm{SO}^o_{3,2}/H$. Indeed, in many places it is more appropriate to consider instead of the four elements $E^o_{\ell|j}$, $e^o_{-\ell|j}$, $\ell = 0, 1$, $j = 1, 2$, the four complex matrices in $(\mathfrak{so}_{3,2})^{\mathbb{C}}$:

$$
(7)\qquad
\begin{array}{ll}
E^o_{\ell(10)} = \frac{1}{2}\left(E^o_{\ell|1} - iE_{\ell|2}\right), & E^o_{\ell(01)} = \overline{E^o_{\ell(10)}}, \\
e^o_{-\ell(10)} = \frac{1}{2}\left(e^o_{-\ell|1} - ie^o_{-\ell|2}\right), & e^o_{-\ell(01)} = \overline{e^o_{-\ell(10)}},
\end{array}
\qquad \ell = 0, 1.
$$

So, in the following, instead of expanding the elements of $\mathfrak{so}_{3,2}$ in terms of the standard basis $\mathcal{B}^o$, we often expand the same elements in terms of the *standard CR basis*

$$
(8)\qquad \mathcal{B}^{CR} = \left(e^o_{-2}, e^o_{-1(10)}, e^o_{-1(01)}, e^o_{0(10)}, e^o_{0(01)}, E^o_{0(10)}, E^o_{0(01)}, E^o_{1(10)}, E^o_{1(01)}, E^o_{2}\right).
$$

Since the elements $X \in \mathfrak{so}_{3,2}$ are real matrices, their expansion in the standard CR basis has the form $X = \sum^A e^o_A + \sum \mu^A E^o_A$, with coefficients satisfying the reality conditions

$$\lambda^{-2}, \mu^{-2} \in \mathbb{R} \qquad \text{and} \qquad \lambda^{-\ell(01)} = \overline{\lambda^{-\ell(10)}},\ \mu^{\ell(01)} = \overline{\mu^{\ell(10)}} \text{ for } \ell = 0,1.$$

A table of all Lie brackets between elements in $\mathcal{B}^{CR}$ can be found in [14].

3.2. The absolute parallelism associated with the standard basis

Consider now a girdled CR manifold $(M, \mathcal{D}, J)$ and its canonical Cartan connection (Q, ϖ) modelled on $M_o = \mathrm{SO}^o_{3,2}/H$. As we discussed in §2, the relation (2) associates with each element e^o_A or E^o_B of the standard basis $\mathcal{B}^o$ of $\mathfrak{so}_{3,2}$ a vector field that we denote by e_A or E_B, respectively. The ordered 10-tuple (e_A, E_B) is the *absolute parallelism corresponding to the basis* $\mathcal{B}^o$.

As we observed, in place of these (real) vector fields, it is often more convenient to consider the collection of (real and complex) vector fields

$$\left(e_{-2}, e_{-1(10)}, e_{-1(01)}, e_{0(10)}, e_{0(01)}, E_{0(10)}, E_{0(01)}, E_{1(10)}, E_{1(01)}, E_2\right), \tag{9}$$

with $e_{-\ell(10)}, e_{-\ell(01)}, E_{\ell(10)}, E_{\ell(01)}$, $\ell = 0,1$, defined by

$$\begin{aligned} &E_{\ell(10)} := \tfrac{1}{2}\left(E_{\ell|1} - iE_{\ell|2}\right), \qquad E_{\ell(01)} := \overline{E_{\ell(10)}}, \\ &e_{-\ell(10)} := \tfrac{1}{2}\left(e_{-\ell|1} - ie_{-\ell|2}\right), \quad e_{-\ell(01)} := \overline{e_{-\ell(10)}}, \end{aligned} \qquad \ell = 0,1. \tag{10}$$

This is the collection of complex vector fields that corresponds to the elements of the standard CR basis $\mathcal{B}^{CR}$ by means of (2). From now, we will use the notation e_A and E_B to indicate just these vector fields.

The vector fields e_A, E_B uniquely determine their dual (real and complex) 1-forms ϑ^A, ω^B, defined by

$$\vartheta^A(e_C) = \delta^A_C, \quad \vartheta^A(E_D) = 0, \quad \omega^B(e_A) = 0, \quad \omega^B(E_D) = \delta^B_D. \tag{11}$$

Note that the $\mathfrak{g}$-valued 1-form ϖ can be written in terms of such 1-forms as

$$\varpi = \sum_A e^o_A \otimes \vartheta^A + \sum_B E^o_B \otimes \omega^B. \tag{12}$$

The vector fields (e_A, E_B) and the dual 1-forms (ϑ^A, ω^B) have several geometric features, which derive from the special step-by-step construction of the Cartan connection ϖ given in [14]. Let us briefly recall them.

First of all, we remind that the girdled CR manifold $(M, \mathcal{D}, J)$ is naturally equipped with a J-invariant, 2-dimensional, involutive subdistribution $\mathcal{E}$ of the distribution $\mathcal{D}$, defined at each point $x \in M$ by (see e.g. [14], §2.1):

$$\begin{aligned} \mathcal{E}_x := \big\{ v \in \mathcal{D}_x : &\text{ there is vector field } X \text{ in } \mathcal{D} \\ &\text{such that } X_x = v \text{ and } [X,Y]_x \in \mathcal{D}_x \text{ for all vector fields } Y \text{ in } \mathcal{D} \big\}. \end{aligned} \tag{13}$$

In other words, $\mathcal{E}$ is the J-invariant distribution of vector spaces, generated by the real vectors that are in the kernels of the Levi forms of $(\mathcal{D},J)$.
In [14], the bundle $\pi : Q \to M$ is obtained as the last step of a tower of three principal bundles, one defined over the other, as in the diagram

$$Q = P^2 \xrightarrow{\pi^2} P^1 \xrightarrow{\pi^1} P^0 \xrightarrow{\pi^0} M, \qquad \text{with } \pi := \pi^0 \circ \pi^1 \circ \pi^2 .$$

In turn, each bundle P^i is defined as a quotient $P^i = P^i_\sharp / N^i_\sharp$ by the action of a special group of matrices $N^i_\sharp$, of an appropriate principal bundle $P^i_\sharp$ of linear frames of the lower order bundle

$$\begin{array}{ccccccc}
\mathcal{F}r(P^1) & & \mathcal{F}r(P^0) & & \mathcal{F}r(M) & & \\
\cup & & \cup & & \cup & & \\
P^2_\sharp & \searrow & P^1_\sharp & \searrow & P^0_\sharp & \searrow & \\
P^2 & \longrightarrow & P^1 & \longrightarrow & P^0 & \longrightarrow & M \\
\| & & \| & & \| & & \\
P^2_\sharp / N^2_\sharp & & P^1_\sharp / N^1_\sharp & & P^0_\sharp / N_\sharp & &
\end{array}$$

The absolute parallelism (e_A, E_B) on $Q = P^2$ is defined as the unique frame field that takes values in a very special trivial subbundle $P^3_\sharp$ of the linear frame bundle $\mathcal{F}r(P^2)$ of P^2.

Each bundle of linear frames $P^i_\sharp \subset \mathcal{F}r(P^{i-1})$, $0 \leq i \leq 3$ (here, we set $M = P^{-1}$), is determined by all linear frames of P^{i-1} that are adapted to the natural distributions of P^{i-1} (for instance, when $P^{i-1} = M$, the frames are adapted to the J-invariant distributions $\mathcal{E}$ and $\mathcal{D}$) and satisfy three sets of conditions:

a) if the base point of the frame is $u = [(f_i)] \in P^{i-1} = P^{i-1}_\sharp / N^{i-1}$, the first vectors of a frame in $P^i_\sharp|_u$ are constrained to project onto one of the linear frames (f_i) in $P^{i-1}_\sharp$, which belong to the equivalence class $u = [(f_i)]$;

b) the other vectors of a frame in $P^i_\sharp|_u$ must be vertical with respect to the projection $\pi^i : P^{i-1} \to P^{i-2}$;

c) any linear frame in $P^i_\sharp|_u$ is constrained by an appropriate set of normalising conditions; such conditions depend on which bundle $P^i_\sharp$ of linear frames we are considering – we refer to [14] for the explicit formulation of such normalising conditions.

Due to (a), the frames in $P^i_\sharp$ not only satisfy the normalising conditions quoted in (c), but also all conditions that are residuals of the three types of conditions for the frames in $P^{i-1}_\sharp$, in $P^{i-2}_\sharp$, etc. In particular, the frame field in $P^3_\sharp$ that gives the absolute parallelism (e_A, E_B) on $Q = P^2$ satisfies a set of conditions that inherits from the three types of constraints on the linear frames of the previous steps. Amongst such conditions one has that

1) (the real and imaginary parts of) the vector fields E_A are the infinitesimal transformations associated with (the real and imaginary parts of) the elements E^o_A, determined by the right action of H on Q; in particular, they are generators of the *vertical distribution* $\mathcal{V}\subset TQ$, i.e. the distribution of the tangent spaces of the fibres of $\pi : Q \to M$;

2) the distribution $\mathcal{H}\subset TQ$, generated by (the real and imaginary parts of) the vector fields e_A, is such that for any $u \in Q$ the projection $\pi_*|_u : T_uQ \to T_xM$, $x = \pi(u)$, gives a linear isomorphism $\pi_* : \mathcal{H}_u \to T_xM$ between $\mathcal{H}_u$ and T_xM;

3) for any $u \in Q$, the complex subspaces of $\mathcal{H}^{\mathbb{C}}_u$

$$\mathcal{D}^{10(\mathcal{H})}_u := \langle e_{-1(10)}|_u, e_{0(10)}|_u\rangle, \quad \mathcal{E}^{10(\mathcal{H})}_u := \langle e_{0(10)}|_u\rangle, \tag{14}$$

$$\mathcal{D}^{01(\mathcal{H})}_u := \langle e_{-1(01)}|_u, e_{0(01)}|_u\rangle, \quad \mathcal{E}^{01(\mathcal{H})}_u := \langle e_{0(01)}|_u\rangle, \tag{15}$$

project isomorphically onto the holomorphic spaces $\mathcal{D}^{10}_x\subset\mathcal{D}^{\mathbb{C}}_x$, $\mathcal{E}^{10}_x\subset\mathcal{E}^{\mathbb{C}}_x$, $x = \pi(u)$, and the antiholomorphic spaces $\mathcal{D}^{01}_x = \overline{\mathcal{D}^{10}_x}$, $\mathcal{E}^{01}_x = \overline{\mathcal{E}^{10}_x}$, respectively.

The other conditions correspond to constraints on the curvature 2-form κ of the Cartan connection and will be discussed in the next section.

3.3. The curvature constraints on the Cartan connection

Consider now the *curvature* 2*-form* κ of the Cartan connection (Q,ϖ), that is the $\mathfrak{so}_{3,2}$-valued 2-form on Q, defined by

$$\kappa := d\varpi + \tfrac{1}{2}[\varpi,\varpi].$$

From basic properties of Cartan connections and the fact that the vector fields E_A are infinitesimal transformations on Q, corresponding to the elements $E^o_A \in \mathfrak{h} = Lie(H)$, the expansion of κ in terms of the pointwise linearly independent 2-forms $(\vartheta^A \wedge \vartheta^B, \vartheta^A \wedge \omega^C, \omega^C \wedge \omega^D)$, determined by the dual coframe (11), has necessarily the form

$$\kappa = d\varpi + \tfrac{1}{2}[\varpi,\varpi] = \sum T^A_{BC} e^o_A \otimes \vartheta^B \wedge \vartheta^C + \sum R^D_{BC} E^o_D \otimes \vartheta^B \wedge \vartheta^C, \tag{16}$$

for appropriate (real and complex) functions T^A_{BC} and R^D_{BC}.

The curvature components T^A_{BC} and R^D_{BC} are determined by the Lie brackets of pairs of vector fields e_A as follows. Let us denote by $(\cdot)^{e^o_A} : \mathfrak{so}_{3,2} \to \langle e^o_A\rangle$ and $(\cdot)^{E^o_B} : \mathfrak{so}_{3,2} \longrightarrow \langle e^o_B\rangle$ the standard projections of $\mathfrak{so}_{3,2}$ along the vectors of the basis $\mathcal{B}^{CR}$ and by $\mathfrak{c}^A_{BC} := ([e^o_B, e^o_C])^{e^o_A}$ and $\mathfrak{d}^D_{BC} := ([e^o_B, e^o_C])^{E^o_D}$ the structure constants of $\mathfrak{so}_{3,2}$ in the basis $\mathcal{B}^{CR}$. Then, by Koszul formula for exterior derivatives and the definition of κ, we have

$$T^A_{BC} = d\vartheta^A(e_B, e_C) + ([e^o_B, e^o_C])^{e^o_A} = -\vartheta^A([e_B, e_C]) + \mathfrak{c}^A_{BC}, \tag{17}$$

$$R^D_{BC} = d\omega^D(e_B, e_C) + ([e^o_B, e^o_C])^{E^o_D} = -\omega^D([e_B, e_C]) + \mathfrak{d}^D_{BC}. \tag{18}$$

By comparison with (1), we immediately see that, modulo the structures constants of $\mathfrak{so}_{3,2}$, *the curvature components* T^A_{BC} *and* R^D_{BC} *of the curvature* 2*-form* κ *are nothing but the structure functions of the absolute parallelism* (e_A, E_B). In fact, this is a well known general fact on Cartan connections.

As mentioned above, besides the conditions (1) – (3) of §3.2, the absolute parallelism (e_A, E_B) is constrained by other normalising conditions. They are conditions on the Lie brackets between the vectors e_A and, through (17) and (18), they can all be expressed in terms of the curvature components T^A_{BC} and R^D_{BC}. For reader's convenience, we give here the complete list of such constraints and we refer to [14] for further details.

Integrability of the complex structure and involutivity of $\mathcal{E}^{\mathbb{C}}$.
From (17) and the fact that the complex distributions $\mathcal{E}^{10(\mathcal{H})} + \mathcal{E}^{01(\mathcal{H})}$ and $\mathcal{D}^{10(\mathcal{H})}$, defined in (14) and (15), project onto the involutive distributions $\mathcal{E}^{\mathbb{C}}$ and $\mathcal{D}^{10}$ of M, one has

$$\begin{aligned} &T^A_{0(10)\,0(01)} = 0 \text{ for } A \in \{-2, -1(10), -1(01)\}, \\ &T^{A'}_{i(10)\,j(10)} = \overline{T^{A'}_{i(10)\,j(10)}} = 0 \text{ for } i, j \in \{-1, 0\},\ A' \in \{-2, -1(01), 0(01)\}. \end{aligned} \tag{19}$$

The distribution $\mathcal{E}^{10}$ *is in the kernel of Levi forms.*
From (17) and the fact that the spaces $\mathcal{E}^{10(\mathcal{H})}_u$, $\mathcal{E}^{01(\mathcal{H})}_u$ project into the kernel of the Levi form, one has

$$T^{-2}_{-1(01)\,0(10)} = T^{-2}_{-1(10)\,0(01)} = 0. \tag{20}$$

Normalising conditions on the frames in $P^0_\sharp$.
From (17) and condition (5.2) in [14], one has

$$T^{-2}_{-1(10)\,-1(01)} = T^{-1(10)}_{-1(01)\,0(10)} = T^{-1(01)}_{-1(10)\,0(01)} = 0. \tag{21}$$

Normalising conditions on the frames of $P^1_\sharp$.
From (17) and the normalising conditions in [14], given by formula (6.7) and condition $\beta_K = 0$ after Lemma 6.5 of that paper, one has

$$\begin{aligned} &T^{-1(10)}_{-1(10)\,0(10)} = T^{-1(10)}_{-1(10)\,0(01)} = T^{-1(01)}_{-1(01)\,0(01)} = T^{-1(01)}_{-1(01)\,0(10)} = 0, \\ &T^{0(10)}_{-1(01)\,0(01)} = T^{0(01)}_{-1(10)\,0(10)} = 0. \end{aligned} \tag{22}$$

Property of the strongly adapted frames in $P^1_\sharp$.
From (17) and Lemma 6.6 (ii) in [14] one has

$$T^{-2}_{-2\,0(10)} = T^{-2}_{-2\,0(01)} = 0. \tag{23}$$

Normalising conditions on the strongly adapted frames in $P^2_\sharp$.
From (17) and the normalising conditions (7.2) on γ_K in [14], one has

$$(24) \qquad T^{0(10)}_{-1(10)\,0(10)} = T^{0(01)}_{-1(01)\,0(01)} = T^{0(10)}_{-1(10)\,0(01)} = T^{0(01)}_{-1(01)\,0(10)} = 0.$$

Normalising conditions on the strongly adapted frames in $P^3_\sharp$.
From (18) and the normalising conditions (8.2) on ε_K in [14], one has

$$(25) \qquad R^{0(10)}_{-2\,0(10)} = R^{0(01)}_{-2\,0(01)} = R^{0(01)}_{-2\,0(10)} = R^{0(10)}_{-2\,0(01)} = 0.$$

Besides (19) – (25), the absolute parallelism is subjected to three further conditions of cohomological nature. They are

a) the condition given in (6.21) of [14], which is equivalent to a system of linear equations on $T^{-2}_{-2\,-1(10)}$, $T^{-2}_{-2\,-1(01)}$ and $T^{-1(10)}_{-1(10)\;-1(01)}$;

b) the condition given in (7.4) in [14], which is equivalent to a system of linear equations on $T^{-1(10)}_{-2\,-1(10)}$, $T^{-1(10)}_{-2\,-1(01)}$, $T^{0(10)}_{-1(10)\,-1(01)}$, $R^{0(10)}_{-1(10)\,-1(01)}$ and their complex conjugates;

c) the condition given in (8.4) in [14], which is equivalent to a system of linear equations on $T^{0(10)}_{-2\,-1(10)}$, $T^{0(10)}_{-2\,-1(01)}$, $R^{0(10)}_{-2(10)\,-1(01)}$, $R^{0(10)}_{-2\,-1(01)}$, $R^{1(10)}_{-1(10)\,-1(01)}$ and their complex conjugates.

The explicit expressions for the linear systems corresponding to the constraints (a), (b), (c) can be determined with straightforward computations. An exposition of such computations, which uses only elementary tools, can be found in the appendix. The result is that the constraints (a), (b) and (c) are equivalent to the linear equations

$$(26) \quad (a) \qquad T^{-2}_{-2\,-1(10)} = T^{-2}_{-2\,-1(01)} = T^{-1(10)}_{-1(10)\,-1(01)} = 0,$$

$$(27) \quad (b) \qquad T^{-1(10)}_{-2\,-1(10)} = T^{-1(10)}_{-2\,-1(01)} = T^{0(10)}_{-1(10)\,-1(01)} = T^{-1(01)}_{-2\,-1(01)} = T^{-1(01)}_{-2\,-1(10)} = \\ = T^{0(01)}_{-1(10)\,-1(01)} = R^{0(10)}_{-1(10)\,-1(01)} = R^{0(01)}_{-1(10)\,-1(01)} = 0,$$

$$(28) \quad (c) \qquad \overline{R^{0(10)}_{-2\,-1(10)}} = -\frac{1}{2}T^{0(10)}_{-2\,-1(10)} - \frac{1}{2}R^{0(10)}_{-2\,-1(01)}, \\ R^{1(10)}_{-1(10)\,-1(01)} = \frac{i}{2}T^{0(10)}_{-2\,-1(10)} - \frac{i}{2}R^{0(10)}_{-2\,-1(01)}$$

and to the equations that follows from (28) by complex conjugation.

3.4. The structure equations of a girdled CR manifold

The projections of the values of the curvature κ along each element of the basis $\mathcal{B}^{CR}$ give explicit expressions for the exterior differentials $d\vartheta^A$ and $d\omega^B$ in terms of the pointwise linearly independent 2-forms $(\vartheta^A\wedge\vartheta^B, \vartheta^A\wedge\omega^C, \omega^C\wedge\omega^B)$, i.e. the *structure equations of the absolute parallelism* (e_A, E_B) (see §2). Here is the complete list of these structure equations, where we set equal to 0 all terms T^A_{BC} that are bound to vanish by the curvature constraints in §3.3.

$$(29)\quad d\vartheta^{-2}+\frac{i}{2}\vartheta^{-1(10)}\wedge\vartheta^{-1(01)}-\left(\omega^{0(10)}+\omega^{0(01)}\right)\wedge\vartheta^{-2}=0,$$

$$(30)\quad d\vartheta^{-1(10)}-\vartheta^{0(10)}\wedge\vartheta^{-1(01)}-\omega^{0(10)}\wedge\vartheta^{-1(10)}+i\omega^{1(10)}\wedge\vartheta^{-2}= \\ =T^{-1(10)}_{-20(10)}\vartheta^{-2}\wedge\vartheta^{0(10)}+T^{-1(10)}_{-20(01)}\vartheta^{-2}\wedge\vartheta^{0(01)},$$

$$(31)\quad d\vartheta^{0(10)}-\left(\omega^{0(10)}-\omega^{0(01)}\right)\wedge\vartheta^{0(10)}+\frac{1}{2}\omega^{1(10)}\wedge\vartheta^{-1(10)}= \\ =T^{0(10)}_{-2-1(10)}\vartheta^{-2}\wedge\vartheta^{-1(10)}+T^{0(10)}_{-2-1(01)}\vartheta^{-2}\wedge\vartheta^{-1(01)}+ \\ +T^{0(10)}_{-20(10)}\vartheta^{-2}\wedge\vartheta^{0(10)}+T^{0(10)}_{-20(01)}\vartheta^{-2}\wedge\vartheta^{0(01)}+T^{0(10)}_{-1(01)0(10)}\vartheta^{-1(01)}\wedge\vartheta^{0(10)}+ \\ +T^{0(10)}_{0(10)0(01)}\vartheta^{0(10)}\wedge\vartheta^{0(01)},$$

$$(32)\quad d\omega^{0(10)}-\vartheta^{0(10)}\wedge\vartheta^{0(01)}+\frac{1}{2}\omega^{1(01)}\wedge\vartheta^{-1(10)}+\omega^{2}\wedge\vartheta^{-2}= \\ =\underbrace{R^{0(10)}_{-2-1(10)}}_{\textbf{constrained by }(28)}\vartheta^{-2}\wedge\vartheta^{-1(10)}+R^{0(10)}_{-2-1(01)}\vartheta^{-2}\wedge\vartheta^{-1(01)}+ \\ +R^{0(10)}_{-1(10)0(10)}\vartheta^{-1(10)}\wedge\vartheta^{0(10)}+R^{0(10)}_{-1(10)0(01)}\vartheta^{-1(10)}\wedge\vartheta^{0(01)}+ \\ +R^{0(10)}_{-1(01)0(10)}\vartheta^{-1(01)}\wedge\vartheta^{0(10)}+R^{0(10)}_{-1(01)0(01)}\vartheta^{-1(01)}\wedge\vartheta^{0(01)}+ \\ +R^{0(10)}_{0(10)0(01)}\vartheta^{0(10)}\wedge\vartheta^{0(01)},$$

$$(33)\quad d\omega^{1(10)}-\omega^{1(01)}\wedge\vartheta^{0(10)}-\omega^{1(10)}\wedge\omega^{0(01)}+i\omega^{2}\wedge\vartheta^{-1(10)}= \\ =R^{1(10)}_{-2-1(10)}\vartheta^{-2}\wedge\vartheta^{-1(10)}+R^{1(10)}_{-2-1(01)}\vartheta^{-2}\wedge\vartheta^{-1(01)}+ \\ +R^{1(10)}_{-20(10)}\vartheta^{-2}\wedge\vartheta^{0(10)}+R^{1(10)}_{-20(01)}\vartheta^{-2}\wedge\vartheta^{0(01)}+ \\ +\underbrace{R^{1(10)}_{-1(10)-1(01)}}_{\textbf{constrained by }(28)}\vartheta^{-1(10)}\wedge\vartheta^{-1(01)}+ \\ +R^{1(10)}_{-1(10)0(10)}\vartheta^{-1(10)}\wedge\vartheta^{0(10)}+R^{1(10)}_{-1(10)0(01)}\vartheta^{-1(10)}\wedge\vartheta^{0(01)}+$$

$$+R^{1(10)}_{-1(01)\,0(10)}\vartheta^{-1(01)}\wedge\vartheta^{0(10)}+R^{1(10)}_{-1(01)\,0(01)}\vartheta^{-1(01)}\wedge\vartheta^{0(01)}+$$
$$+R^{1(10)}_{0(10)\,0(01)}\vartheta^{0(10)}\wedge\vartheta^{0(01)},$$

$$d\omega^2-\frac{i}{2}\omega^{1(10)}\wedge\omega^{1(01)}+\Big(\omega^{0(10)}+\omega^{0(01)}\Big)\wedge\omega^2= \tag{34}$$
$$=R^2_{-2\,-1(10)}\vartheta^{-2}\wedge\vartheta^{-1(10)}+R^2_{-2\,-1(01)}\vartheta^{-2}\wedge\vartheta^{-1(01)}+$$
$$+R^2_{-2\,0(10)}\vartheta^{-2}\wedge\vartheta^{0(10)}+R^2_{-2\,0(01)}\vartheta^{-2}\wedge\vartheta^{0(01)}+$$
$$+R^2_{-1(10)\,-1(01)}\vartheta^{-1(10)}\wedge\vartheta^{-1(01)}+$$
$$+R^2_{-1(10)\,0(10)}\vartheta^{-1(10)}\wedge\vartheta^{0(10)}+R^2_{-1(10)\,0(01)}\vartheta^{-1(10)}\wedge\vartheta^{0(01)}+$$
$$+R^2_{-1(01)\,0(10)}\vartheta^{-1(01)}\wedge\vartheta^{0(10)}+R^2_{-1(01)\,0(01)}\vartheta^{-1(01)}\wedge\vartheta^{0(01)}+$$
$$+R^2_{0(10)\,0(01)}\vartheta^{0(10)}\wedge\vartheta^{0(01)}.$$

3.5. Comparison with other absolute parallelisms

As we already mentioned, other canonical absolute parallelisms for girdled CR manifolds, not associated with Cartan connections, have been recently given in [13, 16]. Note also that the absolute parallelism in [16] is defined only for the girdled CR manifolds admitting no local equivalence with the homogeneous girdled CR manifold M_o.

Let us now focus on the canonical absolute parallelism $(P,(X_i),\widetilde{(\cdot)})$, defined in [13] for an arbitrary girdled CR manifold $(M,\mathcal{D},J)$. There, the bundle $\pi: P\to M$ has 5-dimensional fibers, but it has no natural structure of principal bundle over M. The absolute parallelism $(X_i)_{i=1}^{10}$ on P is associated with a dual coframes field, given by the real and imaginary parts of ten $\mathbb{C}$-valued 1-forms, denoted by $(\omega,\omega^1,\overline{\omega^1},\varphi^2,\overline{\varphi^2},\theta^2,\overline{\theta^2},\varphi^1,\overline{\varphi^1},\psi)$ and with ω and ψ taking only imaginary values.

Since the bundle P and the principal bundle Q of our Cartan connection have the same dimension, if we consider them as mere bundles with no further structures, we may locally identify them. From the construction of P, we may also assume that, under this identification, the 1-form ω is equal to $\omega=-2i\vartheta^{-2}$, where ϑ^{-2} is the 1-form of our parallelism, defined in §3.2.

We now recall that the 1-forms of the absolute parallelism in [13] are characterised by the fact that they satisfy an appropriate set of structure equations. The first two of this set are

$$d\omega=-\omega^1\wedge\omega^{\bar 1}-\omega\wedge(\varphi^2+\varphi^{\bar 2}), \tag{35}$$

$$d\omega^1=\theta^2\wedge\omega^{\bar 1}-\omega^1\wedge\varphi^2-\omega\wedge\varphi^1. \tag{36}$$

Comparing them with our structure equations (29) and (30) and through a tedious but straightforward computation, one can check that the equations (35) and (36) are satisfied by the 1-forms on $P \simeq Q$, defined by

$$
\begin{aligned}
&\omega := -2i\vartheta^{-2},\\
&\omega^1 := \vartheta^{-1(10)} - \overline{T^{-1(10)}_{-20(10)}}\vartheta^{-2},\\
&\varphi^2 := \omega^{0(10)} + \frac{i}{2}\overline{T^{-1(10)}_{-20(10)}}\vartheta^{-1(01)},\\
&\theta^2 := \vartheta^{0(10)} + i\overline{T^{-1(10)}_{-20(10)}}\left(\vartheta^{-1(10)} - \overline{T^{-1(10)}_{-20(10)}}\vartheta^{-2}\right),\\
&\varphi^1 := \frac{1}{2}\omega^{1(10)} - \frac{i}{2}T^{-1(10)}_{-20(01)}\vartheta^{0(01)} - \frac{1}{2}\overline{T^{-1(10)}_{-20(10)}}T^{-1(10)}_{-20(10)}\vartheta^{-1(10)}+\\
&\quad + \frac{1}{4}\overline{T^{-1(10)}_{-20(10)}T^{-1(10)}_{-20(10)}}\vartheta^{-1(01)} - \frac{i}{2}\overline{T^{-1(10)}_{-20(10)}}\omega^{0(01)} - \frac{i}{2}d\overline{T^{-1(10)}_{-20(10)}}.
\end{aligned}
\tag{37}
$$

Now, we expect that if the 1-forms (37) are appropriately modified with additional terms involving $\omega^{1(10)}, \omega^{1(01)}$ and ω^2, they will satisfy not only the first two structure equations of the absolute parallelism in [13], but also all other structure equations of that parallelism.

On the basis of this expectation, the construction in [13] seems to start diverging from ours precisely when the absolute parallelism is required to satisfy (36). In fact, this is a constraint that amounts to impose that the curvature components $T^{-1(10)}_{-20(10)}$ and $T^{-1(10)}_{-20(01)}$ are absorbed into the definition of the vector fields of the absolute parallelism. Since these curvature components are not invariant under the right action of the structure group of $\pi : Q \to M$, the constraint given by (36) is plausibly one of the main reasons for the fact that the constructive process in [13] does not produce a Cartan connection.

An analogous comparison between our canonical Cartan connection and the parallelism in [16] might be done following the same line of arguments. We leave this task to the interested reader.

Appendix

A.1. The Cartan-Killing form of $\mathfrak{so}_{3,2}$

For the following computations, it turns out that the standard basis $\mathcal{B}^o$ of $\mathfrak{so}_{3,2}$, defined in (6), is not very convenient. In place of that basis, it is by far more useful to consider a new basis $\mathcal{B} = (f_\alpha)_{\alpha=1,\dots,10}$, with elements

$$
\text{(A.1)}\quad
\begin{aligned}
&f_1 := \frac{1}{\sqrt{6}}e_{-2}, \quad f_2 := \frac{1}{\sqrt{6}}(e_{-1(10)} + e_{-1(01)}), \quad f_3 := \frac{i}{\sqrt{6}}(e_{-1(10)} - e_{-1(01)}),\\
&f_4 := \frac{1}{\sqrt{12}}(e_{0(10)} + e_{0(01)}), \quad f_5 := \frac{i}{\sqrt{12}}(e_{0(10)} - e_{0(01)}),\\
&f_6 := \frac{1}{\sqrt{12}}(E_{0(10)} + E_{0(01)}), \quad f_7 := \frac{i}{\sqrt{12}}(E_{0(10)} - E_{0(01)}),\\
&f_8 := \frac{1}{\sqrt{6}}(E_{1(10)} + E_{1(01)}), \quad f_9 := \frac{i}{\sqrt{6}}(E_{1(10)} - E_{1(01)}), \quad f_{10} := \frac{1}{\sqrt{6}}E^2.
\end{aligned}
$$

The main motivation for considering such new basis comes from the fact the entries of the Cartan-Killing form $\langle\cdot,\cdot\rangle$ of $\mathfrak{so}_{3,2}$ in this basis are equal to ± 1 or 0. More precisely, using Table 1 in [14], one can check that the components of $\langle\cdot,\cdot\rangle$ in the basis $\mathcal{B}$ are

$$
\text{(A.2)}\quad
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&1&0\\
0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&-1&0&0&0\\
0&1&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0
\end{pmatrix}.
$$

A.2. The space $\ker\partial^*|_{C^2_1(\mathfrak{m},\mathfrak{g})}$

We now want to show that the action of the codifferential ∂^* of $\mathfrak{so}_{3,2}$ on the bilinear maps of shifting degree $+1$ in $\mathrm{Hom}(Ł^2\mathfrak{m}_-,\mathfrak{g})$, $\mathfrak{m}_- := \mathfrak{g}_{-2}+\mathfrak{g}_{-1}$, has trivial kernel. As above, we denote by $\langle\cdot,\cdot\rangle$ the Cartan-Killing form of $\mathfrak{so}_{3,2}$, by $\mathcal{B} = (f_\alpha)$ the basis defined in (A.1) and by $\mathcal{B}^* = (f^\alpha)$ its dual basis. Finally, for each element $f_\alpha \in \mathcal{B}$, we denote by $\widehat{f_\alpha}$ the unique element in $\mathcal{B}$ such that $f^\alpha = \pm\langle\widehat{f_\alpha},\cdot\rangle$ and by $\widehat{f}^\alpha$ the corresponding dual element in $\mathcal{B}^*$. The rest of the notation is taken from [14].
Consider a bilinear map $\tau \in \mathrm{Hom}(Ł^2\mathfrak{m}_-,\mathfrak{g})$ of shifting degree $+1$:

$$
\text{(A.3)}\quad \tau = \tau^1_{12} f_1 \otimes (f^1\wedge f^2) + \tau^1_{13} f_1\otimes(f^1\wedge f^3) + \tau^2_{23} f_2\otimes(f^2\wedge f^3) + \tau^3_{23} f_3\otimes(f^2\wedge f^3).
$$

By definition of ∂^*, this tensor is in $\ker\partial^*$ if and only if

$$
\text{(A.4)}\quad \langle\partial^*\tau, A\rangle = -\langle\tau,\partial A\rangle = 0
$$

for any $A = A^\alpha_\beta f_\alpha\otimes f^\beta \in \mathrm{Hom}(\mathfrak{h},\mathfrak{g})$. From (A.3), equation (A.4) is equivalent to a linear equation on the τ^i_{jk} whose non trivial coefficients are

$$
\widehat{f}^1(\partial A(\widehat{f}_1,\widehat{f}_2)), \quad \widehat{f}^1(\partial A(\widehat{f}_1,\widehat{f}_3)), \quad \widehat{f}^2(\partial A(\widehat{f}_2,\widehat{f}_3)), \quad \widehat{f}^3(\partial A(\widehat{f}_2,\widehat{f}_3)).
$$

The computation of $\widehat{f}^1(\partial A(\widehat{f}_1,\widehat{f}_2))$ is straightforward and gives

$$\begin{aligned}\widehat{f}^1(\partial A(\widehat{f}_1,\widehat{f}_2)) &= f^{10}(\partial A(f_{10},f_8)) = \\ &= f^{10}\Big(f_{10}\cdot A(f_8) - f_8\cdot A(f_{10}) - A([f_{10},f_8])\Big) = \\ &= A^\alpha_8 f^{10}(\mathrm{ad}_{f_{10}}(f_\alpha)) - A^\alpha_{10} f^{10}(\mathrm{ad}_{f_8}(f_\alpha)) - f^{10}(A([f_{10},f_8])) = \\ &= -\frac{1}{\sqrt{3}}A^6_8 + \frac{1}{\sqrt{6}}A^9_{10}.\end{aligned}$$

Similar computations give all other coefficients of the equations and (A.4) reduces to

$$\begin{aligned}&\tau^1_{12}\big(-\frac{1}{\sqrt{3}}A^6_8+\frac{1}{\sqrt{6}}A^9_{10}\big)+\tau^1_{13}\big(-\frac{1}{\sqrt{3}}A^6_9-\frac{1}{\sqrt{6}}A^8_{10}\big)+\\ &+\tau^2_{23}\big(-\frac{1}{2\sqrt{3}}A^6_9-\frac{1}{2\sqrt{3}}A^4_9+\frac{1}{2\sqrt{3}}A^7_8+\frac{1}{2\sqrt{3}}A^5_8+\frac{1}{\sqrt{6}}A^8_{10}\big)+\\ &+\tau^3_{23}\big(\frac{1}{2\sqrt{3}}A^7_9-\frac{1}{2\sqrt{3}}A^5_9+\frac{1}{2\sqrt{3}}A^6_8-\frac{1}{2\sqrt{3}}A^4_8+\frac{1}{\sqrt{6}}A^9_{10}\big)=0.\end{aligned} \tag{A.5}$$

By arbitrariness of A, it follows that $\tau\in\ker\partial^*$ if and only if $\tau^1_{12}=\tau^1_{13}=\tau^2_{23}=\tau^3_{23}=0$, meaning that $\ker\partial^*|_{C^2_1(\mathfrak{m},\mathfrak{g})}=0$, as claimed.

A.3. The space $(\partial\mathfrak{l}^1)^\perp$

We recall that, according to Lemma 6.5 in [14] and the definition of the (abelian) group L^1, the abelian Lie algebra $\mathfrak{l}^1 = Lie(L^1)$ can be identified with the real vector space generated by the linear maps

$$\begin{aligned}&B_1 := e_{-1(10)}\otimes e^{-2} + e_{-1(01)}\otimes e^{-2},\\ &B_2 := ie_{-1(10)}\otimes e^{-2} - ie_{-1(01)}\otimes e^{-2},\\ &B_3 := \big(e_{0(10)}-E_{0(01)}\big)\otimes e^{-1(10)} + \big(e_{0(01)}-E_{0(10)}\big)\otimes e^{-1(01)},\\ &B_4 := i\big(e_{0(10)}-E_{0(01)}\big)\otimes e^{-1(10)} - i\big(e_{0(01)}-E_{0(10)}\big)\otimes e^{-1(01)},\\ &B_5 := i\big(e_{0(10)}+E_{0(01)}\big)\otimes e^{-1(10)} - i\big(e_{0(01)}+E_{0(10)}\big)\otimes e^{-1(01)},\\ &B_6 := \big(e_{0(10)}+E_{0(01)}\big)\otimes e^{-1(10)} - \big(e_{0(01)}+E_{0(10)}\big)\otimes e^{-1(01)},\\ &B_7 := \big(E_{0(10)}+E_{0(01)}\big)\otimes e^{-1(10)} + \big(E_{0(10)}+E_{0(01)}\big)\otimes e^{-1(01)},\\ &B_8 := i\big(E_{0(10)}+E_{0(01)}\big)\otimes e^{-1(10)} - i\big(E_{0(10)}+E_{0(01)}\big)\otimes e^{-1(01)}.\end{aligned} \tag{A.6}$$

We also recall that the elements $\tau^1\in Tor^1(\mathfrak{m})$ have the form

$$\begin{aligned}\tau^1 = \tau^{-2}_{-2-1(10)}e_{-2}\otimes e^{-2}\wedge e^{-1(10)} + \overline{\tau^{-2}_{-2-1(10)}}e_{-2}\otimes e^{-2}\wedge e^{-1(01)}+\\ +\tau^{-1(10)}_{-1(10)-1(01)}e_{-1(10)}\otimes e^{-1(10)*}\wedge e^{-1(01)*}+\\ +\overline{\tau^{-1(01)}_{-1(10)-1(01)}}e_{-1(01)}\otimes e^{-1(10)*}\wedge e^{-1(01)*}.\end{aligned}$$

Now, let us choose as $\mathrm{ad}_{E_2^0}$-invariant inner product on the space $Tor^1(\mathfrak{m})$ the sum of an arbitrary inner product on $\mathfrak{m}_{-2}$ and the standard Hermitian inner product of $\mathbb{C}\simeq\mathfrak{m}_{-1}$. This implies that, in order to determine the subspace $(\partial\mathfrak{l}^1)^\perp\subset Tor^1(\mathfrak{m})$, the only relevant components of the generators ∂B_i of $\partial\mathfrak{l}^1$ are the components

$$(\partial B_i)^{-2}_{-2-1(10)} := e^{-2}\left(\partial B_i(e_{-2},e_{-1(10)})\right),$$
$$(\partial B_i)^{-1(10)}_{-1(10)-1(01)} := e^{-1(10)}\left(\partial B_i(e_{-1(10)},e_{-1(01)})\right).$$

We now observe that

$$(\partial B_3)^{-2}_{-2-1(10)} = e^{-2}\left([e_{-2},B_3(e_{-1(10)})]-[e_{-1(10)},B_3(e_{-2})]\right) = -1,$$
$$(\partial B_4)^{-2}_{-2-1(10)} = e^{-2}\left([e_{-2},B_4(e_{-1(10)})]-[e_{-1(10)},B_4(e_{-2})]\right) = -i,$$

meaning that $\partial\mathfrak{l}^1$ contains the 2-dimensional real subspace generated by

$$e_{-2}\otimes e^{-2}\wedge e^{-1(10)}+\overline{e_{-2}\otimes e^{-2}\wedge e^{-1(10)}},$$
$$ie_{-2}\otimes e^{-2}\wedge e^{-1(10)}-\overline{ie_{-2}\otimes e^{-2}\wedge e^{-1(10)}}.$$

This yields that if $\tau^1\in(\partial\mathfrak{l}^1)^\perp$, then $\tau^{-2}_{-2-1(10)}=\overline{\tau^{-2}_{-2-1(10)}}=0$. Similar computations show that

$$(\partial B_1)^{-1(10)}_{-1(10)-1(01)} = -\frac{i}{2},\qquad (\partial B_2)^{-1(01)}_{-1(10)-1(01)} = -\frac{1}{2},$$

hence that $\partial\mathfrak{l}^1$ contains the 2-dimensional real subspace generated by

$$e_{-1(10)}\otimes e^{-1(10)}\wedge e^{-1(01)}+\overline{e_{-1(10)}\otimes e^{-1(10)}\wedge e^{-1(01)}},$$
$$ie_{-1(10)}\otimes e^{-1(10)}\wedge e^{-1(01)}-\overline{ie_{-1(10)}\otimes e^{-1(10)}\wedge e^{-1(01)}},$$

and therefore that if $\tau^1\in(\partial\mathfrak{l}^1)^\perp$, then $\tau^{-1(10)}_{-1(10)-1(01)}=\overline{\tau^{-1(10)}_{-1(10)-1(01)}}=0$. We therefore conclude that $(\partial\mathfrak{l}^1)^\perp=0$.

Since $\ker\partial^*|_{C^2_1(\mathfrak{m},\mathfrak{g})}$ is trivial as well (see §A.2), condition (6.21) in [14] is equivalent to requiring that *the c-torsion c^1_K is identically equal to* 0.

A.4. The space $\ker\partial^*|_{C^2_2(\mathfrak{m}_-,\mathfrak{g})}$

Here, we want show that the space of the bilinear maps in $\mathrm{Hom}(Ł^2\mathfrak{m}_-,\mathfrak{g})$ of shifting degree $+2$ that are in $\ker\partial^*|_{C^2_2(\mathfrak{m},\mathfrak{g})}$, is trivial. This amount to say that condition (7.4) of [14] reduces to $c^2_K=0$.

A bilinear map $\tau\in\mathrm{Hom}(Ł^2\mathfrak{m}_-,\mathfrak{g})$ of shifting degree $+2$ has the form

$$\tau=\tau^2_{12}f_2\otimes(f^1\wedge f^2)+\tau^3_{12}f_3\otimes(f^1\wedge f^2)+\tau^2_{13}f_2\otimes(f^1\wedge f^3)+\tau^3_{13}f_3\otimes(f^1\wedge f^3)+$$
$$+\tau^4_{23}f_4\otimes(f^2\wedge f^3)+\tau^5_{23}f_5\otimes(f^2\wedge f^3)+\tau^6_{23}f_6\otimes(f^2\wedge f^3)+\tau^7_{23}f_7\otimes(f^2\wedge f^3). \tag{A.7}$$

As in §A.2, this tensor is in $\ker\partial^*$ if and only if $\langle\partial^*\tau, A\rangle = -\langle\tau, \partial A\rangle = 0$ for any element $A = A^{\alpha}_{\beta} f_{\alpha}\otimes f^{\beta} \in \mathrm{Hom}(\mathfrak{h},\mathfrak{g})$. By (A.7), this corresponds to a linear equation on the components τ^i_{jk} with coefficients

$$\widehat{f}^2(\partial A(\widehat{f}_1,\widehat{f}_2)),\quad \widehat{f}^3(\partial A(\widehat{f}_1,\widehat{f}_2)),\quad \widehat{f}^2(\partial A(\widehat{f}_1,\widehat{f}_3)),\quad \widehat{f}^3(\partial A(\widehat{f}_1,\widehat{f}_3)),$$
$$\widehat{f}^4(\partial A(\widehat{f}_2,\widehat{f}_3)),\quad \widehat{f}^5(\partial A(\widehat{f}_2,\widehat{f}_3)),\quad \widehat{f}^6(\partial A(\widehat{f}_2,\widehat{f}_3)),\quad \widehat{f}^7(\partial A(\widehat{f}_2,\widehat{f}_3)).$$

With the same computations of §A.2, we compute all these coefficients and get that the equation $\langle\tau, \partial A\rangle = 0$ has the explicit expression

$$\text{(A.8)}\quad \begin{aligned}&\tau^2_{12}\left(-\frac{1}{\sqrt{6}}A^3_8+\frac{1}{2\sqrt{3}}A^4_{10}+\frac{1}{2\sqrt{3}}A^6_{10}\right)+\tau^3_{12}\left(\frac{1}{\sqrt{6}}A^2_8+\frac{1}{2\sqrt{3}}A^5_{10}-\frac{1}{2\sqrt{3}}A^7_{10}\right)+\\&+\tau^2_{13}\left(-\frac{1}{\sqrt{6}}A^3_9+\frac{1}{2\sqrt{3}}A^5_{10}+\frac{1}{2\sqrt{3}}A^7_{10}\right)+\tau^3_{13}\left(\frac{1}{\sqrt{6}}A^2_9-\frac{1}{2\sqrt{3}}A^4_{10}+\frac{1}{2\sqrt{3}}A^6_{10}\right)+\\&+\tau^4_{23}\left(\frac{1}{2\sqrt{3}}A^2_9+\frac{1}{2\sqrt{3}}A^3_8+\frac{1}{\sqrt{6}}A^4_{10}\right)+\tau^5_{23}\left(\frac{1}{2\sqrt{3}}A^3_9-\frac{1}{2\sqrt{3}}A^2_8+\frac{1}{\sqrt{6}}A^5_{10}\right)+\\&+\tau^6_{23}\left(\frac{1}{2\sqrt{3}}A^2_9-\frac{1}{2\sqrt{3}}A^3_8+\frac{1}{\sqrt{6}}A^6_{10}\right)+\tau^7_{23}\left(-\frac{1}{2\sqrt{3}}A^3_9-\frac{1}{2\sqrt{3}}A^2_8-\frac{1}{\sqrt{6}}A^7_{10}\right)=0.\end{aligned}$$

Since this has to be satisfied for each A, factoring the components of A we get that $\tau\in\ker\partial^*|_{C^2_2(\mathfrak{m},\mathfrak{g})}$ if and only if its components satisfy the system

$$\begin{aligned}&\tau^3_{12}-\frac{1}{\sqrt{2}}\tau^5_{23}-\frac{1}{\sqrt{2}}\tau^7_{23}=0,\quad \tau^2_{12}-\frac{1}{\sqrt{2}}\tau^4_{23}+\frac{1}{\sqrt{2}}\tau^6_{23}=0,\\&\tau^3_{13}+\frac{1}{\sqrt{2}}\tau^4_{23}+\frac{1}{\sqrt{2}}\tau^6_{23}=0,\quad \tau^2_{13}-\frac{1}{\sqrt{2}}\tau^5_{23}+\frac{1}{\sqrt{2}}\tau^7_{23}=0,\\&\tau^4_{23}+\frac{1}{\sqrt{2}}\tau^2_{12}-\frac{1}{\sqrt{2}}\tau^3_{13}=0,\quad \tau^5_{23}+\frac{1}{\sqrt{2}}\tau^3_{12}+\frac{1}{\sqrt{2}}\tau^2_{13}=0,\\&\tau^6_{23}+\frac{1}{\sqrt{2}}\tau^2_{12}+\frac{1}{\sqrt{2}}\tau^3_{13}=0,\quad \tau^7_{23}+\frac{1}{\sqrt{2}}\tau^3_{12}-\frac{1}{\sqrt{2}}\tau^2_{13}=0.\end{aligned}$$

A simple check shows that this system has just the trivial solution. This means that $\ker\partial^*|_{C^2_2(\mathfrak{m},\mathfrak{g})}=0$ and that (7.4) of [14] is equivalent to $c^2_K = 0$.

A.5. The space $\ker\partial^*|_{C^2_3(\mathfrak{m}_-,\mathfrak{g})}$

In this section we determine explicitly the bilinear maps in $\mathrm{Hom}(Ł^2\mathfrak{m}_-,\mathfrak{g})$ of shifting degree $+3$ that are in $\ker\partial^*$.

A bilinear map $\tau\in\mathrm{Hom}(Ł^2\mathfrak{m}_-,\mathfrak{g})$ of shifting degree $+3$ has the form

$$\text{(A.9)}\quad \begin{aligned}\tau=\;&\tau^4_{12}f_4\otimes(f^1\wedge f^2)+\tau^5_{12}f_5\otimes(f^1\wedge f^2)+\tau^6_{12}f_6\otimes(f^1\wedge f^2)+\tau^7_{12}f_7\otimes(f^1\wedge f^2)+\\&+\tau^4_{13}f_4\otimes(f^1\wedge f^3)+\tau^5_{13}f_5\otimes(f^1\wedge f^3)+\tau^6_{13}f_6\otimes(f^1\wedge f^3)+\tau^7_{13}f_7\otimes(f^1\wedge f^3)+\\&+\tau^8_{23}f_8\otimes(f^2\wedge f^3)+\tau^9_{23}f_9\otimes(f^2\wedge f^3).\end{aligned}$$

As in the previous sections, this is in $\ker\partial^*$ if and only if $\langle\partial^*\tau, A\rangle = -\langle\tau, \partial A\rangle = 0$ for any $A = A^{\alpha}_{\beta} f_{\alpha} \otimes f^{\beta} \in \mathrm{Hom}(\mathfrak{h}, \mathfrak{g})$. From (A.9), this condition is a linear equation in the components τ^{i}_{jk} with coefficients

$$\widehat{f}^4(\partial A(\widehat{f}_1,\widehat{f}_2)),\quad \widehat{f}^5(\partial A(\widehat{f}_1,\widehat{f}_2)),\quad \widehat{f}^6(\partial A(\widehat{f}_1,\widehat{f}_2)),\quad \widehat{f}^7(\partial A(\widehat{f}_1,\widehat{f}_2)),$$
$$\widehat{f}^4(\partial A(\widehat{f}_1,\widehat{f}_3)),\quad \widehat{f}^5(\partial A(\widehat{f}_1,\widehat{f}_3)),\quad \widehat{f}^6(\partial A(\widehat{f}_1,\widehat{f}_3)),\quad \widehat{f}^7(\partial A(\widehat{f}_1,\widehat{f}_3)),$$
$$\widehat{f}^8(\partial A(\widehat{f}_2,\widehat{f}_3)),\quad \widehat{f}^9(\partial A(\widehat{f}_2,\widehat{f}_3)).$$

We explicitly compute these coefficients with the same standard computations indicated in §A.2. Then we obtain that the condition $\langle \tau, \partial A\rangle = 0$ has the explicit form

$$\begin{aligned}
&\tau^4_{12}\left(-\frac{1}{2\sqrt{3}}A^2_{10}\right)+\tau^5_{12}\left(-\frac{1}{2\sqrt{3}}A^3_{10}\right)+\tau^6_{12}\left(\frac{1}{\sqrt{3}}A^1_8-\frac{1}{2\sqrt{3}}A^2_{10}\right)+\\
&+\tau^7_{12}\left(\frac{1}{2\sqrt{3}}A^3_{10}\right)+\tau^4_{13}\left(\frac{1}{2\sqrt{3}}A^3_{10}\right)+\tau^5_{13}\left(-\frac{1}{2\sqrt{3}}A^2_{10}\right)+\\
&+\tau^6_{13}\left(\frac{1}{\sqrt{3}}A^1_9-\frac{1}{2\sqrt{3}}A^3_{10}\right)+\tau^7_{13}\left(-\frac{1}{2\sqrt{3}}A^2_{10}\right)+\\
&+\tau^8_{23}\left(\frac{1}{\sqrt{6}}A^1_8-\frac{1}{\sqrt{6}}A^2_{10}\right)+\tau^9_{23}\left(\frac{1}{\sqrt{6}}A^1_9-\frac{1}{\sqrt{6}}A^3_{10}\right)=0.
\end{aligned}\tag{A.10}$$

Since this needs to hold for each A, factoring the components of A we get that $\tau \in \ker\partial^*|_{C^2_3(\mathfrak{m},\mathfrak{g})}$ if and only if its components satisfy the system for

$$\begin{aligned}
&\frac{1}{2\sqrt{3}}\tau^4_{12}+\frac{1}{2\sqrt{3}}\tau^6_{12}+\frac{1}{2\sqrt{3}}\tau^5_{13}+\frac{1}{2\sqrt{3}}\tau^7_{13}-\frac{1}{\sqrt{6}}\tau^8_{23}=0,\\
&-\frac{1}{2\sqrt{3}}\tau^5_{12}+\frac{1}{2\sqrt{3}}\tau^7_{12}+\frac{1}{2\sqrt{3}}\tau^4_{13}-\frac{1}{2\sqrt{3}}\tau^6_{13}+\frac{1}{\sqrt{6}}\tau^9_{23}=0,\\
&\frac{1}{\sqrt{3}}\tau^6_{12}+\frac{1}{\sqrt{6}}\tau^8_{23}=0,\qquad \frac{1}{\sqrt{3}}\tau^6_{13}+\frac{1}{\sqrt{6}}\tau^9_{23}=0.
\end{aligned}$$

Using the last two equations to simplify the first two, the system reduces to

$$\begin{aligned}
&\tau^4_{12}+\tau^5_{13}=-3\tau^6_{12}-\tau^7_{13},\qquad \tau^4_{13}-\tau^5_{12}=3\tau^6_{13}-\tau^7_{12},\\
&\tau^8_{23}=-\sqrt{2}\,\tau^6_{12},\qquad \tau^9_{23}=-\sqrt{2}\,\tau^6_{13}.
\end{aligned}\tag{A.11}$$

This means that the space $\ker\partial^*|_{C^2_3(\mathfrak{m},\mathfrak{g})}$ is 6-dimensional and that condition (8.4) of [14] correspond to a system of linear equations on the curvature components

$$T^{0(10)}_{-2\,-1(10)}\ \ T^{0(10)}_{-2\,-1(01)},\ \ R^{0(10)}_{-2\,-1(10)},\ \ R^{0(10)}_{-2\,-1(01)},\ \ R^{1(10)}_{-1(10)\,-1(01)}.$$

In order to make explicit these equations, we have to convert the system (A.11) on the components of τ in the basis $\mathcal{B}$ into a system on the components of τ in the standard CR basis $\mathcal{B}^{CR}$. For this purpose, we recall that

$$\begin{aligned}
&e_{-2}=\sqrt{6}f_1,\quad e_{-1(10)}=\frac{\sqrt{6}}{2}(f_2-if_3),\quad e_{-1(01)}=\frac{\sqrt{6}}{2}(f_2+if_3),\\
&e_{0(10)}=\frac{\sqrt{12}}{2}(f_4-if_5),\quad e_{0(01)}=\frac{\sqrt{12}}{2}(f_4+if_5),\quad E_{0(10)}=\frac{\sqrt{12}}{2}(f_6-if_7),\\
&e_{0(01)}=\frac{\sqrt{12}}{2}(f_6+if_7),\quad E_{1(10)}=\frac{\sqrt{6}}{2}(f_8-if_9),\quad E_{1(01)}=\frac{\sqrt{6}}{2}(f_8+if_9)
\end{aligned}$$

and that, for the dual vectors,

$$e^{0(10)} = \frac{1}{\sqrt{12}}(f^4 + if^5), \quad e^{0(01)} = \frac{1}{\sqrt{12}}(f^4 - if^5), \quad E^{0(10)} = \frac{1}{\sqrt{12}}(f^6 + if^7),$$
$$E^{0(01)} = \frac{1}{\sqrt{12}}(f^6 - if^7), \quad E^{1(10)} = \frac{1}{\sqrt{6}}(f^8 + if^9), \quad E^{1(01)} = \frac{1}{\sqrt{6}}(f^8 - if^9).$$

From this we get that

(A.12)
$$T^{0(10)}_{-2\,-1(10)} = \frac{\sqrt{3}}{2}(f^4 + if^5)(\tau(f_1, f_2) - i\tau(f_1, f_3)) = \\ = \frac{\sqrt{3}}{2}(\tau^4_{12} + \tau^5_{13}) + i\frac{\sqrt{3}}{2}(-\tau^4_{13} + \tau^5_{12}),$$

(A.13)
$$T^{0(10)}_{-2\,-1(01)} = \frac{\sqrt{3}}{2}(f^4 + if^5)(\tau(f_1, f_2) + i\tau(f_1, f_3)) = \\ = \frac{\sqrt{3}}{2}(\tau^4_{12} - \tau^5_{13}) + i\frac{\sqrt{3}}{2}(\tau^4_{13} + \tau^5_{12}),$$

(A.14)
$$R^{0(10)}_{-2\,-1(10)} = \frac{\sqrt{3}}{2}(f^6 + if^7)(\tau(f_1, f_2) - i\tau(f_1, f_3)) = \\ = \frac{\sqrt{3}}{2}(\tau^6_{12} + \tau^7_{13}) + i\frac{\sqrt{3}}{2}(-\tau^6_{13} + \tau^7_{12}),$$

(A.15)
$$R^{0(10)}_{-2\,-1(01)} = \frac{\sqrt{3}}{2}(f^6 + if^7)(\tau(f_1, f_2) + i\tau(f_1, f_3)) = \\ = \frac{\sqrt{3}}{2}(\tau^6_{12} - \tau^7_{13}) + i\frac{\sqrt{3}}{2}(\tau^6_{13} + \tau^7_{12}),$$

(A.16)
$$R^{1(10)}_{-1(10)\,-1(01)} = \frac{\sqrt{3}}{\sqrt{2}}(f^8 + if^9)(i\tau(f_2, f_3)) = \sqrt{\frac{3}{2}}(\tau^8_{23} + i\tau^9_{23}).$$

From this we see that

$$\frac{1}{\sqrt{3}}(\overline{R^{0(10)}_{-2\,-1(10)}} + R^{0(10)}_{-2\,-1(01)}) = \tau^6_{12} + i\tau^6_{13}, \; -i\frac{\sqrt{2}}{\sqrt{3}}R^{1(10)}_{-1(10)\,-1(01)} = \tau^8_{23} + i\tau^9_{23},$$

which yields that the last equation in (A.11) is equivalent to

(A.17)
$$R^{1(10)}_{-1(10)\,-1(01)} = -i\overline{R^{0(10)}_{-2\,-1(10)}} - iR^{0(10)}_{-2\,-1(01)}.$$

On the other hand, since

$$\frac{2}{\sqrt{3}}T^{0(10)}_{-2\,-1(10)} = (\tau^4_{12} + \tau^5_{13}) - i(\tau^4_{13} - \tau^5_{12}),$$
$$\frac{2}{\sqrt{3}}\left(\overline{R^{0(10)}_{-2\,-1(10)}} + R^{0(10)}_{-2\,-1(01)}\right) + \frac{2}{\sqrt{3}}\overline{R^{0(10)}_{-2\,-1(10)}} = 3\tau^6_{12} + \tau^7_{13} + i(3\tau^6_{13} - \tau^7_{12}),$$

we immediately see that the first two equations in (A.11) are equivalent to

$$T^{0(10)}_{-2\,-1(10)} = -2\overline{R^{0(10)}_{-2\,-1(10)}} - R^{0(10)}_{-2\,-1(01)}. \tag{A.18}$$

Rearranging in an appropriate way the equations (A.17) and (A.18), we conclude that condition (8.4) in [14] is equivalent to the following equalities:

$$\begin{aligned} \overline{R^{0(10)}_{-2\,-1(10)}} &= -\frac{1}{2}T^{0(10)}_{-2\,-1(10)} - \frac{1}{2}R^{0(10)}_{-2\,-1(01)},\\ R^{1(10)}_{-1(10)\,-1(01)} &= \frac{i}{2}T^{0(10)}_{-2\,-1(10)} - \frac{i}{2}R^{0(10)}_{-2\,-1(01)}. \end{aligned} \tag{A.19}$$

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AMS Subject Classification: 32V05, 32V35, 53C10, 53C05

Costantino MEDORI,
Dipartimento di Matematica e Informatica,
Università di Parma,
Parma, ITALY
e-mail: `costantino.medori@unipr.it`

Andrea SPIRO,
Scuola di Scienze e Tecnologie,
Università di Camerino,
Camerino, ITALY
e-mail: `andrea.spiro@unicam.it`

Lavoro pervenuto in redazione il 26.10.2015 e accettato il 27.01.2016.